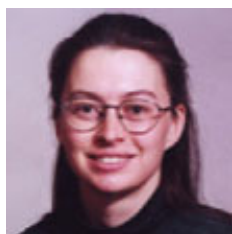# As the Planimeter's Wheel Turns:
Planimeter Proofs for Calculus Class*

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Tanya Leise (tleise@amherst.edu) received her Ph.D. from Texas A&M University and is captivated by applied mathematics in many contexts, from accelerating cracks to biological clocks. Of particular interest to her recently has been the mathematics of mechanical devices such as astrolabes and planimeters, which are also great fun to collect. When not doing math, she enjoys cycling and sewing stuffed dragons for her daughter.

A classic example of Green's theorem in action is the planimeter, a device that measures the area enclosed by a curve. Most familiar may be the polar planimeter (see Figure 1), for which a nice geometrical explanation can be found in the book by Jennings [4] and a direct constructive proof using Green's theorem is given by Gatterdam [2]. Other types include the rolling planimeter, which is particularly suited to a vector calculus course for both ease of use and simplicity of proof, and radial planimeters that integrate functions plotted on circular charts (that is, the function is in polar form, $r = f(\theta)$). In this article, we present simple proofs using Green's theorem for the rolling and polar planimeters, followed by an analysis of how to design radial planimeters that calculate a desired integral, such as that of the square root of a function marked on a circular chart. These proofs are suitable for use in a vector calculus course and avoid the awkward trigonometric and algebraic calculations required by proofs like that in [2]. While the proofs in this article are probably not new (though the author has not seen them elsewhere), they are not readily available, and so these planimeter proofs are presented with the aim of providing calculus instructors a wonderful supplement for their courses. Other planimeter proofs can be found on the web. For example, see [6] for a geometric analysis and [5] for a vector analysis of the polar planimeter, and see [1] for an explanation of the radial planimeter.

Both rolling and polar planimeters are available in mechanical and electronic versions for commercial use (a quick web search will reveal several manufacturers). For classroom

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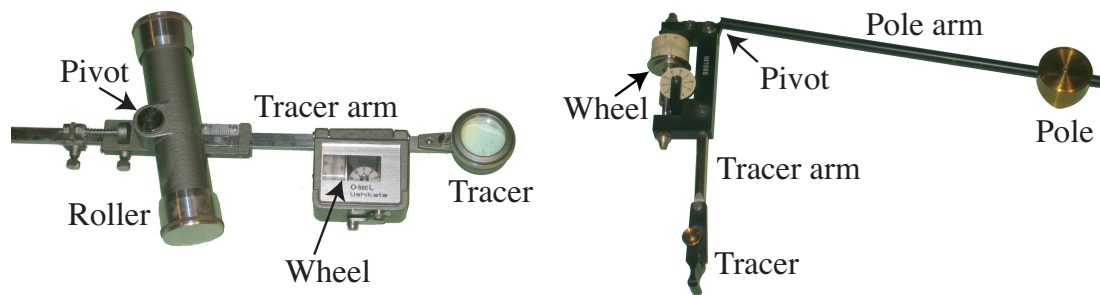
Figure 1. An Ushikata rolling planimeter on the left and a Keuffel and Esser polar planimeter on the right (from author's collection).

demonstrations, relatively inexpensive used polar and radial planimeters are available via web auction sites and sellers of antique instruments. Unfortunately, rolling planimeters and square root planimeters tend to be more difficult to procure.

Rolling planimeter

The proofs for the rolling and polar planimeters are quite similar, and we start by treating the slightly simpler rolling planimeter.

Let C be a positively oriented, piecewise smooth, simple closed curve. Recall that Green's theorem states that, given functions $P(x,y)$ and $Q(x,y)$ whose partial derivatives are continuous on an open set containing the region R enclosed by the curve C, we have

$$\int_C P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

In particular, if $P = 0$ and $Q = x$, we obtain the identity

$$\int_C x\ dy = \iint_R dA = \text{Area of the region } R. \tag{1}$$

We will show how (1) forms the vital link between tracing the curve and finding the enclosed area.

The coordinates (x, y) will represent points on the curve C, while $(0, Y)$ will describe the position of the pivot of the rolling planimeter (see Figure 2). It is crucial to recognize that as the pointer traces out the curve C, the planimeter's roller can only roll forward and backward (the roller cannot turn), so it is as though the pivot were fixed to a straight line, which we make our y-axis. As the planimeter traces out the curve, the roller moves up and down the y-axis, while the tracer arm rotates on the pivot. Hence the rolling planimeter is really a convenient form of linear planimeter (as opposed to polar planimeters in which the pivot traces out a circular arc – see the next section). Also note that the tracer arm's

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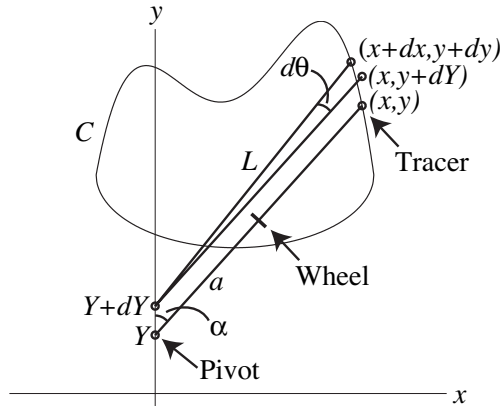
Figure 2. The motion of the tracer arm of a rolling planimeter as it traverses a curve C counterclockwise. The tracer arm is attached to a roller which rolls along the y-axis. The tracer arm may pivot where it attaches to the roller at $(0, Y)$ as the tracer at its opposite end traces out the curve with coordinates (x, y).

rotation is limited and it may not swing past the roller.

Consider the motion of the tracer as it moves along a small portion of the curve C, from a point (x, y) counterclockwise to $(x + dx, y + dy)$. The pivot will have a corresponding displacement from position $(0, Y)$ to a new position $(0, Y + dY)$. We wish to determine how much the measuring wheel on the tracer arm will turn as a result of this small motion, which can be decomposed into two parts. First roll the pivot along the y-axis from position $(0, Y)$ to $(0, Y + dY)$ so that the tracer arm maintains a fixed angle α with the y-axis and the tracer ends up at $(x, y + dY)$. Next rotate the tracer arm by an angle $d\theta$ (without moving the roller) so that the tracer ends up at $(x + dx, y + dy)$. During this operation, the wheel on the tracer arm will roll a distance of $\sin \alpha \ dY + a \ d\theta = \frac{x}{L} \ dY + a \ d\theta$, since only the component of the motion perpendicular to the tracer arm will result in the wheel turning. The planimeter returns to its original placement after traversing C and so the total angle of rotation of the tracer arm will be zero ($\oint_C d\theta = 0$). Therefore, the total rolling distance of the tracer arm wheel is

$$\text{Total wheel roll} = \frac{1}{L} \oint_C x \ dY. \tag{2}$$

We need to relate (2) to the identity in (1). Observe that $x^2 + (y - Y)^2 = L^2$. Since the tracer arm cannot rotate past the roller, we have a unique value of Y for each point (x, y): $Y = y - \sqrt{L^2 - x^2}$ (given the orientation of the planimeter as shown in Figure 2, the tracer must always be above the pivot, that is, $Y > y$ must hold). This implies that $dY = dy + \frac{x}{\sqrt{L^2 - x^2}} dx$. After applying Green's theorem to see that $\oint_C \frac{x}{\sqrt{L^2 - x^2}} dx = 0$, we find that

$$\text{Total wheel roll} = \frac{1}{L} \oint_C \left[x \ dy + \frac{x^2}{\sqrt{L^2 - x^2}} dx \right] = \frac{1}{L} \oint_C x \ dy.$$

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Hence, the area enclosed by the curve C equals the length L of the tracer arm times the total wheel roll.

As a practical matter, many planimeters have arms with adjustable lengths as a way to account for the scale of graph, which could be part of a map or pressure chart. According to our formula, doubling the length L cuts the vernier reading of the wheel roll in half, while changing the position of the wheel on the tracer arm (that is, the length a) does not directly affect the reading (although it likely affects the accuracy as a practical matter). For example, a tracer arm length of 15 cm on the rolling planimeter shown in Figure 1 leads to a vernier reading of 1 corresponding to 100 cm^2. Extending the arm to 30 cm leads to a vernier reading of 1 corresponding to 200 cm^2.
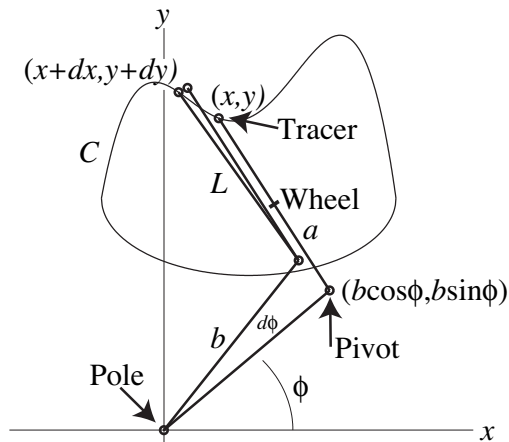
Polar planimeter

Surprisingly, despite the mechanical differences between rolling and polar planimeters, the proofs of why they work are quite similar. A proof for the polar planimeter may be constructed by replacing the coordinates $(0, Y)$ with the coordinates $(b\cos\phi, b\sin\phi)$ to reflect the circular motion of the pivot. Consider the motion of the tracer of the polar planimeter as it moves along a small portion of the curve C, from a point (x, y) counterclockwise to $(x + dx, y + dy)$. The pivot will have a corresponding displacement from position $(b\cos\phi, b\sin\phi)$ to a new position $(b\cos(\phi + d\phi), b\sin(\phi + d\phi))$. Since we are considering an infinitesimal displacement, we can linearize the new coordinates to be $(b\cos(\phi) - b\sin(\phi)d\phi, b\sin(\phi) + b\cos(\phi)d\phi)$. As before, we decompose this small motion into two parts. First swing the pivot along the arc to its new position, keeping the tracer arm parallel to its original orientation, thereby moving the tracer to $(x - b\sin(\phi)d\phi, y + b\cos(\phi)d\phi)$ (see Figure 3). During this operation, only the component of the motion perpendicular to the tracer arm will result in the wheel turning, and so the wheel will rotate a distance equal to the dot product of the displacement vector with the unit vector orthogonal to the tracer arm (chosen so that tracing the curve counterclockwise yields a positive value of wheel roll). Next rotate the tracer arm by an angle $d\theta$ (without changing the pivot's position) so that the tracer ends up at $(x + dx, y + dy)$ and the wheel rolls a distance of $a\ d\theta$. The wheel will cover a combined distance during these two small motions of

$$\frac{1}{L}\langle b\sin\phi - y, x - b\cos\phi\rangle \cdot \langle -b\sin\phi, b\cos\phi\rangle d\phi + a\ d\theta = \frac{b}{L}(x\cos\phi + y\sin\phi - b)d\phi + a\ d\theta.$$

The planimeter returns to its original placement after traversing C (and cannot do a complete rotation of 360°), so the total angle $\oint_C d\theta$ of rotation of the tracer arm will be zero, as will be the total angle $\oint_C d\phi$ of rotation of the pole arm. Therefore the total rolling distance of the tracer arm wheel is

$$\text{Total wheel roll} = \frac{b}{L}\oint_C (x\cos\phi + y\sin\phi)\ d\phi. \tag{3}$$

Polar coordinates, not surprisingly, simplify the evaluation of the integral in (3). We first observe that $(x - b\cos\phi)^2 + (y - b\sin\phi)^2 = L^2$ (compare this to the expression $x^2 + (y - Y)^2 = L^2$ for the rolling or linear planimeter), and then substitute $x = r\cos\theta$ and $y = r\sin\theta$ to

Figure 3. The motion of the tracer arm of a polar planimeter as it traverses a curve C counterclockwise. The tracer arm is attached via a pivot to an arm with a fixed pole at the origin. This pivot traces out a circular arc of radius b as the planimeter traces out the curve.

find that

$$r\cos(\theta - \phi) = \frac{r^2 + b^2 - L^2}{2b} \quad \text{and} \quad d\phi = d\theta + \frac{r^2 - b^2 + L^2}{r\sqrt{4b^2r^2 - (r^2 + b^2 - L^2)^2}}dr.$$

Rewriting (3) in polar coordinates and applying Green's theorem with respect to r and θ yields

$$\begin{aligned}
\text{Total wheel roll} &= \frac{b}{L}\oint_{C'}(r\cos\theta\cos\phi + r\sin\theta\sin\phi)\,d\phi \\
&= \frac{b}{L}\oint_{C'}r\cos(\theta - \phi)\,d\phi \\
&= \frac{b}{L}\oint_{C'}\frac{r^2 + b^2 - L^2}{2b}\left(d\theta + \frac{r^2 - b^2 + L^2}{r\sqrt{4b^2r^2 - (r^2 + b^2 - L^2)^2}}dr\right) \\
&= \frac{b}{L}\iint_R\frac{\partial}{\partial r}\left[\frac{r^2 + b^2 - L^2}{2b}\right]dr\,d\theta \\
&= \frac{1}{L}\iint_R r\,dr\,d\theta = \frac{1}{L}\cdot\text{Area of the region }R,
\end{aligned}$$

where C' is the curve in the $r\theta$-plane corresponding to C (which lies in the xy-plane).

So again we find that the area of the region enclosed by the curve C equals the length L of the tracer arm times the total wheel roll, using calculations that closely parallel those for the rolling or linear planimeter, but with the use of polar coordinates.

Next we take a look at a very different family of planimeters that display an amazing versatility in that they can be designed to integrate a function of the graphed data, for example, to calculate the mean square root of a graphed function.
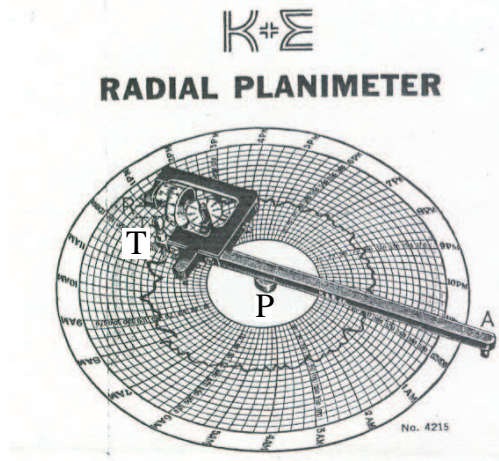
Figure 4. Diagram from the instructions to a Keuffel and Esser radial planimeter (shown with a circular chart) in the author's collection. P marks the pivot and T marks the tracer. The tracer arm AT can slide as well as turn on the pivot.

Radial planimeter

A radial planimeter measures the mean height of a polar graph $f(\theta), 0 \leq \theta \leq 2\pi$. It has a simple design, consisting of a tracer arm with a pivot, a tracer and a measuring wheel (which is placed near the tracer with axis parallel to the tracer arm). As the tracer follows the curve, the tracer arm swings on the pivot, which runs along a track inside the tracer arm, allowing the tracer arm to slide back and forth as needed to trace out the curve (see Figure 4). The planimeter rotates completely around the pivot after doing a full circuit of the circular diagram. One can quickly see that a small motion of the tracer arm can be decomposed into a small angle $d\theta$ plus a sliding motion dr. The wheel will turn an amount $f(\theta)d\theta$ (dr is perpendicular to the wheel and so does not contribute), and so after a complete rotation the wheel will record $\int_0^{2\pi} f(\theta)d\theta$, which, if we divide by 2π, equals the mean value of f.

During the first half of the 20th century, radial planimeters were used to calculate the mean flow rate from circular pressure charts [3]. Certain flow meters recorded the pressure as it varied over a 24-hour period on a circular chart. The mean flow is proportional to the square root of the pressure, so one could use a radial planimeter to find the average pressure and then take its square root $\sqrt{\frac{1}{2\pi} \int_0^{2\pi} f(\theta)d\theta}$ to use in calculating the mean flow. But this isn't quite right, as the correct quantity is the mean of the square root of the pressure: $\frac{1}{2\pi} \int_0^{2\pi} \sqrt{f(\theta)}d\theta$. If the function $f(\theta)$ varies little over the period of interest, then these two quantities are nearly the same, and so one can find a good approximation using a radial planimeter. The proper tool, however, is a planimeter that integrates the square root of the function, and, in fact, such a planimeter exists!

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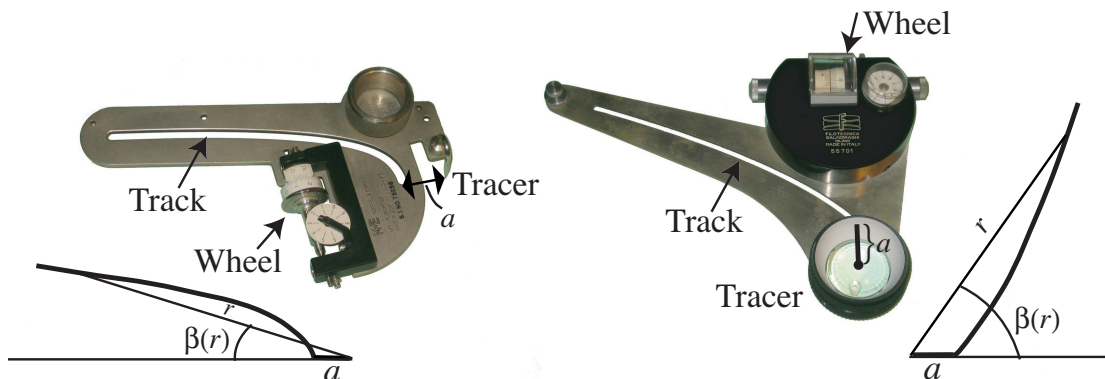
Figure 5. On the left is a square root planimeter (calculates the mean square root of a graph on a circular chart). On the right is a planimeter similar in form to the square root planimeter, but with a very different track. The previous owner did not know the purpose of this "mystery" planimeter, which motivated the author to derive (5) in order to deduce its function. The pivot pin is not shown here; it is a simple piece that can be affixed to the chart's center and has a small rod that fits into the track and allows the planimeter to slide back and forth.

Square root planimeters and beyond

One can do even better than to design a planimeter that calculates the mean of the square root of a function. We will derive a more general formula for a planimeter that calculates the value of $\int_0^{2\pi} (g \circ f)(\theta)d\theta$ for a specified function $g(r)$, such as $g(r) = \sqrt{r}$ or $g(r) = r^{3/2}$.

We will assume that we are given the graph of $r = f(\theta) + a$ for some positive constant a, in contrast to the regular radial planimeter for which $a = 0$. This offset of the graph gives us some wiggle room that makes design of the desired planimeter possible.

Where the radial planimeter has a straight track, the new design will have a curved track, described in polar coordinates by $\beta = \beta(r)$ for $r \geq a$, as shown in Figure 5. The path of the tracer as it follows the graph of $r = f(\theta) + a$ is

$$\mathbf{c}(\theta) = (f(\theta) + a)\langle\cos\theta, \sin\theta\rangle,$$

and a tangent vector to this path is given by

$$\mathbf{c}'(\theta) = (f(\theta) + a)\langle-\sin\theta, \cos\theta\rangle + f'(\theta)\langle\cos\theta, \sin\theta\rangle.$$

Let $\mathbf{w}(\theta) = \langle\cos(\theta + \beta), \sin(\theta + \beta)\rangle$ be the unit vector parallel to the wheel (see Figure 6). Note that the angle $\beta = \beta(r)$ is a function of the radial distance from the pivot, which is $\beta(r) = \beta(f(\theta) + a)$ for a point on the curve. The wheel roll due to the tracer moving an infinitesimal distance $\mathbf{c}'(\theta)d\theta$ along the curve is given by

$$\mathbf{w}(\theta) \cdot \mathbf{c}'(\theta)d\theta = (f(\theta) + a)\sin\beta(f(\theta) + a)d\theta + f'(\theta)\cos\beta(f(\theta) + a)d\theta.$$

We must also account for the return of the tracer to the starting position along the radial
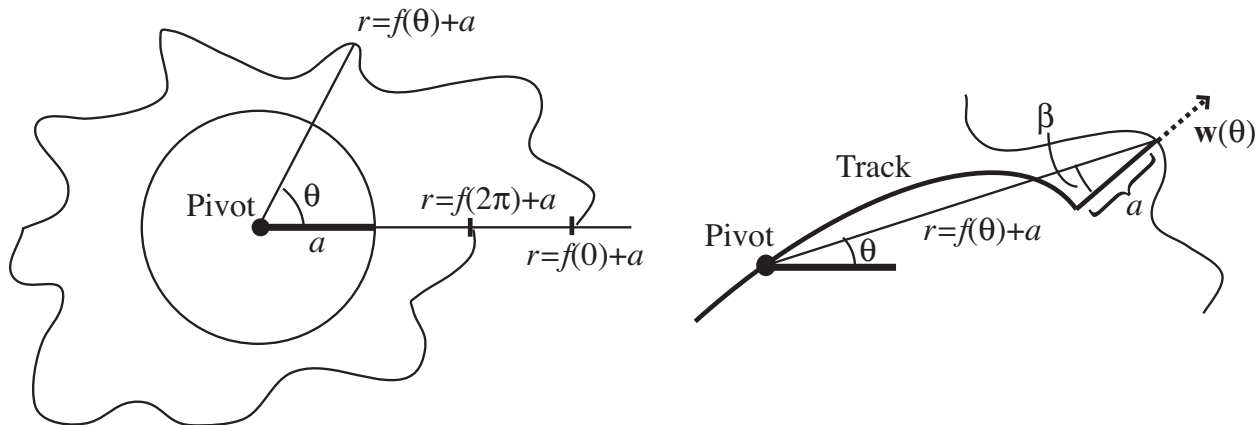
Figure 6. Diagram for designing a more general planimeter like those shown in Figure 5. In general, r is the distance from the pivot to a point on the curve.

line $\theta = 0$ if $f(2\pi) \neq f(0)$. Observe that the wheel makes an angle $\beta(r)$ with the radial line, so a small displacement dr results in the wheel rolling a distance $\cos \beta(r)dr$. Adding these parts leads to the following expression for the total wheel roll:

$$\text{Total wheel roll} = \int_0^{2\pi} (f(\theta) + a) \sin \beta(f(\theta) + a)d\theta$$

$$+ \int_0^{2\pi} f'(\theta) \cos \beta(f(\theta) + a)d\theta + \int_{f(2\pi)+a}^{f(0)+a} \cos \beta(r)dr. \quad (4)$$

Let $G(z) = \int_a^{z+a} \cos \beta(r)dr$. We can greatly simplify (4) by observing that

$$\int_0^{2\pi} f'(\theta) \cos \beta(f(\theta) + a)d\theta = G(f(2\pi)) - G(f(0))$$

and

$$\int_{f(2\pi)+a}^{f(0)+a} \cos \beta(r)dr = G(f(0)) - G(f(2\pi)).$$

Then (4) reduces to

$$\text{Total wheel roll} = \int_0^{2\pi} (f(\theta) + a) \sin[\beta(f(\theta) + a)]d\theta. \quad (5)$$

We will use (5) to design the track, that is, to determine the $\beta(r)$ that yields a desired planimeter function. For example, to obtain a square root planimeter that calculates $\int_0^{2\pi} \sqrt{f(\theta)}d\theta$, we observe that we need $r \sin \beta(r) = \sqrt{r - a}$, which implies that $\beta(r) = \sin^{-1} \frac{\sqrt{r-a}}{r}$. This design matches the track of the square root planimeter shown in Figure 5 with $a = 2$cm.

We can also use this approach to discover what the "mystery planimeter" shown in Figure 5 does. By comparing its track to the graphs of $\beta(r)$ corresponding to $(g \circ f)(\theta) = f(\theta)^p$
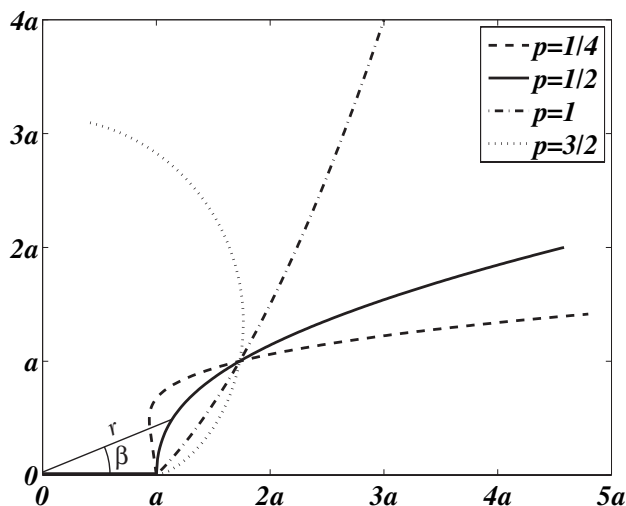
Figure 7. Graphs of tracks for planimeters that will integrate $(g \circ f)(\theta) = f(\theta)^p$, corresponding to $\beta(r) = \sin^{-1} \frac{(r-a)^p}{r}$.

for various powers p (see Figure 7), we find that $p = 1$ yields a match. This planimeter is an alternate form of the radial planimeter with $a = 3$cm (the regular radial planimeter has a straight line track with $a = 0$). Mystery solved!

References

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