Persistence of Network Synchronization under Nonidentical Coupling Functions[*]

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Abstract. We investigate the persistence of synchronization in networks of diffusively coupled oscillators when the coupling functions are nonidentical. Under mild conditions, we uncover the influence of the network interaction structure on the mismatches of the coupling function. We show that Erdös–Rényi random graphs support large perturbations in the coupling function. In contrast scale-free graphs do not allow large perturbations in the coupling function, that is, as the network size n goes to infinity it forces the coupling functions to be identical.

Key words. networks, synchronization, coupling function, persistence

AMS subject classifications. 34D06, 34D10

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1. Introduction. Recent empirical studies of real complex systems have led to a deep understanding of the structure of networks and of the coupling function. In particular, experimental findings revealed that the interaction between diffusively coupled oscillators can be mediated by different coupling functions [1]. For instance, the cardio-respiratory coupling function can be decomposed into a number of independent functions of a time-varying nature [2]. Moreover, different time-varying coupling functions can be used in the context of networks with time-varying topology [3]. The mathematical theory for synchronization in the presence of nonidentical coupling functions remains elusive [4, 5, 6]. A natural question in this context concerns the stability of the synchronized motion for nonidentical coupling functions.

We provide some conditions for the persistence of synchronized motion when the coupling functions are nonidentical. We consider undirected, simple, and connected networks; see Ref. [7] for details. The dynamics of the n identical oscillators diffusively coupled is described as

$$(1.1) \qquad \dot{x}_i = f(x_i) + \alpha \sum_{j=1}^{n} A_{ij} H_{ij}(t, x_j - x_i), \quad i = 1, \ldots, n,$$

where $\alpha > 0$ is the overall coupling strength, the adjacency matrix $A = (A_{ij})_{i,j=1}^{n}$ describes

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the interaction structure of the network ($A_{ij} = 1$ if i is connected to j and 0 otherwise), the function $f : \mathbb{R}^q \to \mathbb{R}^q$ describes the isolated node dynamics, and the family of coupling functions $H_{ij} : \mathbb{R}_+ \times \mathbb{R}^q \to \mathbb{R}^q$ describes an interaction akin to diffusion between nodes $i, j = 1, \ldots, n$.

Consider the mismatches between coupling functions $H_{ij} - H_{kl}$. Our main contribution provides sufficient conditions on the mismatches to guarantee stable synchronization. Loosely speaking, our results show that

– Erdös–Rényi (ER) networks support large mismatches;

– scale-free networks force the mismatches to converge to zero as $n \to \infty$.

In order words, the heterogeneity in the degrees $g_i = \sum_{j=1}^{n} A_{ij}$ determines the mismatches size. The precise statement of our results can be found in section 2 and the numerical results in section 3.

2. Main results. Since our main goal is to study the effect of the coupling function on the synchronization behavior, we keep the vector field f identical for all nodes. If the vector field is nonidentical we can use the approach developed in [5] to obtain the collective behavior of the model.

To fix notation, throughout the manuscript we use the norm $\|x_i\| = \max_j |x_{ij}|$ for $x_i = (x_{i1}, \ldots, x_{iq})$. When dealing with operators we use the induced operator norm. For instance, if $C = (C_{ij})_{i,j=1}^{n}$ is a matrix then $\|C\| = \max_i \sum_j |C_{ij}|$. When using the Euclidean norm, we represent it as $\| \cdot \|_2$. To state our results we proceed with some assumptions on the vector field f and coupling functions H_{ij}.

Assumption 1. The function f is continuous differentiable and there exists an inflowing invariant open ball $U \subset \mathbb{R}^q$ with

$$\|Df(x)\| \leq \varrho \quad \text{for all } x \in U$$

for some $\varrho > 0$.

Assumption 2 (coupling function perturbation). We assume that every coupling function can be factorized as

$$H_{ij}(t, x) = H(x) + \widetilde{P}_{ij}(t, x)$$

satisfying

1. $H_{ij}(t, 0) = 0$;
2. H is differentiable and $DH(0) = \Gamma$ has eigenvalues γ_i satisfying

$$(2.1) \qquad\qquad\qquad \gamma = \gamma(\Gamma) = \min_{1 \leq i \leq q} \Re(\gamma_i) > 0;$$

3. the perturbations $\tilde{P}_{ij} : \mathbb{R}_+ \times \mathbb{R}^q \to \mathbb{R}^q$ are continuous matrices satisfying

$$\widetilde{P}_{ij}(t, x) = P_{ij}(t)x,$$

$$(2.2) \qquad\qquad \sup_{t>0, x \in U} \|P_{ij}(t)\| \leq \delta \quad \text{for all } i, j \in \{1, \ldots, n\},$$

where U is given in Assumption 1.

Because of the diffusive nature of the coupling, if all oscillators start with the same initial condition, then the coupling term vanishes identically. This ensures that the globally synchronized state $x_i(t) = s(t)$ for all $i = 1, 2, \ldots, n$ is an invariant state for all coupling strengths α and all choices of coupling functions H_{ij}. We call the subset

$$\mathcal{S} := \{x_i \in U \subset \mathbb{R}^q \text{ for } i \in \{1, \ldots, n\} : x_1 = \cdots = x_n\}$$

the synchronization manifold. The local stability of $\mathcal{S}$ is determined by the spectral properties of the combinatorial Laplacian L. Consider the diagonal matrix $D = \operatorname{diag}(g_1, \ldots, g_n)$, where again $g_i = \sum_{j=1}^{n} A_{ij}$ denotes the degree of the vertex i. Then the Laplacian matrix reads $L = D - A$. As we are considering symmetric networks, the eigenvalues of L are all real and they can be arranged of the form

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n.$$

The second eigenvalue λ_2, known as algebraic connectivity of the graph, plays an important role in the stability analysis of the synchronization manifold. Our main result determines the perturbation size δ (in Assumption 2) in terms of the network structure. All proofs of the following results are placed in Appendix A.

Theorem 2.1 (persistence). *Consider the model in* (1.1) *satisfying Assumptions* 1 *and* 2 *on a connected network. Then, there exist constants* $\eta = \eta(f, \Gamma)$ *and* $K = K(\Gamma)$ *such that for all coupling strengths satisfying*

$$(2.3) \qquad \alpha > \frac{\eta}{\lambda_2 \gamma}$$

and perturbations of the coupling function satisfying

$$(2.4) \qquad \delta < \frac{\lambda_2 \gamma - \eta/\alpha}{K\|L\|},$$

where γ *is given by* (2.1) *and* $\lambda_2 = \lambda_2(L)$ *is the algebraic connectivity, the synchronization manifold is locally exponentially stable. That is, there exist constants* $\rho_0 > 0$ *and* $C > 0$ *such that if* $\|x_j(t_0) - x_i(t_0)\|_2 \leq \rho_0$ *for all* $i, j = 1, \ldots, n$, *then*

$$\|x_j(t) - x_i(t)\|_2 \leq Ce^{-\nu(t-t_0)}\|x_j(t_0) - x_i(t_0)\|_2$$

for all $t \geq t_0$ *and all* $i, j = 1, \ldots, n$, *where*

$$(2.5) \qquad \nu = \alpha \lambda_2 \gamma - \eta - \delta K\|L\| > 0.$$

One challenge to be overcome in the proof of the above result is to show that the constant K is independent of the network size. Hence, the network contribution can be factored in terms of the spectral gap λ_2 and the spectral radius via $\|L\|$. This is only true for undirected networks. For digraphs, K can depend badly on the network size n and no persistence result may be possible.

Notice that the parameter ν in (2.5) provides the decaying rate towards synchronization. Moreover, the decay rate given by (2.5) is proportionally related to λ_2. In the absence of coupling function perturbations, λ_2 dictates the speed of convergence towards synchronization; see, e.g., [8] for a discussion on the role of network structure on the convergence rate. Hence, our decay rate towards synchronization captures the structural properties of the network in terms of the spectral gap and the spectral radius $\|L\|$.

The perturbation slows down the synchronization by a factor proportional to δ. In other words, the perturbations on the coupling function slows down the convergence towards synchronization. If the perturbations are nonlinear then the speed can be kept the same as discussed in Theorem 2.1.

Remark 1. If $\widetilde{P}_{ij}(t, x)$ are nonlinear operators, in x, satisfying

$$\|\widetilde{P}_{ij}(t, x)\| \leq M\|x\|^{1+c}$$

for some uniform constant $M > 0$ and any $c > 0$, then δ can be taken arbitrarily small, that is, nonlinear perturbations do not affect the decay rate.

Theorem 2.1 ensures that the solutions of (1.1) with initial conditions in an open neighborhood of the synchronization manifold are attracted to it uniformly and exponentially fast. Moreover, it ensures that there is no synchronization loss or bubbling bifurcations [9, 10].

The persistent condition (2.4) relates the size of the mismatch to the network structure. We will explore the relation between network structure and δ in Corollaries 2.2 and 2.3. We relate δ to the graph structure for two important examples of complex networks:

– *Homogeneous networks*, where the disparity in the node degrees is small. A paradigmatic example is the ER random network: Starting with n nodes the graph is constructed by connecting nodes randomly. Each edge is included in the graph with probability p independent from every other edge. If $p \gg \log n / n$ then all degrees are nearly the same [11].
– *Heterogeneous networks*, where a few nodes are massively connected (theses nodes are called *hubs*) while most of the nodes have only a few connections. A typical example of such networks is the Barabási–Albert (BA) random tree. To construct the graph we start with a single edge. Then at each step, we start a new edge from one of the nodes created so far to a new node. The starting node is chosen at random in such a way that the probability to choose a given node is proportional to its degree.

Illustrations of ER random networks (homogeneous) and BA scale-free networks (heterogeneous) can be seen in Figure 1. These graphs are random so we want to characterize events in the large network limit. We say that an event holds asymptotically almost surely if the probability tends to 1 as $n \to \infty$.

Corollary 2.2 (ER networks). *Consider Theorem* 2.1 *for an ER network with* $p \gg \log n / n$. *Then asymptotically almost surely there exists* $K_0 = K_0(\Gamma)$ *such that in the limit of large coupling parameter* α *and network size* n *the perturbation condition* (2.4) *reduces to*

$$\delta < K_0.$$

Figure 1. *Illustrations of a BA network (left) and an ER network (right), both with $n = 100$. BA networks have a high heterogeneity in the node's degrees as represented by the four bold nodes. While the mean degree of such a BA network is around 4, each of the bold nodes has over 25 connections.*

That is, the perturbation size on ER networks can always be bounded away from zero. Hence, ER networks support large mismatches in the coupling function.

In contrast to homogeneous networks, heterogeneous networks such as BA networks will support increasingly smaller mismatches in the large n-limit.

Corollary 2.3 (BA networks). *Consider Theorem 2.1 for a BA scale-free network. Then asymptotically almost surely there exists $K_1 = K_1(\Gamma)$ such that in the limit of large coupling parameter α and network size n the perturbation condition (2.4) reduces to*

$$\delta < K_1 n^{-1/2}.$$

Now, turning back to Theorem 2.1, if further information is given on the behavior of the perturbations $P_{ij}(t)$ the bounds can be improved, that is, even if $\|P_{ij}(t)\|$ is large, synchronization can be attained. For instance, consider that the perturbations $P_{ij}(t)$ are taken from a family $\mathcal{P}$ of periodic matrix functions of one parameter ω (called frequency) and with mean value zero. For the sake of simplicity, let us omit the indexes ij in the next definition.

Definition 2.4 (fast oscillation). *We say that $P(t) \in \mathcal{P}$ oscillates fast if for any $c, h > 0$ there is a frequency $\omega_0 = \omega_0(c, h)$ such that for all $\omega > \omega_0$*

$$\left\| \int_{t_1}^{t_2} P(\omega t) dt \right\| \leq c \quad \text{for any } t_1 < t_2 < t_1 + h.$$

For this class of perturbation, synchronization is attained and the effect of $P_{ij}(t)$ can be neglected even it is large in magnitude.

Theorem 2.5 (fast limit). *Consider the model in* (1.1) *satisfying Assumptions* 1 *and* 2. *Regardless of the values of* $\sup_t \|P_{ij}(t)\| \leq \delta$, *if the perturbations* $P_{ij}(t) \in \mathcal{P}$ *oscillate fast enough then for all* α *satisfying* (2.3), *the synchronization manifold is locally exponentially stable. Moreover, the decaying rate towards synchronization is not affected.*

3. Illustrations. We present in this section two illustrations for the presented results. The first illustration is a simple but rich illustration that covers Theorems 2.1 and 2.5 and the second illustration approaches Corollaries 2.2 and 2.3. For both illustrations, we make use of the Lorenz system

$$\begin{aligned}
\dot{x}_{i1} &= 10(x_{i2} - x_{i1}), \\
\dot{x}_{i2} &= x_{i1}(28 - x_{i3}) - x_{i2}, \\
\dot{x}_{i3} &= x_{i1}x_{i2} - (8/3)x_{i3},
\end{aligned} \tag{3.1}$$

as the dynamics for the individual node $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t)) \in \mathbb{R}^3$.

The Lorenz system has an absorbing domain, that is, there is a compact subset $U \subset \mathbb{R}^3$ to which the solutions of (3.1) will converge [12]. Therefore, the solutions of this system exist globally and Assumption 1 follows. Moreover, inside U the system (3.1) is chaotic for the chosen parameters [13]. If the coupling parameter α is larger than the critical coupling equation (2.3) the Lorenz systems will synchronize and have a chaotic dynamics.

For the perturbed coupling functions we set

$$H_{ij}(t, x) = x + P_{ij}(t)x$$

such that H_{ij} is a perturbation of the identity, where

$$P_{ij}(t) = \delta \cos(t)\, \mathrm{R}_{ij}, \quad i, j \in \{1, \ldots, n\}, \tag{3.2}$$

where R_{ij} is a random matrix picked independently from an orthogonal Gaussian ensemble for each i and j normalized according to $\|\mathrm{R}_{ij}\| = 1$. By construction, $\sup_t \|P_{ij}(t)\| = \delta$ for every i and j, which agrees with Assumption 2 and makes δ a perturbation control parameter.

We numerically integrate (1.1) using the sixth order Runge–Kutta method with fixed integration step for all illustrations that follows. The initial conditions for each vector state $x_i(0)$ were, also for all experiments that follow, $x_i(0) = (-7, -10, 5) + \varepsilon_i$, where ε_i is a random variable in the interval $(0, 0.1)$ with a uniform distribution.

3.1. Synchronization tongue. Let us consider the first illustration, namely, when $n = 2$ (two coupled oscillators). Our Theorem 2.1 provides a *synchronization tongue*. The persistence condition (2.4) is of the form

$$\delta < c_1 - c_2/\alpha, \tag{3.3}$$

where $c_1 = \lambda_2 \gamma/(K\|L\|)$ and $c_2 = \eta/(K\|L\|)$.

Considering the Lorenz system (3.1) and perturbations according to (3.2) we perform the numerical computation of (1.1) for combinations of parameters α (coupling) and δ (perturbation) and compute the synchronization error $\|x_2(t) - x_1(t)\|$.
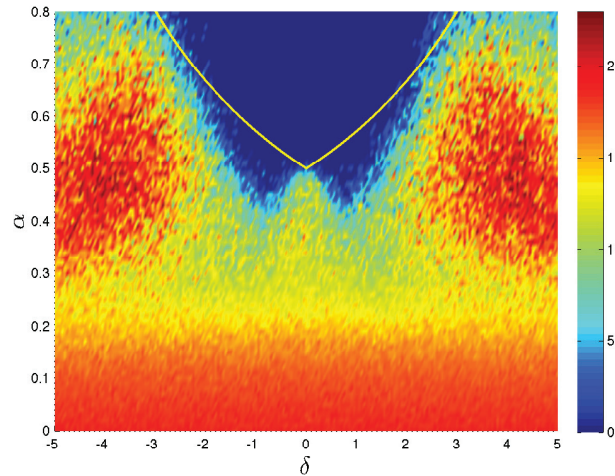
Figure 2. *Color map for the mean synchronization error. It has a total of* 201 *(horizontal)* × 101 *(vertical)* = 20301 *points. The color scale represents the mean synchronization error* E_a. *The yellow curve represents the theoretical bound* (2.4) *with outcome fitting* $\delta = 8 - 4/\alpha$.

We regard the first integration time $\tau = 1000$ as a transient and discard it. Then at the next time $T = 2000$ we compute the mean synchronization error

$$E(\alpha, \delta) = \frac{1}{T - \tau} \int_{\tau}^{T} \|x_2(t) - x_1(t)\| dt.$$

Moreover, for each fixed α and δ we average E over an ensemble of initial conditions (20 distinct initial conditions chosen uniformly as discussed above). We denote this averaged synchronization error by E_a. We use the triple (α, δ, E_a) to produce a color map where the color level represents the synchronization error E_a for the combination (α, δ).

The color map is depicted in Figure 2 and we call it a synchronization tongue because of its particular shape. Note that if $\delta = 0$ (no perturbation at all) we find that for $\alpha > 0.5$ the Lorenz oscillators synchronize. Using the theoretical equation (3.3) and using the data provided by the numerical simulation one obtain $\delta < 8 - 4/\alpha$. This equation is drawn in Figure 2 as a yellow solid line.

The synchronization tongue reveals a nontrivial transition to synchronization. The tongue appears to be a superposition of two synchronization tongues. This phenomenon is model dependent and it is not captured by our approach. We provide a sufficient condition on the local stability of the synchronization manifold, therefore with (2.5) we can guarantee that above the level given by this equation, the synchronization manifold is locally exponentially stable. Indeed, typically, sufficient conditions capture the scaling properties of the bounds (in terms of the network properties [14]) but it does not provide a sharp bound.

One possible way to tackle this effect and gain information on the tongue is to employ the master stability function approach [4]. This approach must be used with caution. It uses the theory of Lyapunov exponents and it may happen that the Lyapunov exponents are not continuous functions of the perturbation [15]. This implies that nonlinear perturbations
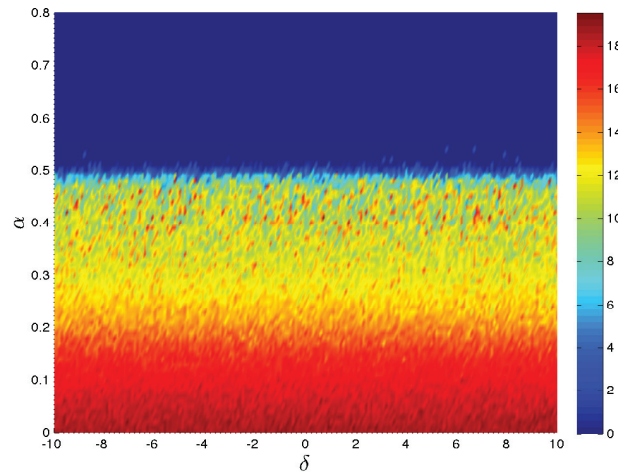
Figure 3. *Color map for the mean synchronization error for the fast oscillating coupling with $\omega = 1000$. The synchronization tongue flattens out meaning that coupled oscillators do not feel the oscillating perturbation in the coupling.*

can destroy the synchronization. Indeed, it can lead to loss of synchronization [10, 16] or incomplete synchronization [17].

3.2. Fast oscillations. Let us illustrate the behavior of the synchronization tongue when perturbations oscillate very fast. To this end we consider

$$(3.4) \qquad\qquad P_{ij}(t) = \delta \cos(\omega t)\, \mathrm{R}_{ij}, \quad i,j \in \{1,\ldots,n\},$$

with R_{ij} chosen as in the previous example. In the limit $\omega \gg 1$ the perturbations that we are considering (3.4) fulfill the fast oscillation condition (Definition 2.4).

Indeed, it easy to compute

$$\left\| \int_{t_1}^{t_2} P_{ij}(\omega t)dt \right\| = \left\| \int_{t_1}^{t_2} \delta \cos(\omega t)\, \mathrm{R}_{ij}\, dt \right\| = |\delta| \left| \frac{\sin(t_2\omega) - \sin(t_1\omega)}{\omega} \right| \le \frac{2|\delta|}{\omega}$$

for any $t_2 > t_1$. So, for any $c > 0$ there is an $\omega_0 = 2|\delta|/c$ so that for every $\omega > \omega_0$ we have

$$\left\| \int_{t_1}^{t_2} P_{ij}(\omega t)dt \right\| \le c \quad \text{for any } t_2 > t_1.$$

The color map of Figure 2 was produced using the perturbations in (3.4) with $\omega = 1$. Now, from Theorem 2.5, we know that in the large limit of ω the synchronization tongue in Figure 2 will flatten at the level $\alpha = 0.5$, which is the level that produces synchronization in a scenario of no perturbations in coupling function ($\delta = 0$). Figure 3 shows the numerical results of this property.

The result on fast oscillations is based on the fact that sufficiently fast oscillations of the mismatches in the coupling function "average out." Here we consider perturbations in the coupling function. If the network structure A_{ij} changes quickly in time, one can use averaging theory to obtain synchronization conditions [3, 18].
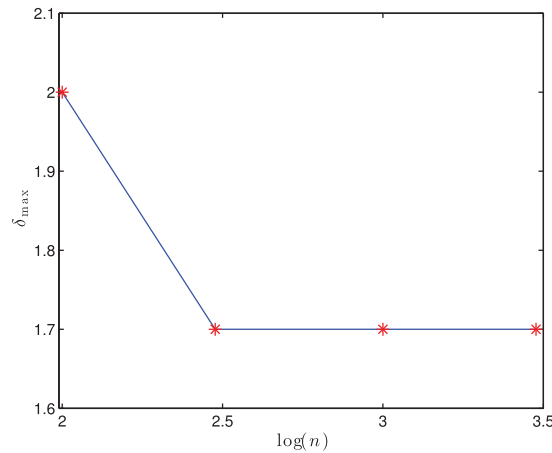
Figure 4. *Effect of the network size on perturbation size δ for ER random networks. In the large n-limit the ER network tolerates large mismatches. Moreover, the size of the mismatch does not decrease with the network size as predicted. The values of $\delta_{\max}$ have been rounded to the second decimal place.*

3.3. Network size effects on perturbations.

Corollaries 2.2 and 2.3 predict interesting system size effects on the perturbations. Here, we wish to illustrate these effects. Hence, we perform numerics experiments considering connected networks with n nodes, the Lorenz system (3.1) as a model of isolated dynamics, and coupling functions according to (3.4) with $\omega = 1$.

We determine the effect of the network size n on the perturbation norm δ as follows. For each fixed network size n we start the problem with large coupling α such that the system synchronizes at $\delta = 0$ (no perturbation). Then we increase δ in (3.4). When synchronization is lost at a given $\delta_{\max}$ we stop the simulation. That is, the value $\delta_{\max}$ stands for the maximal perturbation value that the network synchronization can bare before being destroyed for any $\delta > \delta_{\max}$. We consider that the synchronization is lost when the synchronization mean error $E_a > 10$.

Again, we perform numerical simulations using the sixth order Runge–Kutta method to evolve the dynamics of (1.1) using $\alpha = 5 \gg \eta/(\lambda_2\gamma)$. For each fixed n we use the fixed step size $\Delta\delta = 0.01$ to increase the values of δ. For each fixed value of δ we let (1.1) evolve for a transient time $\tau = 1000$ and then we compute the synchronization mean error E_a as before.

ER networks: We generated ER networks[1] with fixed probability $p = 0.3$ so that the assumptions in Corollary 2.2 holds for n large enough. Our numerical simulations show excellent agreement with Corollary 2.2—the effect of the network size n on δ is nearly constant for large n and $\delta_{\max}$ is always bounded away from zero, which can be seen in Figure 4 where $\log(n) = \log_{10}(n)$.

BA scale-free networks: Corollary 2.3 says that, if one perturbs the coupling function, it will be hard to synchronize BA scale-free networks in the large limit of α and n because in

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[1]We used the software Network Workbench (NWB) to generate all networks used in this paper. NWB is free and it is available at nwb.cns.iu.edu.
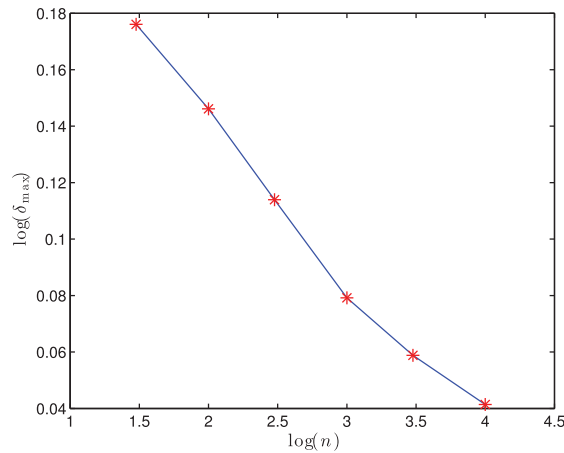
Figure 5. *Effect of the network size on the perturbation size δ for BA scale-free networks. As predicted in the large n-limit BA networks do not tolerate large perturbations. In fact, it forces the perturbations to decay.*

this case we have

$$\delta < O(n^{-1/2}).$$

To check this prediction we generated BA scale-free networks with parameter $m_0 = 2$ (links set by new node) unchanged for each network of size n. The effect of n on δ, as expected from Corollary 2.3 can be observed in Figure 5 where, again, $\log = \log_{10}$.

Performing a fitting on the data of Figure 5 we obtain

$$\delta_{\max} \propto n^{-\beta}, \text{ with } \beta \approx 0.1.$$

Our conditions on δ for the persistence of the network synchronization are sufficient conditions. Regarding BA networks, our experiments have shown that the actual decay may be slower than the predicted one, that is slower than $n^{-1/2}$.

It remains an open question how to obtain conditions that are sufficient and necessary. These experiments suggest that the relation between sufficient and necessary conditions may be nontrivial.

Appendix A. The proofs. The proof of Theorem 2.1 is given in terms of perturbation theory, in particular, the roughness of exponential dichotomies by combining results from [5] and [19]. More precisely, first we consider the case $\delta = 0$ and we use results from [5] to give conditions on α. Then we consider the case when $\delta \neq 0$ via the roughness of exponential dichotomies. Theorem 2.5 is also a corollary of Theorem 2.1 when considering the fast oscillation Definition 2.4.

We start with some auxiliary results. The results of Lemma A.1 are valid for any operator norm.

Lemma A.1. *Suppose $A(t)$ is a bounded continuous matrix function on an interval J and the evolution operator Φ of the equation*

(A.1) $$\dot{X} = A(t)X$$

satisfies the inequality

(A.2)
$$\|\Phi(t, t_0)\| \leq K e^{-\nu(t-t_0)} \quad \text{for } t \geq t_0.$$

If $B(t)$ is a continuous matrix function such that $\|B(t)\| \leq \theta$ for all $t \in J$ then the evolution operator Ψ of the perturbed equation

(A.3)
$$\dot{Y} = [A(t) + B(t)]Y$$

satisfies the inequality

$$\|\Psi(t, t_0)\| \leq K e^{\beta(t-t_0)} \quad \text{for } t \geq t_0,$$

with

(A.4)
$$\beta = -\nu + \theta.$$

The Lemma A.1 is classical and its proof can be found in [19].

Next, we wish to obtain variational equations for perturbations in a vicinity of the synchronization manifold. To this end, we write the solutions of (1.1) in block form

$$X(t) = \text{vec}[x_1(t), x_2(t), \ldots, x_n(t)] \in \mathbb{R}^{nq},$$

where vec denotes the stacking of the columns of $[x_1(t), x_2(t), \ldots, x_n(t)]$ into one long vector [20]. Likewise, one can rewrite the whole vector field as

$$F(X) = \text{vec}[f(x_1), f(x_2), \ldots, f(x_n)],$$

where $F : \mathbb{R}^{nq} \to \mathbb{R}^{nq}$. Near to the synchronization manifold, one can rewrite the vector solution $X(t) \in \mathbb{R}^{nq}$ as

$$X(t) = \mathbf{1} \otimes s(t) + \xi(t),$$

where the vector $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ is the eigenvector of the Laplacian matrix L associated with the eigenvalue 0, $\otimes$ stands for the Kronecker product, $s(t)$ is the synchronous solution satisfying $\dot{s} = f(s)$, and $\xi(t) = \text{vec}[\xi_i(t), \ldots, \xi_n(t)] \in \mathbb{R}^{nq}$ is a perturbation of the synchronized state.

Consider (1.1) with linearizations in the vector field and coupling function near to the synchronous manifold, that is, writing $x_i(t) = s(t) + \xi_i(t)$ we have

(A.5)
$$\dot{\xi}_i = Df(s)\xi_i + \alpha \sum_{j=1}^{n} A_{ij}(\Gamma + P_{ij}(t))(\xi_j - \xi_i) + R(\xi), \quad i = 1, \ldots, n,$$

where $Df(s(t))$ is the Jacobian matrix of the isolated vector field f along the synchronous solution $s(t)$ and $R : \mathbb{R}^{nq} \to \mathbb{R}^q$ is such that $\|R(\xi)\| = O(\|\xi\|^2)$ stands for the Taylor remainder of the expansions of the vector field and the coupling function. As we are considering the local stability of the synchronous solution we regard $\|R(\xi)\|$ as being so small that we can neglect it. Then, putting (A.5) into block form, the following Lemma A.2 holds.

Lemma A.2. *Near to the synchronous manifold the first variational equation of ξ is*

(A.6) $\dot{\xi} = [\mathrm{I}_n \otimes Df(s(t)) - \alpha(L \otimes \Gamma) + \alpha P(t)]\xi,$

where $P : \mathbb{R} \times \mathbb{R}^{nq} \to \mathbb{R}^{nq}$ satisfies

$$\|P(t)\| \le \|L\|\delta,$$

where δ is given according to Assumption 2.

Proof of Lemma A.2. We will omit the dependency in t of $P_{ij}(t)$ for the sake of simplicity. Using the fact that the elements of the Laplacian matrix read $L_{ij} = \delta_{ij}g_i - A_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, the network model (A.5) reads

(A.7) $\dot{\xi}_i = Df(s)\xi_i + \alpha \sum_{j=1}^{n}(\delta_{ij}g_i - L_{ij})(\Gamma + P_{ij})(\xi_j - \xi_i).$

Note that all diagonal terms $\delta_{ij}g_i$ vanish because when $i = j$ then $\xi_j - \xi_i = 0$. Therefore, (A.7) can be written in terms of the Laplacian matrix and we have

$$\dot{\xi}_i = Df(s)\xi_i - \alpha \sum_{j=1}^{n} L_{ij}\Gamma(\xi_j - \xi_i) - \alpha \sum_{j=1}^{n} L_{ij}P_{ij}(\xi_j - \xi_i)$$

(A.8) $= Df(s)\xi_i \underbrace{- \alpha \sum_{j=1}^{n} L_{ij}\Gamma(\xi_j)}_{I} + \underbrace{\alpha \sum_{j=1}^{n} L_{ij}\Gamma(\xi_i)}_{=0} \underbrace{- \alpha \sum_{j=1}^{n} L_{ij}P_{ij}(\xi_j)}_{II} + \underbrace{\alpha \sum_{j=1}^{n} L_{ij}P_{ij}(\xi_i)}_{III}.$

In block form, each portion $Df(s)\xi_i$ stands for the ith block of $[\mathrm{I}_n \otimes Df(s)]\xi$. It is easy to see that portion I of (A.8) stands for the ith block of $(L \otimes \Gamma)\xi$. For portion II, note that it stands for the ith block of

$$-\alpha \begin{pmatrix} L_{11}P_{11} & L_{12}P_{12} & \cdots & L_{1n}P_{1n} \\ L_{21}P_{21} & L_{22}P_{22} & \cdots & L_{2n}P_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ L_{i1}P_{i1} & L_{i2}P_{i2} & \cdots & L_{in}P_{in} \\ \vdots & \vdots & \cdots & \vdots \\ L_{n1}P_{n1} & L_{n2}P_{n2} & \cdots & L_{nn}P_{nn} \end{pmatrix}_{nq \times nq} \xi.$$

For portion III, note that it stands for the ith block of

$$\alpha \begin{pmatrix} \sum_{j=1}^{n} L_{1j}P_{1j} & 0 & \cdots & & 0 \\ 0 & \ddots & \cdots & & 0 \\ \vdots & \vdots & \sum_{j=1}^{n} L_{ij}P_{ij} & & \vdots \\ \vdots & \vdots & \cdots & & \ddots \\ 0 & 0 & \cdots & & \sum_{j=1}^{n} L_{nj}P_{nj} \end{pmatrix}_{nq \times nq} \xi.$$

Therefore, adding up the all the portions we end up with the first variational equation for ξ (A.6), where $P(t)$ is a Laplacian-like big perturbation matrix.

Regarding the computation of $\|P(t)\|$ we have

$$\|P(t)\| = 2\max_i \left\| \sum_{j=1, j\neq i}^n L_{ij} P_{ij}(t) \right\| \leq \left(2\max_i \sum_{j=1, j\neq i}^n |L_{ij}| \right) \sup_t \|P_{ij}(t)\| = \|L\|\delta,$$

where $\sup_t \|P_{ij}(t)\| \leq \delta$ according to Assumption 2. ■

A.1. The proof of Theorem 2.1 (persistence). The aim now is to give conditions on α so that the trivial solution $\xi(t) = 0$ of (A.6) is exponentially stable. This can be achieved in terms of exponential dichotomies. The case when $P(t) = 0$ was already studied in [5].

Now, we split the proof of Theorem 2.1 into two steps. In Step 1, we check that the assumptions of Theorem 2.1 satisfy the hypothesis of [5, Theorem 1] (when $P(t) = 0$); we also discuss the dichotomy parameters and in Step 2 we use the persistence Lemma A.1 to conclude the result.

Step 1 (*estimates on dichotomy parameters*): Let us consider the case with no perturbation on the coupling function, that is, $P(t) = 0$. As we said, this case was already studied in [5]. For completeness we discuss the main steps. Consider the variational equation

$$\dot{\varphi} = [\mathrm{I}_n \otimes Df(s(t)) - \alpha(L \otimes \Gamma)]\varphi.$$

Since, L is undirected it also has a diagonal representation $L = R^{-1}\Lambda R$. In this setting the change of coordinates

$$\varphi = [R \otimes I_q]^{-1}\zeta$$

block diagonalizes the variational equation

$$\dot{\zeta} = \bigoplus_i (Df(s(t)) - \alpha\lambda_i\Gamma)\zeta,$$

and since φ is not parallel to the synchronization manifold, the eigenvalue $\lambda_1 = 0$ does not contribute to the evolution of ζ. In [5] it was shown that under Assumption 1 if one defines

$$\sigma = \min_{1\leq i\leq q,\, 2\leq j\leq n} \Re(\lambda_j\gamma_i) > 0,$$

and considers the coupling strength threshold given by

(A.9) $$\alpha > \frac{\eta}{\sigma}$$

then

$$\|\Phi_\zeta(t, t_0)\|_2 \leq K e^{-(\alpha\sigma-\eta)(t-t_0)},$$

where Φ_ζ is the evolution operator of ζ, $K = K(\Gamma)$ is a constant independent of the network (because of the block structure of the equation), and $\eta = \eta(\Gamma, f)$. For the evolution operator for original variables, φ reads as

$$\Phi_\varphi(t, t_0) = [R \otimes I_q]\Phi_\zeta(t, t_0)[R \otimes I_q]^{-1},$$

hence

$$\|\Phi_\varphi(t, t_0)\|_2 \leq \kappa_2(R \otimes I_q)\|\Phi_\zeta(t, t_0)\|_2,$$

where κ is the condition number. Since $\kappa_2(R \otimes I_q) = \kappa_2(R)$ and as R is orthogonal $\kappa_2(R) = 1$ we obtain

$$\|\Phi_\varphi(t, t_0)\|_2 \leq Ke^{-(\alpha\sigma - \eta)(t - t_0)},$$

where K is independent of the network structure. So for every α above the threshold the synchronization manifold is locally exponentially stable.

In our setting we need to check that

$$\sigma = \min_{1 \leq i \leq q,\, 2 \leq j \leq n} \Re(\lambda_j \gamma_i) > 0.$$

Note that we are considering only symmetric and connected networks, so the Laplacian matrix L itself is symmetric and its eigenvalues can be ordered as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ and they are real. So,

$$\sigma = \min_{1 \leq i \leq q,\, 2 \leq j \leq n} \lambda_j \Re(\gamma_i) = \lambda_2 \min_{1 \leq i \leq q} \Re(\gamma_i).$$

Furthermore, we are considering Assumption 2, so $\gamma = \gamma(\Gamma) = \min_{1 \leq i \leq q} \Re(\gamma_i) > 0$, therefore

$$\sigma = \min_{1 \leq i \leq q,\, 2 \leq j \leq n} \Re(\gamma_i \lambda_j) = \lambda_2 \gamma > 0$$

and (A.9) translates to

(A.10) $$\alpha > \frac{\eta}{\lambda_2 \gamma}.$$

Then, the Euclidean norm of the evolution operator Φ_φ of (A.6), with $P(t) = 0$, reads

$$\|\Phi_\varphi(t, t_0)\|_2 \leq Ke^{-(\alpha\lambda_2\gamma - \eta)(t - t_0)}.$$

Step 2 (*persistence*): Considering now the perturbed equation (A.6), note that this equation has a linear perturbation term $\alpha P(t)$. So, we can use Lemma A.1 to study the stability of the synchronous manifold under this perturbation.

Using Lemmas A.1 and A.2, we ensure that if $\alpha \sup_t \|P(t)\|_2 = \theta$ then there are constants $K > 0$ (the same K in Step 1) and β (see (A.4)) such that the evolution operator Ψ_ξ of the perturbed equation (A.6) reads

$$\|\Psi_\xi(t, t_0)\|_2 \leq Ke^{\beta(t - t_0)}$$

with

$$\beta = -(\alpha\lambda_2\gamma - \eta) + \theta K.$$

In order to guarantee that $\beta < 0$ we must have

$$\theta < \frac{\alpha\lambda_2\gamma - \eta}{K}.$$

But, by Lemma A.2 we have $\sup_t \|P(t)\| \le \delta\|L\|$, then

$$\theta = \alpha \sup_t \|P(t)\|_2 \le \alpha \sqrt{\sup_t \|P(t)\|_1 \sup_t \|P(t)\|} \le \alpha\sqrt{\delta\|L\|_1 \delta\|L\|} = \alpha\delta\sqrt{\|L\|_1\|L\|},$$

where $\|\cdot\|_1$ stands for the matrix 1-norm. As L is symmetric then $\|L\|_1 = \|L\|$ and hence $\theta \le \alpha\delta\|L\|$.

Therefore, the sufficient condition

$$(A.11) \qquad \delta < \frac{\lambda_2\gamma - \eta/\alpha}{K\|L\|}$$

ensures that the synchronization manifold is locally exponentially stable.

Note that we always can write the solution of (A.6) as $\xi(t) = \Psi_\xi(t, t_0)\xi(t_0)$. It implies that

$$(A.12) \qquad \|\xi(t)\|_2 \le Ke^{\beta(t-t_0)}\|\xi(t_0)\|_2.$$

As we are using the representation $X(t) = \mathbf{1} \otimes s(t) + \xi(t)$, then $\|\xi(t)\|_2 = \|X(t) - \mathbf{1} \otimes s(t)\|_2$. In a component format the convergence of $\|x_j(t) - x_i(t)\|_2$ will have the same convergence as $\|X(t) - \mathbf{1} \otimes s(t)\|_2$ and then, in (A.12), we can replace $\|\xi(t)\|_2$ by $\|x_j(t) - x_i(t)\|_2$ by norm equivalence. Therefore, if one takes α according to (A.10) and δ according to (A.11) the synchronization manifold is locally exponentially stable and the statement of Theorem 2.1 holds.

A.2. Proof of Corollary 2.2 (ER networks). In the limit of large coupling α, (2.4) reduces to

$$(A.13) \qquad \delta < \frac{\lambda_2\gamma}{K\|L\|} = \frac{\lambda_2\gamma}{2K \max_i g_i}.$$

The algebraic connectivity λ_2 of an ER random graph of n vertices, where the edges are chosen with probability $p = p_0(\log n)/n$ for some constant $p_0 > 1$, follows (see [21, Theorem 1.1] for more details).

Lemma A.3 (see [21]). *Consider an ER random graph on n vertices, where the edges are connected with probability*

$$p = p_0\frac{\log n}{n}$$

for $p_0 > 1$, constant in n. Then the algebraic connectivity λ_2 is

$$\lambda_2 \sim np\left(a(p_0) + O\left(\frac{1}{\sqrt{np}}\right)\right) \quad \text{as } n \to \infty,$$

where $a = a(p_0) \in (0, 1)$ denotes the solution of $p_0 - 1 = ap_0(1 - \log a)$.

As a remark, note that $a \to 1^-$ as $p_0 \to \infty$; in other words, if one considers $p_0 \gg 1$ then $a(p_0) \sim 1$. From concentration of degrees (see, e.g., [22]) we have the following asymptotic behavior.

Lemma A.4. *Consider an ER graph with p chosen as before, then with high probability*

$$\|L\| = np(1 + O(n^{1/2+\varepsilon}))$$

for any $\varepsilon > 0$.

Therefore, for ER networks with $p_0 \gg 1$ and for large n one has

(A.14) $$\frac{\lambda_2}{\|L\|} = 1(1 + o(1)),$$

where $o(1)$ stands for the little o notation. Using (A.14) in (A.13) we obtain

(A.15) $$\delta < \frac{\gamma}{K} = K_0.$$

The fact that δ is always bounded away from zero holds because of the fact that the condition (A.15) is a sufficient condition. So, for instance, we could take

$$0 < \frac{K_0}{2} < \delta < K_0$$

satisfying the statement of Corollary 2.2.

A.3. Proof of Corollary 2.3 (BA networks). Again, in the limit of large coupling α, (2.4) reduces to (A.13). For any connected network, the algebraic connectivity λ_2 fulfills the following bound.

Lemma A.5 (see [23]). *Let G be an undirected graph. Then the second smallest eigenvalue λ_2 of Laplacian L satisfies*

(A.16) $$\lambda_2 \leq \left(\frac{n}{n-1}\right) g_{\min},$$

where $g_{\min}$ is the minimal degree of the graph.

As we are considering BA networks, the minimal degrees equals m_0—the number of initial edges a node is given at each step. This number is bounded and independent of the network size n. Therefore, for a BA network

$$\lambda_2 < m$$

for some constant $m > 0$.

The bound on $\|L\|$ follows from [24, Theorem 3.1], which we state for completeness.

Lemma A.6 (see [24]). *Write $g_{\max} = \max_i g_i$. With probability 1 we have*

(A.17) $$\lim_{n \to \infty} n^{-1/2} g_{\max} = \mu;$$

the limit is almost surely positive and finite, and it has an absolutely continuous distribution.

Using (A.16) and (A.17) in (A.13) we obtain

$$\delta < \frac{\gamma \tilde{m}}{2K\mu n^{1/2}} = K_1 n^{-1/2}$$

with $K_1 = \gamma \tilde{m}/(2\mu K) = K_1(\Gamma)$, where we absorbed the dependence of α in the constant $\tilde{m}$.

A.4. The proof of Theorem 2.5 (fast limit). The following Lemma A.7, adapted from [19, Proposition 6], contains almost all the proof of Theorem 2.5.

Lemma A.7. Consider Lemma A.1 and let $A(t)$ and $B(t)$ be bounded continuous matrix functions, where $B(t)$ is periodic with zero mean and oscillates fast in the sense of Definition 2.4.

Suppose that the evolution operator Φ of (A.1) satisfies the inequality (A.2). Then, there is an $\epsilon > 0$ so that the evolution operator Ψ of the perturbed equation (A.3) satisfies

$$\|\Psi(t, t_0)\| \leq (1 + c)Ke^{\tau(t - t_0)} \quad for \ t \geq t_0,$$

where $\tau = -\nu + \epsilon$.

The result stated in Lemma A.7 is valid for any operator norm. In our case, that is, considering (A.6), the operators $A(t)$ and $B(t)$ are, respectively, $A(t) = \mathrm{I}_n \otimes Df(s(t)) - \alpha(L \otimes \Gamma)$ and $B(t) = \alpha P(t)$. Due to Assumptions 1 and 2 both these operators are bounded for all $t \geq 0$.

From [19] we can see that

$$\epsilon = 3KMc + h^{-1}\log[(1 + c)K],$$

where $M = \max\{\sup_t \|A(t)\|_2, \sup_t \|B(t)\|_2\}$ and $h = t_2 - t_1$ as in Definition 2.4. Even if $M = \sup_t \|\alpha P(t)\|_2 \leq \delta\|L\|$ is large, we can always make $\epsilon < \nu$ if one take h large enough and c small enough and it is always possible because $P(t)$ oscillates fast, that is, there will always be an $\omega_0 = \omega_0(c, h)$ that satisfies the condition $\epsilon < \nu$ (or $\tau < 0$).

Therefore, as $B(t) = \alpha P(t)$ is periodic and oscillates fast, Lemma A.7 can be applied and the evolution operator Ψ_ξ of (A.6) satisfies

$$\|\Psi_\xi(t, t_0)\|_2 \leq (1 + c)Ke^{\tau(t - t_0)}.$$

It implies that

$$\|\xi(t)\|_2 \leq (1 + c)Ke^{\tau(t - t_0)}\|\xi(t_0)\|_2,$$

and in a component format the convergence of $\|x_j(t) - x_i(t)\|_2$ will have the same convergence of $\|X(t) - \mathbf{1} \otimes s(t)\|_2 = \|\xi(t)\|_2$ and then we can replace $\|\xi(t)\|_2$ by $\|x_j(t) - x_i(t)\|_2$ without loss of generality. That is, there are constants $\omega_0 = \omega_0(c, h) > 0$, $\rho_0 > 0$, $K > 0$, and $\epsilon = \epsilon(c) > 0$ such that if $\omega > \omega_0$ and $\|x_j(t_0) - x_i(t_0)\|_2 \leq \rho_0$ then

$$\|x_j(t) - x_i(t)\|_2 \leq (1 + c)Ke^{\tau(t - t_0)}\|x_j(t_0) - x_i(t_0)\|_2$$

with $\tau = -(\alpha\lambda_2\gamma - \eta) + \epsilon < 0$. Therefore, the synchronization manifold is locally exponentially stable and the decaying rate towards synchronization is not affected since we take $\omega > \omega_0$ large enough making c and ϵ as small as we want.

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