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Upper bounds in phase synchronous weak coherent chaotic attractors

M.S. Baptista*, T. Pereira, J. Kurths

Universität Potsdam, Institut für Physik Am Neuen Palais 10, D-14469 Potsdam, Germany

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Abstract

An approach is presented for coupled chaotic systems with weak coherent motion, from which we estimate the upper bound value for the absolute phase difference in phase synchronous states. This approach shows that synchronicity in phase implies synchronicity in the time of events, a characteristic explored to derive an equation to detect phase synchronization, based on the absolute difference between the time of these events. We demonstrate the potential use of this approach for the phase coherent and the funnel attractor of the Rössler system, as well as for the spiking/bursting Rulkov map.

Keywords: Chaotic phase synchronization; Phase of chaotic attractors

1. Introduction

This work deals with the phenomenon of phase synchronization (PS) [1,2] in coupled chaotic systems, which describes interacting systems that have a bounded phase difference, despite the fact that their amplitudes may be uncorrelated. PS was found in many natural and physical systems [1,2], being experimentally observed in electronic circuits [3], electrochemical oscillators [4], Chua's circuit [5], and spatio-temporal systems [6]. There is also evidence of PS in communication processes in the human brain [7,8] and neural networks [9].

In the case of two coupled systems, PS exists [1] if

$$|\phi_1 - q\phi_2| \leq \varrho, \tag{1}$$

where $\phi_{1,2}$ are the phases calculated from a projection of the attractor onto appropriate subspaces $\mathcal{X}_{1,2}$, in which the trajectory has coherent properties [1,10]. The rational constant q [11] is the frequency ratio between the average phase growing, and ϱ is a finite constant to be determined, bounded away from zero.

The purpose of this work is to give an upper bound value for the absolute phase difference in Eq. (1) in phase synchronous states, in terms of a defined phase [13]. This is equivalent to determining an inferior bound value for the constant ϱ. We show that this minimal value, namely $\langle r \rangle$, can be estimated as the average growing of the phase, calculated for typical trajectories, in one of the subspaces. Particularly, $\langle r \rangle = \langle W \rangle \times \langle T \rangle$, where $\langle W \rangle$ is the average angular frequency associated to a subspace $\mathcal{X}$, and $\langle T \rangle$ is the average returning time of trajectories in this same subspace, calculated from the recurrence of events of the chaotic trajectory. Similarly to periodic oscillating systems, in which it is valid to say that an angular frequency ω is related to the period T by $\omega = 2\pi/T$, for chaotic systems it is valid to say that $\langle W \rangle = \langle r \rangle / \langle T \rangle$.

In the derivation of the constant $\langle r \rangle$, we obtain a series of inequalities that can be used to check for the existence of PS. A particular interesting one is suitable for systems where the only available information is a series of time events. We also introduce the phase of a chaotic attractor to be given by the amount of rotation of the tangent vector of the flow.

These results are shown to be valid to coupled chaotic systems that present weakly coherent attractors. By weakly coherent attractors we mean, following Ref. [10], attractors in which it is possible to define a Poincaré section or a threshold that defines an event, such that for the time between two events

* Corresponding address: Institut fur Physik der Universität Potsdam, Arbeit gruppe Arbeitsgruppe Nichtlineare Dynamik, Am Neuen Palais 10, Haus 19, D-14469 Potsdam, Brandenburg, Germany. Tel.: +49 30 1977 1432; fax: +49 30 1977 1142.

E-mail address: murilo@agnld.uni-potsdam.de (M.S. Baptista).

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τ, it is true that $|\tau - \langle T\rangle| < \kappa$, where $\langle T\rangle$ is the average returning time between two successive events, and $\kappa < \langle T\rangle$ is a rather small constant. So, our results are extended to attractors whose trajectories might not have a clear rotation point, but still presenting a weak coherent property in the time between events, e.g. bursting/spiking dynamics.

For illustrating our ideas, we use two coupled Rössler oscillators in two situations: when they produce a phase coherent attractor and when they produce the funnel attractor. We also work with two coupled neuron models from the Rulkov map [14]. This last example was chosen because we want to demonstrate that PS can be detected by only knowing the time at which bursts occur (events).

2. A minimal bound for the constant ϱ

We start by developing some ideas to give a minimum bound for c in Eq. (1). For simplicity, we eliminate the rational constant q [11], given by $q = \frac{\langle W_1\rangle}{\langle W_2\rangle}$, by a changing of variable, $\phi_2(t)' = q\phi_2(t)$. With a slight abuse of notation, from now on, we omit the $'$ symbol in the phase. Note however that such a changing of variable does not change the fact that PS exists or not.

Having two oscillators S_1 and S_2 that are coupled forming a chaotic attractor, we define the subspaces $\mathcal{X}_j$ to be a special projection in the variables of $\S_j$. This projection is such that the attractor in these subspaces presents the coherent properties defined in [10], such that one can calculate the phase on them.

Next, we define a time series of events, where events here are the crossing of the trajectory to a given Poincaré section or the crossing of one variable to some threshold. Being τ_j^i the time at which the ith event happens in $\mathcal{X}_j$, we consider the average return time, $\langle T_j\rangle$, of the subspace $\mathcal{X}_j$ to be the average of time intervals $T_j^i = \tau_j^i - \tau_j^{i-1}$ between two events in $\mathcal{X}_j$, for N events. So,

$$\langle T_j\rangle = \frac{\sum_{i=1}^{N} T_j^i}{N} = \frac{\tau_j^N}{N}. \tag{2}$$

We introduce the phase as the amount of rotation of the unitary tangent vector, $\vec{\mathcal{A}}_j(t)$. Being $|\vec{\mathcal{A}}_j(t+\delta t) - \vec{\mathcal{A}}_j(t)|$ a a small displacement of the phase for the time interval δt, calculated on the subspaces $\mathcal{X}_j$, and making $\delta t \to 0$, we arrive at

$$\phi_j(t) = \int_0^t |\dot{\vec{\mathcal{A}}}_j| \mathrm{d}t. \tag{3}$$

So, $\phi_j(t)$ measures how much the tangent vector of the flow, projected on the subspaces $\mathcal{X}_j$, rotates in time. This equation also suggests that $|\dot{\vec{\mathcal{A}}}_j|$ can be seen as an angular frequency, more precisely $W_j = |\frac{\mathrm{d}\vec{\mathcal{A}}_j}{\mathrm{d}t}|$ [15] and the average angular frequency is simply

$$\langle W_j\rangle = \frac{1}{T}\int_0^T W_i \mathrm{d}t. \tag{4}$$

We introduce the quantity $r_1^i = \int_{\tau_2^i}^{\tau_2^{i+1}} W_1 \mathrm{d}t$, which is the evolution of the phase from the time τ_2^i (when the ith event happens in $\mathcal{X}_2$) until the time τ_2^{i+1} (when the $(i+1)$th event happens in $\mathcal{X}_2$). Thus, $\langle r_1\rangle = (\sum_{i=0}^{N} r_1^i)/N$, or in a continuous form, after the Nth event, this average is calculated as $\langle r_1\rangle = \sum_i \int_{\tau_2^i}^{\tau_2^{i+1}} W_1 \mathrm{d}t/N$ which is equal to

$$\langle r_1\rangle = \frac{\int_0^{\tau_2^N} W_1 \mathrm{d}t}{N}. \tag{5}$$

Using that, for $N \to \infty$, it is valid to say that $\tau_2^N \approxeq N\langle T_1\rangle$, in Eq. (4), for $T = \tau_1^N$, we have that $\langle r_1\rangle = \frac{1}{N\langle T_1\rangle}\int_0^{\tau_2^N} W_1 \mathrm{d}t = \frac{1}{\langle T_1\rangle}\left(\frac{1}{N}\int_0^{\tau_2^N} W_1 \mathrm{d}t\right)$, which using Eq. (5), can be written as

$$\langle W_1\rangle = \frac{\langle r_1\rangle}{\langle T_1\rangle}. \tag{6}$$

These calculations can be done for $\langle r_2\rangle$, however, if PS exists, i.e. Eq. (1) is satisfied, one should have that $\langle W_1\rangle = \langle W_2\rangle$, $\langle r_1\rangle = \langle r_2\rangle$, and $\langle T_1\rangle = \langle T_2\rangle$. Thus, in Eq. (6) we can use the index j.

Synchronicity of events: The number of events at a given time for synchronous oscillators is not always the same, but can differ by unity. This occurs because the Nth event in $\mathcal{X}_1$ and $\mathcal{X}_2$ may not be simultaneous, resulting in a difference of unity between the number N_1 and N_2 of events, in $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. So, we can say that the number of events in PS are always related by

$$|N_1(t) - N_2(t)| \le 1. \tag{7}$$

The inequality in Eq. (7) is another variant for the condition already used to detect phase synchronization [1]. In that equation, every time an event occurs, like the crossing of the trajectory through a threshold, the phase is assumed to grow 2π. And PS is considered to happen if the phase difference is always smaller than or equal to 2π. Note that Eq. (7) can also be used to detect synchronous events in maps, in the case an event can be well specified. As an example, one can observe the occurrence of local maxima in the trajectory [16].

Synchronicity in the time of events: Using Eq. (2) in Eq. (7), we arrive at:

$$\left|\sum_{i=0}^{N}(T_1^i - T_2^i)\right| \le \langle T_1\rangle. \tag{8}$$

This equation is related to the weak coherence in the dynamics. The more phase coherent the attractors are the more the amount $|\sum_i(T_1^i - T_2^i)|$ approaches to zero. As a consequence, the value $\langle T_1\rangle$ overestimates the maximum difference in the time intervals between events. To overcome this, we introduce a physical parameter, namely γ, which brings us information about the coherence of a specific system. Thus, we put Eq. (8) as

$$\left|\sum_{i=0}^{N}(T_1^i - T_2^i)\right| \le \gamma\langle T_1\rangle. \tag{9}$$

It is important to notice that γ also brings some information about the projection and about the section in which the events are defined, once the difference in the time intervals depends on the projection and on the Poincaré section definition.

Multiplying both sides of Eq. (9) by $\langle W_1\rangle$, we can relate the time of events with the averaging growing of the phase:

$$\langle W_1\rangle \left|\sum_{i=0}^{N}(T_1^i - T_2^i)\right| \leq \gamma\langle r_1\rangle. \tag{10}$$

Synchronicity of the phase: Next, we represent Eq. (1) at the time the Nth event happens in $\mathcal{X}_1$ by

$$\left|\sum_{i=0}^{N-1}(r_1^i - r_2^i) + \xi(N)\right| \leq \varrho, \tag{11}$$

where

$$\xi(N) = \int_{\tau_2^{N-1}}^{\tau_1^N} W_1 \mathrm{d}t - \int_{\tau_1^{N-1}}^{\tau_1^N} W_2 \mathrm{d}t. \tag{12}$$

The term $\sum_{i=0}^{N-1}(r_1^i - r_2^i)$ represents the phase in $\mathcal{X}_1$ at the moment the $(N-1)$th event happens in $\mathcal{X}_2$ minus the phase in $\mathcal{X}_2$ at the moment the $(N-1)$th event happens in $\mathcal{X}_1$. The term $\xi(N)$ represents the difference between the evolution of the phase from the event $N-1$ in $\mathcal{X}_2$ till the time at which the Nth event happens at the subspace $\mathcal{X}_1$, minus the evolution of the phase at the subspace $\mathcal{X}_2$ from the $(N-1)$th event in $\mathcal{X}_1$ until the time at which the event N happens in $\mathcal{X}_1$. This term establishes a bridge between the continuous-time formulation of the phase difference [Eq. (1)] and the phase difference between events.

From Eq. (10), one sees that the smaller (bigger) is the time difference $|T_1^i - T_2^i|$ the more (the less) synchronous is the system, which means that the phase difference $|r_1^i - r_2^i|$ also gets smaller (bigger). So, it is suggestive to consider that the difference $(r_1^i - r_2^i)$ is linearly related to $(T_1^i - T_2^i)$ as

$$(r_1^i - r_2^i) = \beta\langle W_1\rangle(T_1^i - T_2^i) + \sigma(i), \tag{13}$$

with β being a constant, and $\sigma(i)$ brings non-linear terms.

To obtain the value of the constant β in Eq. (13), we imagine that PS is about to be lost, by a small parameter change, and so $|T_2^N - T_1^N|$ approaches $\gamma\langle T_1\rangle$. Analogously, at this situation, the phase difference $|r_1^i - r_2^i|$ has the ability to grow one typical cycle, i.e., $\langle r_1\rangle$, and therefore the term $\sigma(i)$ in Eq. (13) becomes very small and can be neglected. Thus, from Eq. (13) we have that $\beta\langle W_1\rangle\gamma\langle T_1\rangle \approx \langle r_1\rangle$, and we arrive at $\beta \approx \frac{1}{\gamma}$. That means that the larger is the time difference between two events (γ), the smaller is the linear growing of the phase difference (β). In other words, the more non-coherent (coherent) the two coupled oscillators are, the smaller (larger) is the linear growing of the phase difference.

For coherent attractors, whose trajectories spiral around an equilibrium point, β is approximately 2. This result is discussed in the appendices. In Appendix A, we discuss how to construct maps using the time events τ_j^i, and in Appendix B, we explain how to use these maps in order to obtain that $\beta \approx 2$.

Knowing the constant β, we put Eq. (13) in Eq. (11), and we get that

$$\left|\beta\langle W_1\rangle \sum_{i=0}^{N-1}(T_1^i - T_2^i) + \sum_{i=0}^{N-1}\sigma(i) + \xi(N)\right| \leq \varrho. \tag{14}$$

Using the triangular inequality and the fact that ϱ at this moment is considered to be an arbitrary constant, with a threshold (minimal) value, we write that

$$\left|\beta\langle W_1\rangle \sum_{i=0}^{N-1}(T_1^i - T_2^i)\right| + \left|\sum_{i=0}^{N-1}\sigma(i) + \xi(N)\right| \leq \varrho. \tag{15}$$

Eq. (15) can be written as $|\sum_{i=0}^{N-1}\sigma(i) + \xi(N)| \leq \varrho - |\beta\langle W_1\rangle\sum_{i=0}^{N-1}(T_1^i - T_2^i)|$. At a specific event, may the term $|\sum_{i=0}^{N-1}\sigma(i) + \xi(N)|$ reach the permitted maximum value; this implies that the term $|\beta\langle W_1\rangle\sum_{i=0}^{N-1}(T_1^i - T_2^i)|$ gets close to zero. At this situation, $|\sum_{i=0}^{N-1}\sigma(i) + \xi(N)| \leq \varrho$. Using the same arguments we arrive at $|\beta\langle W_1\rangle\sum_{i=0}^{N-1}(T_1^i - T_2^i)| \leq \varrho$, which implies that $\langle r_1\rangle \leq \varrho$. Since $|\sum_{i=0}^{N-1}\sigma(i) + \max\xi(N)| \leq \varrho$ we also have straightforwardly that $|\sum_{i=0}^{N-1}\sigma(i)| \leq \varrho$.

These results show that the upper bound for the phase difference is given by the constant $\langle r_1\rangle = \langle W\rangle \times \langle T\rangle$. This means that the arbitrary constant ϱ in Eq. (1) is always greater than or equal to $\langle r_1\rangle$, in other words, $\langle r_1\rangle$ is our threshold. The physical meaning is obvious. If $\langle r_1\rangle$ is the bound for phase difference, given a number $\kappa \geq 1$, the value $\kappa\langle r_1\rangle$ is also a bound, but it is not a minimal one. Thus, we fix the constant ϱ as

$$\varrho = \langle r_1\rangle. \tag{16}$$

From Eqs. (15) and (1), we have the following inequalities

$$\left|\sum_{i=0}^{N-1}\sigma(i)\right| \leq \langle r_1\rangle \tag{17}$$

$$\beta\langle W_1\rangle\left|\sum_{i=0}^{N-1}(T_1^i - T_2^i)\right| \leq \langle r_1\rangle \tag{18}$$

$$\left|\sum_{i=0}^{N-1}(r_1^i - r_2^i)\right| \leq \langle r_1\rangle \tag{19}$$

$$|\phi_1(t) - \phi_2(t)| \leq \langle r_1\rangle. \tag{20}$$

If one wants to use the inequality in Eq. (20) [or Eq. (19)] to detect phase synchronization, it is required that the phase is an available information. For that, one needs to have access to a continuous measuring of at least one variable. The inconvenience of using this approach becomes evident when either one has an experimental system where the only available information is a time series of events, like the dripping faucet experiment [18], or the signal is so corrupted by noise that one can really only measure spikes in neurons [19]. In these two cases one should use the inequality in Eq. (9) [or Eq. (18)]. The only inconvenience in the use of this inequality is that one should be careful with the type of event chosen. If the specified event is the spiking times, one might not see PS in the bursting

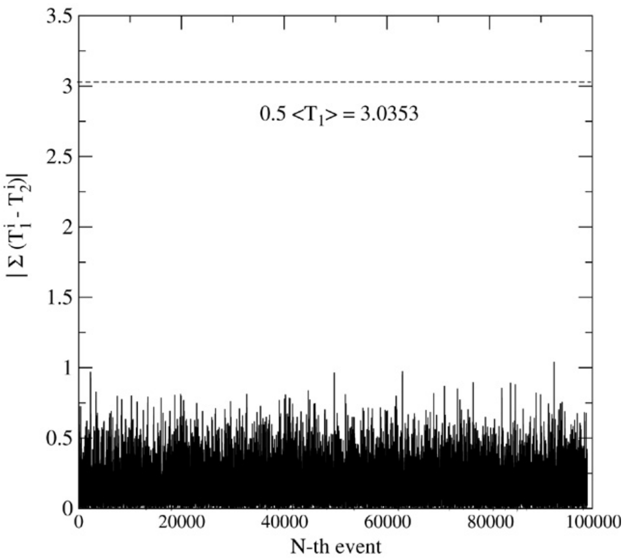

Fig. 1. The fluctuation $|\sum_{i=0}^{N}(T_1^i - T_2^i)|$, in Eq. (9). Note that $|\sum_{i=0}^{N}(T_1^i - T_2^i)| \le 0.5\langle T_1\rangle$. $\delta\alpha = 0.001$ and $\epsilon = 0.01$. The phase is calculated from Eq. (3).

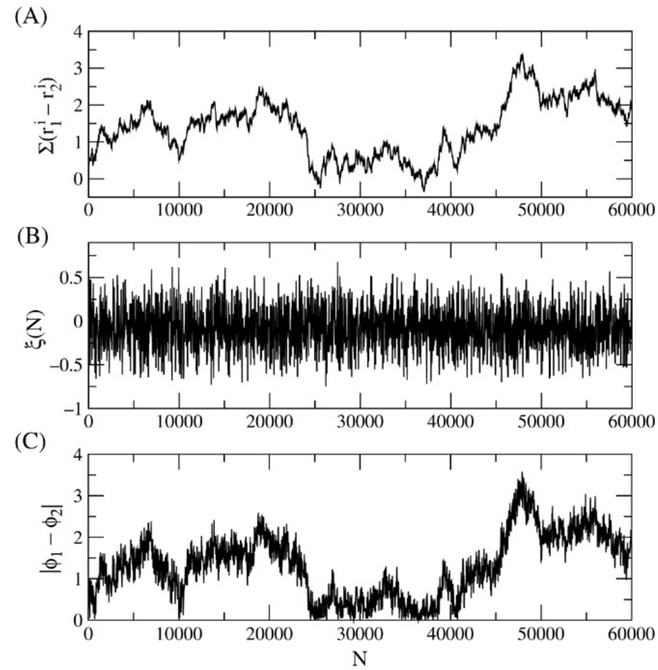

Fig. 2. In (A), we show the phase difference at the time the Nth event happens in both subsystems. In (B) we show $\xi(N)$, and in (C), we show the phase difference at the time that the Nth event happens in $\mathcal{X}_1$. So, the number of events in $\mathcal{X}_2$, N_2, can assume either one of the following values $(N-1, N, N+1)$. $\delta\alpha = 0.001$ or $\epsilon = 0.01$. The phase is calculated from Eq. (3).

time (and vice versa). In detecting PS in large networks, it might be computationally costly to check for all the phase difference or event time difference among each pair of subsystems. In this case, one could check the validation of inequality in Eq. (7), but having in mind that such a condition is only a necessary condition for PS.

3. Phase synchronization in the two coupled Rössler oscillators

To illustrate our approach we consider two non-identical coupled Rössler oscillators, given by

$$\begin{aligned} \dot{x}_{1,2} &= -\alpha_{1,2}y_{1,2} - z_{1,2} + \epsilon[x_{2,1} - x_{1,2}] \\ \dot{y}_{1,2} &= \alpha_{1,2}x_{1,2} + a y_{1,2} \\ \dot{z}_{1,2} &= b + z_{1,2}(x_{1,2} - d), \end{aligned} \tag{21}$$

with $\alpha_1 = 1$, and $\alpha_2 = \alpha_1 + \delta\alpha$. First, the constants $a = 0.15$, $b = 0.2$, and $d = 10$ are chosen such that we have a chaotic attractor in a phase coherent regime, whose attractor projections can be seen in Fig. A.1. The subspace where the phase is computed is given by $\mathcal{X}_1 = (x_1, y_1)$ and $\mathcal{X}_2 = (x_2, y_2)$. The time series that defines the events in $\mathcal{X}_j$, are defined as follows. τ_1^i is constructed measuring the time the trajectory crosses the plane $y_2 = 0$ in $\mathcal{X}_2$; τ_2^i is constructed measuring the time the trajectory crosses the plane $y_1 = 0$ in $\mathcal{X}_1$. In Fig. 1, we show the coupled Rössler oscillator for the parameters $\delta\alpha = 0.001$ and $\epsilon = 0.01$. We show that the inequality in Eq. (9) is always satisfied (for 10^5 pairs of events), i.e., $|\sum_{i=0}^{N}(T_1^i - T_2^i)| \le \langle T_1\rangle/2$, with $\langle T_1\rangle/2 = 3.0353$, and therefore, there is PS.

In Fig. 2(A), we show that the inequalities in Eqs. (17), (19) and (20) are satisfied and therefore, there is PS. In (A), we show the phase difference at the time the Nth event happens in both systems, i.e. the term $\sum_{i=0}^{N}(r_1^i - r_2^i)$ in Eq. (11). Note that the time that the Nth event happens in $\mathcal{X}_1$ is different from the time the Nth event happens in $\mathcal{X}_2$. In (B), we show $\xi(N)$ in Eq. (12), and in (C), we show the phase difference, at the time that the Nth event happens in $\mathcal{X}_1$. Note that the phase difference in (C) is just the phase difference for the same number of events [in (A)] plus the term $\xi(N)$ [in (B)].

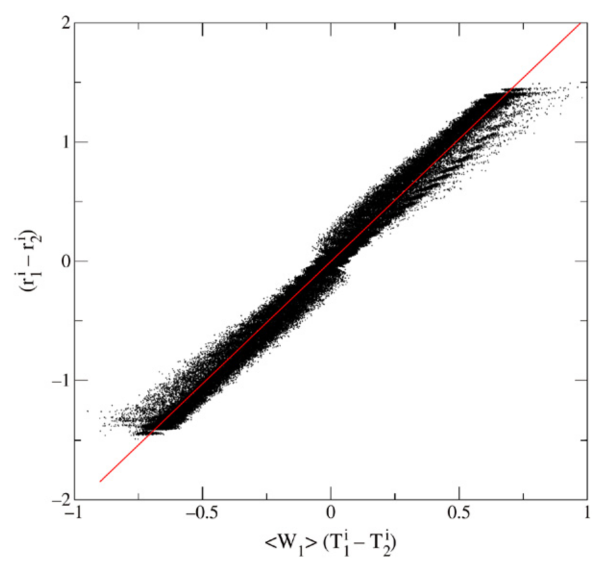

Fig. 3. The variable $(r_1^i - r_2^i)$ versus $\langle W_1\rangle(T_1^i - T_2^i)$. We find that $r_1^i - r_2^i \simeq \beta\langle W_1\rangle(T_1^i - T_2^i)$, with $\beta = 2.0512(3)$.

Then, we show in Fig. 3 that the linear hypothesis between $(r_1^i - r_2^i)$ and $\langle W_1\rangle(T_1^i - T_2^i)$ done in Eq. (13) stands and $\beta = 2.0512 \pm 0.0003$. If PS is not present, such a linear scale is not anymore found for the system considered.

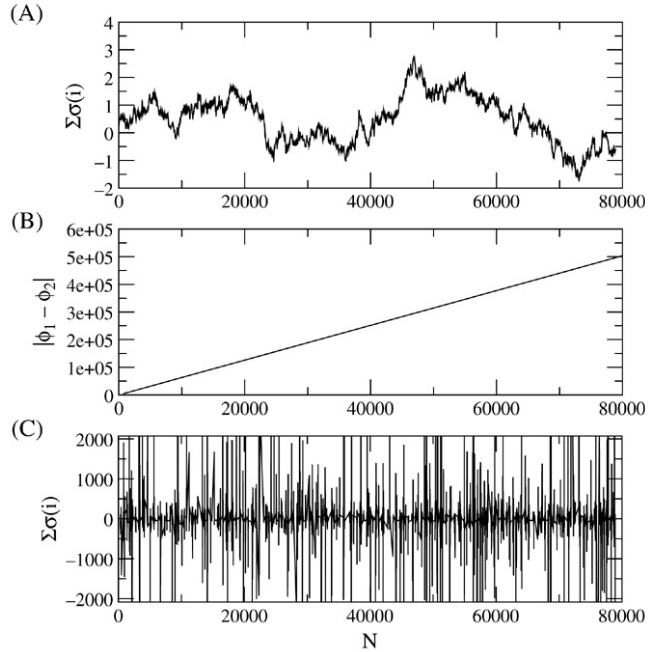

Fig. 4. In (A), we show the quantity $\sum \sigma$ in Eq. (17) for a situation when PS exists. As we decrease the coupling, Eq. (20) is not anymore satisfied as shown in (B), as well as Eq. (17), as shown in (C). In (C) we have made a zoom in of the vertical axis. In (A), $\delta\alpha = 0.001$ and $\epsilon = 0.01$ and in (B)–(C), $\delta\alpha = 0.001$ and $\epsilon = 0.000001$.

In Fig. 4(A), we show the quantity σ in Eq. (17) for a situation that PS exists. As we decrease the coupling, Eq. (20) is not anymore satisfied as shown in (B), as well as, Eq. (17). In (C) we make a zoom in of the vertical axis. Note the different nature of the fluctuations of the phase difference in (B) and the term $\sum \sigma$ in Eq. (17).

In order to compare the phase as defined in Eq. (3) (for $\delta\alpha = 0.001$ and $\epsilon = 0.01$), and the phase as defined in [2], e.g. $\theta = tg(y/x)$, we compare the function $\langle r_j \rangle$, as calculated for both definitions. For the phase, as defined in Eq. (3), we arrive at $\langle r_j \rangle = 6.2984$ and so, $\langle r_j \rangle > 2\pi$. Other quantities are $\langle W_j \rangle = 1.0375$, and $\langle T_j \rangle = 6.07097$. On the other hand, the phase as defined in [2] is a function that grows on average 2π each time the trajectory crosses some Poincaré section, which gives $\langle r_j \rangle = 2\pi$. So, the phase definitions arrive at two different quantities, but Eq. (20) is valid in order to detect PS and Eq. (6) is valid to measure the average angular frequency of the attractor, using both phase definitions.

To illustrate the generality of the phase definition in Eq. (3) in order to detect the phenomena of PS also in non-coherent attractors, we consider Eqs. (21) with the following set of parameters, $a = 0.3$, $b = 0.4$, $d = 7.5$, such that we have the funnel attractor shown in Fig. 5(A). This attractor has a non-coherent phase character [9,10]. For a parameter mismatch of $\delta\alpha = 0.0002$, and for a null coupling, $\epsilon = 0$, both Rössler oscillators (presenting the funnel attractor) are not phase synchronized as one can check in Fig. 6(A), which shows the absolute discrete phase difference in Eq. (19). At this situation, one has different angular frequencies $\langle W_1 \rangle = 1.2763$ and $\langle W_1 \rangle = 1.2994$, with the angular frequencies calculated from Eq. (4).

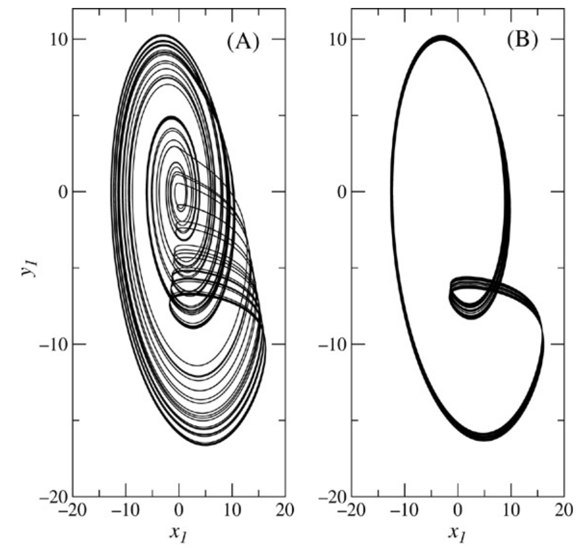

Fig. 5. Chaotic attractors projected on the variables x_1 and y_1. (A) The coupling is null and therefore, there is no PS. Here one sees the non-coherent funnel attractor. (B) The coupling induces PS, creating a coherent dynamics.

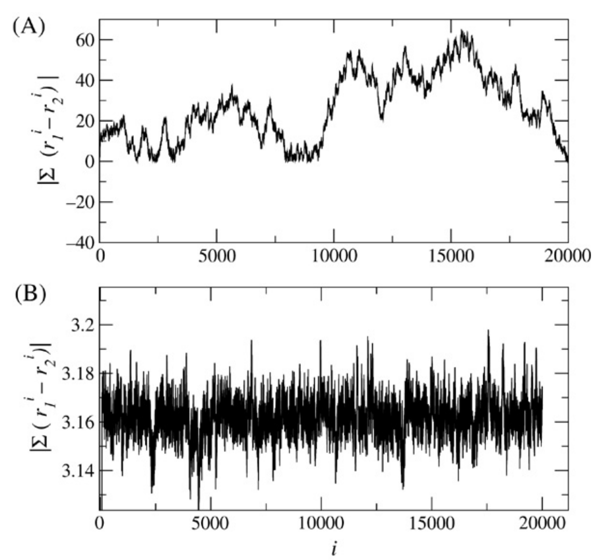

Fig. 6. Discrete phase difference $|\sum_{i=0}^{N-1}(r_1^i - r_2^i)|$ for no coupling (A) in which there is no PS and for a coupling $\epsilon = 0.00535$ (B) responsible to induce PS.

As we introduce the coupling $\epsilon = 0.00535$, the oscillators present PS with a weak coherent motion [see Figs. 5(B) and 6(B)], and as it should be expected, with equal angular frequencies $\langle W_1 \rangle = 1.4785$ and $\langle W_1 \rangle = 1.4785$, calculated with Eq. (4), i.e., using the phase as defined in Eq. (3). At the present situation, the attractor has a weak coherent character, and therefore, it is possible to measure the average time of recurrence, which is $\langle T_j \rangle = 8.4992\ldots$. Since $\langle r_j \rangle = 12.5667\ldots$, one can check that indeed, Eq. (6) is valid.

4. Phase synchronization in two coupled neurons

Now, we give an example for the detection of PS without knowledge of the state equations, but instead only using a

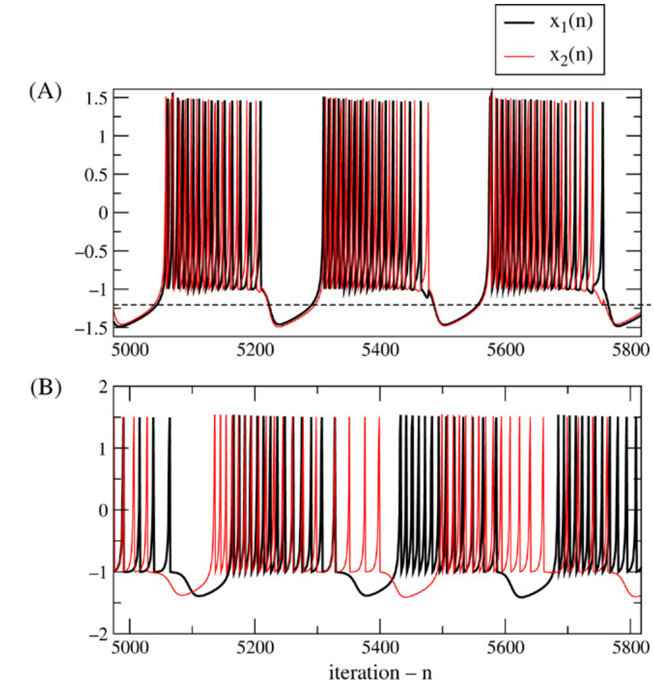

Fig. 7. A sample of the variables $x_1(n)$ and $x_2(n)$ from the subspaces that correspond to both neurons, for a situation where there is PS, for $\epsilon = 0.03$ (A), and for a situation where there is not PS, for $\epsilon = 0.001$ (B). In (A), Eq. (9) is satisfied, and in (B) it is not. In (A), we show three bursts, which are basically a sequence of spiking.

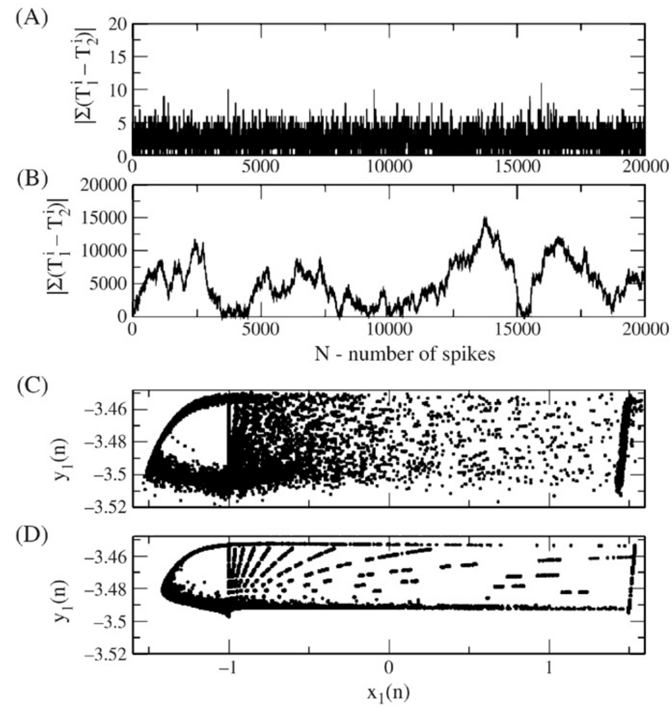

Fig. 8. In (A)–(B), we show the absolute difference between the time of the Nth burst, in both subspaces (that represent the two neurons). In (A), Eq. (9) is satisfied with $\langle T_1 \rangle = 259.028$ and, in (B), Eq. (9) is not satisfied (there is not PS). For (A) and (C), $\alpha_1 = 4.99$ and $\epsilon = 0.03$. For (B) and (D), $\alpha_1 = 4.99$ and $\epsilon = 0.001$.

time series of bursting events. We consider two non-identical coupled neurons described by the Rulkov map

$$x_j(n+1) = f[x_j(n), y_j(n) + \beta_j(n)] \tag{22}$$
$$y_j(n+1) = y_j(n) - \theta(x_j(n)+1) + \theta\sigma_j + \theta\beta_j(n), \tag{23}$$

which produces a chaotic attractor, for $\theta = 0.001$, $\alpha_2 = 5$, $\sigma_1 = 0.240$, and $\sigma_2 = 0.241$. The subspaces are defined as $\mathcal{X}_j = (x_j, y_j)$. $\beta_{1,2}(n) = g[x_{2,1}(n) - x_{1,2}(n)]$. The function f is given by

$$\begin{aligned} f &= \alpha_j/[1 - x_j(n)] + y_j(n), \quad x \le 0 \\ f &= \alpha_j + y_j(n), \quad 0 < x < \alpha_j(n) + y \\ f &= -1, \quad x \ge \alpha + y_j(n). \end{aligned} \tag{24}$$

The control parameters are α_1 and g, with $|\alpha_1 - \alpha_2|$ being the parameter mismatch and g the coupling amplitude (cf. [14]). The time at which events occur is defined by measuring the time instants in which the variable x_j, of subsystem $\mathcal{X}_j$, is equal to $x_j = -1.2$ (the event is the occurrence of a burst) [20], and N_j is the number of bursts of the subsystem $\mathcal{X}_j$. In this example, PS exists if Eq. (9) is satisfied, which also means that Eq. (7) is satisfied.

In Fig. 7, we show the variables $x_1(n)$ and $x_2(n)$ for a situation where there is PS (A), and for a situation where there is not PS (B). Note that in (A), although the neurons are phase synchronized, the difference between the number of bursts in the variable $x_1(n)$ minus the number of bursts in the variable $x_2(n)$ might be different to zero (for a short moment), as the hypothesis done in Eq. (7). In (A), we also represent by the dashed line the threshold, $x_j = -1.2$, from which the events are specified.

In Fig. 8(A)–(B), we show the absolute difference between the time of the Nth burst, in both neurons. In (A), Eq. (9) is satisfied (there is PS) with $\langle T_1 \rangle = 259.028$, and therefore $0.5\langle T_1 \rangle = 124.5014$, much bigger than the maximum fluctuation in (A). In (B), there is no PS. In (C) and (D), we show a projection of the attractor on the variables (x_1, y_1), for the parameters in (A) and (B) respectively.

Note that although the attractor of these neurons does not have the dynamics of a limit cycle, presenting a very complicated geometry in the phase space (as one can see in Fig. 8), it is still possible to well define events as well as the average period of the spiking times by the use of the threshold shown in Fig. 7, a characteristic that defines this attractor to be of the weak coherent type.

5. Conclusions

We estimate the upper bound value of the absolute phase difference between two coupled chaotic systems, in order to verify the existence of phase synchronization. Our approach shows that this bound value $\langle r \rangle$ is given by the average evolution of the phase, calculated in a subspace of the attractor, for a series of pairs of events in this same subspace. These events can be the number of local maxima or minima in the trajectory, the crossing of the trajectory to some Poincaré section, or the occurrence of a burst/spike.

The advantage of looking at the phase difference at these discrete times, instead of looking at the continuous phase difference, is that this approach allows us to detect phase synchronization by looking for a bounded time difference between events. This is helpful for chaotic systems from which there is no available information about the state equations, and therefore, this work is helpful in the experimental detection of PS in chaotic oscillators, in the laboratory or by data analyses.

In searching for a bound value of the absolute phase difference, we have shown that the larger is the time difference between two events (proportional to γ), the smaller is the linear growing of the phase difference (proportional β). In other words, the more non-coherent (coherent) two coupled chaotic oscillators are, the smaller (larger) is the linear growing of the phase difference.

All our results are extended to coupled chaotic systems that present weak coherent properties, i.e., it is possible to define an average time between two events $\langle T \rangle$, such that for each time interval between two events τ, it is true that $|\tau - \langle T \rangle| < \kappa$, with κ being a small constant.

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Appendix A. Constructing PS-sets from the event time series

The event time series τ_j^i can be used to construct maps of the attractor, whose geometrical properties states whether there is PS. These maps are constructed following simple rules:

- At the time τ_2^i, a point of the trajectory in $\mathcal{X}_1$ is collected.
- At the time τ_1^i, a point of the trajectory in $\mathcal{X}_2$ is collected.

So, as a result of measuring the trajectories in $\mathcal{X}_1$ (resp. $\mathcal{X}_2$) at the times τ_2^i (resp. τ_1^i) we have a discrete set of points $\mathcal{D}_1$ (resp. $\mathcal{D}_2$).

In PS, these sets $\mathcal{D}_j$ will be localized, not spreading out to the whole attractor. In this case $\mathcal{D}_j$ is called a PS-set. The theory for characterizing and constructing these sets is presented in [21]. In short, what happens is the following: when phase synchronization occurs, the time difference between events in both subspaces becomes smaller. As a consequence, the points in the $\mathcal{D}_j$ sets are more localized. For a non-synchronous phase dynamics, the sets $\mathcal{D}_j$ spreads over $\mathcal{X}_j$. Thus, detecting a PS-set offers an alternative way of detecting PS [22].

In the following, we give examples of the PS-sets in the coupled Rössler oscillators and in the Rulkov map, that mimic the neuronal dynamics presenting spiking/bursting behavior.

A.1. PS-sets for the coupled Rössler oscillators

The time series that defines the events in $\mathcal{X}_j$ are defined as follows. τ_1^i is obtained measuring the time the trajectory crosses the plane $y_2 = 0$ in $\mathcal{X}_2$. The discrete set of points is called $\mathcal{D}_2$. τ_2^i is obtained measuring the time the trajectory crosses the plane $y_1 = 0$ in $\mathcal{X}_1$. The discrete set of points is called $\mathcal{D}_1$.

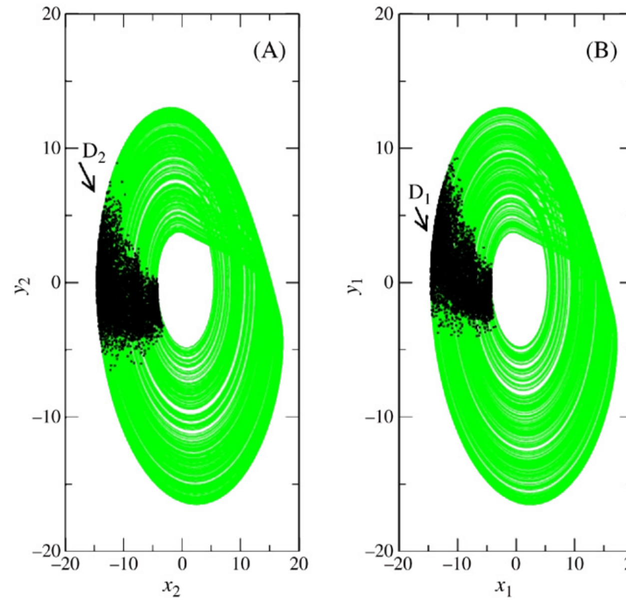

Fig. A.1. Bidimensional projection of the attractor (gray) and of the projections $\mathcal{D}_j$ of the PS-set (black) on the subspaces $\mathcal{X}_j$. The PS-set projection $\mathcal{D}_2$, in (A) and $\mathcal{D}_1$, in (B). $\delta\alpha = 0.001$ and $\epsilon = 0.01$.

In Fig. A.1, we show the coupled Rössler oscillators for a situation where PS exists. In this figure, we show bidimensional projections on the variables of subsystem $\mathcal{X}_2$ (A) and $\mathcal{X}_1$ (B). In gray, we show the attractor projection, and in black, projections of the PS-set $\mathcal{D}_2$ (A) and $\mathcal{D}_1$ (B). Note that the PS-sets, do not visit everywhere $\mathcal{X}_j$, rather are localized structures.

A.2. PS-sets for the coupled Rulkov map

In the neuronal dynamics it is not possible to define a Poincaré section, due to the non-coherence of the attractor. However, it is possible to define an event where the dynamics is weak coherent. This event is the ending or the beginning of the bursts, and in here we choose the beginning of the burst. Hence, we construct our time series by measuring the crossing of the trajectory with the threshold, $x_j = -1.2$.

In Fig. A.2 we show a projection of the attractor on the variables (x_1, y_1), in black points, and the subsets $\mathcal{D}_j$, in gray points. In (A), where we have phase synchronization the set $\mathcal{D}_1$ does not fulfill the whole attractor, but is rather localized, whereas in (B), where PS is not present the set $\mathcal{D}_1$ spreads over the attractor $\mathcal{X}_1$.

Appendix B. Estimation of β for coherent coupled oscillators

In this section, we explain why in coherent attractors, e.g. Rössler-type, the constant β is approximately 2.

That is so, because we compare the phase difference at the time events occurrence. Let us just remember that we are measuring the phase in one subsystem at the times that events in the other subsystem happen. Hence, at the time events happens in $\mathcal{X}_1$ [resp. $\mathcal{X}_2$], we collect points in $\mathcal{X}_2$ [resp. $\mathcal{X}_1$],

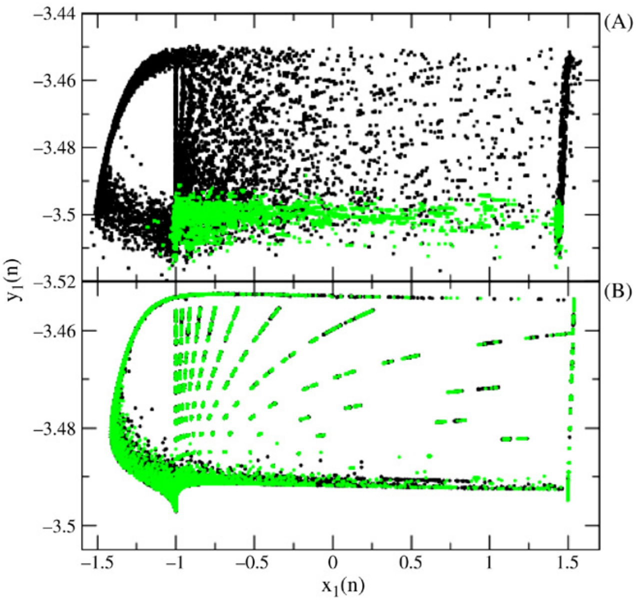

Fig. A.2. In (A), Eq. (9) is satisfied (there is PS), with $\langle T_j \rangle = 259.028$ and, in (B), Eq. (9) is not satisfied (there is not PS). While in (A) the set $\mathcal{D}_1$ (gray) is a PS-set, in (B) there is not a PS-set. In (A), $\alpha_1 = 4.99$ and $\epsilon = 0.03$ and in (B) $\alpha_1 = 4.99$ and $\epsilon = 0.001$.

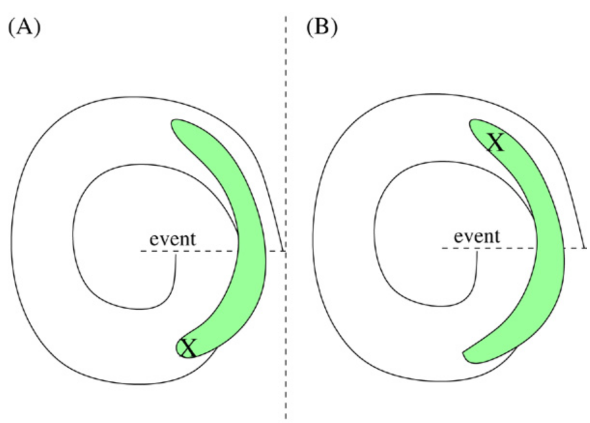

Fig. B.1. Pictorial visualization of a situation where PS exists. We represent by the full line in (A) the trajectory in the subspace $\mathcal{X}_1$, and (B) the trajectory in the subspace $\mathcal{X}_2$. The filled regions in (A) [(B)] represent trajectory positions at the time the Nth event happens in $\mathcal{X}_2$ [$\mathcal{X}_1$] for a situation when PS exists. An event is considered to happen when the trajectory crosses the dotted line.

obtaining the gray filled region in Fig. B.1(B) [resp. (A)], which represents a PS-set.

In particular, when the Nth event happens in $\mathcal{X}_2$, the trajectory on $\mathcal{X}_1$ is indicated by the cross in (A). At this time, τ_2^N, we record the phase in $\mathcal{X}_1$, namely $\phi_1(\tau_2^N)$. As the time goes on, the trajectory (in an counter-clockwise direction of rotation) in X reaches the event line in $\mathcal{X}_1$ at the time τ_1^N. At this time, the trajectory in $\mathcal{X}_2$ is at the cross in (B), and the phase is $\phi_2(\tau_1^N)$. Since these are typical events, we can say that $|\tau_2^N - \tau_1^N| \approx \langle T \rangle/4$, for the particular case represented in this figure. That is so because the time difference is approximately given by the time that the trajectory in $\mathcal{X}_1$ spends from the cross in (A) till the event line, which is approximately $1/4$ of the average period $\langle T \rangle$.

The phase difference, at which the same number N of events happen, is $|\phi_1(\tau_1^N) - \phi_1(\tau_2^N)| \approx \langle r \rangle/2$, since this phase difference is basically given by the displacement of the phase in $\mathcal{X}_1$ from the cross in (A) till the event line, plus the displacement of the phase in $\mathcal{X}_2$ from the event line till the cross in (B). But that is approximately given by $1/2$ of the average increasing of the phase $\langle r \rangle$, which was shown to be equal to $\langle W \rangle \times \langle T \rangle$. Therefore, $|\phi_1(\tau_1^N) - \phi_1(\tau_2^N)| \approx 2 \times |\tau_2^N - \tau_1^N|$, which consequently leads to $\beta \approx 2$.

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