SINGULARITIES OF EQUIDISTANTS AND GLOBAL CENTRE SYMMETRY SETS OF LAGRANGIAN SUBMANIFOLDS

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ABSTRACT. We define the Global Centre Symmetry set (GCS) of a smooth closed m-dimensional submanifold $M \subset \mathbb{R}^n$, $n \leq 2m$, which is an affinely invariant generalization of the centre of a k-sphere in $\mathbb{R}^{k+1}$. The GCS includes both the centre symmetry set defined by Janeczko [16] and the Wigner caustic defined by Berry [3]. We develop a new method for studying generic singularities of the GCS which is suited to the case when M is lagrangian in $\mathbb{R}^{2m}$ with canonical symplectic form. The definition of the GCS, which slightly generalizes one by Giblin and Zakalyukin [10]-[12], is based on the notion of affine equidistants, so, we first study singularities of affine equidistants of Lagrangian submanifolds, classifying all the stable ones. Then, we classify the affine-Lagrangian stable singularities of the GCS of Lagrangian submanifolds and show that, already for smooth closed convex curves in $\mathbb{R}^2$, many singularities of the GCS which are affine stable are not affine-Lagrangian stable.

1. INTRODUCTION

A circle is usually defined as the set of all points on a plane which are equidistant to a fixed point. Naturally, this point is called the centre of the circle or, equivalently, the centre of symmetry of the circle. Similarly, a 2-sphere in $\mathbb{R}^3$ has a unique point of $\mathbb{R}^3$ as its centre, or centre of symmetry, and the same applies for any k-sphere in $\mathbb{R}^{k+1}$.

When trying to generalize this notion of centre of symmetry of a smooth closed m-dimensional submanifold of $\mathbb{R}^n$, one finds that there seems to be more than one way of doing it. Coming back to the circle on the plane, or even an ellipse, its center can also be defined as the set (in this case consisting of a single element) of midpoints of straight lines connecting pairs of points on the curve with parallel tangent vectors.

For a generic smooth convex closed curve, this set, of midpoints of straight lines connecting pairs of points on the curve with parallel

W. Domitrz was supported by FAPESP/Brazil, during his stay in ICMC-USP, São Carlos, and by Polish MNiSW grant no. N N201 397237. P. de M. Rios received partial support by FAPESP for his visit to Warsaw.

tangent vectors, is not a single point, but forms a curve with an odd number of cusps, in the interior of the smooth original curve.

This singular inner curve has been known as the "Wigner caustic" of a smooth curve since the work of Berry in the 70's, because of its prominent appearance in the semiclassical limit of the "Wigner function" of a pure quantum state whose classical limit corresponds to the given smooth curve in $\mathbb{R}^2$ (with canonical symplectic structure) [3].

Therefore, this "Wigner caustic" of a smooth closed curve on the plane is a natural affine-invariant generalization of the centre of symmetry of a circle, or an ellipse, which extends to higher dimensional smooth closed submanifolds of $\mathbb{R}^n$.

On the other hand, the centre of a circle or an ellipse in $\mathbb{R}^2$ can also be described as the "envelope" of all straight lines connecting pairs of points on the curve with parallel tangent vectors.

For a generic smooth convex closed curve, this set, the envelope of all straight lines connecting pairs of points on the curve with parallel tangent vectors, is not a single point, but forms a curve with an odd number of cusps, in the interior of the smooth original curve.

This singular inner curve has been known as the "centre symmetry set" of a smooth closed curve on the plane since the work of Janeczko, over a decade ago, and is a natural affine-invariant generalization of the centre of symmetry of a circle, or an ellipse, which extends to higher dimensional smooth closed submanifolds of $\mathbb{R}^n$ [16].

However, except for circles or ellipses, when both symmetry sets are the same point, the Wigner caustic and the centre symmetry set of a smooth convex closed curve are not the same singular curve. Instead, the Wigner caustic is interior to the centre symmetry set and the cusp points of the inner curve touches the outer one in its smooth part.

A new, more complicated curve, containing the Wigner caustic and the centre symmetry set, can be defined in a single way and this affine-invariant definition extends to an arbitrary smooth closed m-dimensional submanifold M of $\mathbb{R}^n$, for $n \leq 2m$. We call this new set, the "Global Centre Symmetry" set of M.

In fact, our definition is only a very slight modification of a definition already introduced and used by Giblin and Zakalyukin [10]-[12] to study singularities of centre symmetry sets of hypersurfaces. A key notion in their definition is that of an affine λ-equidistant to the smooth submanifold, of which the Wigner caustic is the case $\lambda = 1/2$. The singularities of these λ-equidistants are then fundamental to characterize the Global Centre Symmetry set and its singularities.

In this paper, we present a new method that is suitable for studying the singularities of affine λ-equidistants $E_\lambda(M)$, $\forall\lambda \in \mathbb{R}$, and the affine-invariant $GCS(M)$ of a smooth closed submanifold $M^m \subset \mathbb{R}^n$, $n \leq 2m$. However, the more general study shall be published elsewhere [7]. Here, we focus on the extreme case $n = 2m$. More particularly, in this paper we focus on the case when L is a smooth closed *Lagrangian* submanifold of the affine symplectic space $(\mathbb{R}^{2m}, \omega)$, where ω denotes the canonical symplectic form. This Lagrangian case is particular in various respects.

From a physical standpoint, this is the setting where Wigner caustics were first defined, from the semiclassical limit of Wigner functions, which are important in semiclassical dynamics [3][17][19][21]. It is therefore natural to investigate in detail the singularities of the Wigner caustic of a closed smooth Lagrangian submanifold L of arbitrary dimension, particularly the yet little studied case of a Lagrangian surface. Then, given the neat geometrical character of the full Global Centre Symmetry set, it is natural to extend these investigations, when L is Lagrangian in $(\mathbb{R}^{2m}, \omega)$, to the singularities of $GCS(L)$.

From a mathematical standpoint, because this is the extreme case $n = 2m$, the notion of a pair of points on L^m with parallel tangent subspaces is more amply generalized and we can study all the cases of "degree of parallelism", running from 1 to m. Also, this is the setting where generating functions and generating families are more naturally defined, but, because we have to cope with a symplectic structure on $\mathbb{R}^{2m}$ and use generating families, the correct definition of an equivalence relation for the singularities of $GCS(L)$ is more subtle.

This paper is organized as follows. In section 2 we present the definition of the Global Centre Symmetry set. This section also contains the basic definitions of degree of parallelism, affine equidistant, Wigner caustic, centre symmetry caustic and criminant. In section 3 we define λ-chord transformations which are used to define a general characterization and classification for affine equidistants.

In section 4 we define the generating families for these affine equidistants and relate their general classification to the well known classification by Lagrangian equivalence [2]. This is then used in section 5 to obtain the classification of all stable singularities of all affine equidistants of any generic Lagrangian submanifold.

Thus, theorem 5.1 states that any caustic of stable Lagrangian singularity is realizable as $E_\lambda(L)$, for some Lagrangian $L \subset (\mathbb{R}^{2m}, \omega)$, and corollaries 5.2 and 5.3 specialize this theorem to the cases when L is a curve or a surface, respectively. In the first case, generic singularities are cusps, while, in the second case, they can be cusps, swallowtails, butterflies, or hyperbolic, elliptic and parabolic umbilics.

The following three sections are devoted to the singularities of the Global Centre Symmetry set. In section 6 we give a geometric characterization for the criminant of $GCS(L)$ similar to results in [10]-[12] for hypersurfaces. In section 7 we introduce the equivalence relation (also as an equivalence of generating families) that allows for a complete affine-symplectic-invariant classification of the stable singularities of $GCS(L)$. We show that only singularities of the criminant, the smooth part of the Wigner caustic, or tangent union of both, are stable.

Finally, section 8 is devoted to the study of the GCS of Lagrangian curves. First, we state two theorems for the GCS of convex curves in $\mathbb{R}^2$ when no symplectic structure is considered. The results presented in theorem 8.1 are not new ([3], [16], [9]-[13]), but they are proved in the appendix using a method that is entirely original and twin to the method used in the Lagrangian case. In the second theorem, the inequality on the number of cusps of the CSS and the Wigner caustic, although straightforward from the results in [9], had not been mentioned before. Pictures illustrate these theorems. Then, we specialize the results of section 7 to the case of Lagrangian curves, showing that most of the singularities which were affine stable when no symplectic structure was considered are not affine-Lagrangian stable. In other words, there is a breakdown of their stabilities due to the presence of a symplectic form, similarly to some results presented in [4]-[6].

Acknowledgements: We specially thank M.A.S. Ruas for many stimulating discussions and invaluable remarks that greatly contributed to this paper. We also thank P. Giblin and S. Janeczko for stimulating discussions. W. Domitrz thanks M.A.S. Ruas for invitation and hospitality during his stay in São Carlos. P. de M. Rios thanks S. Janeczko for invitation and hospitality during his visit to Warsaw.

2. Definition of the Global Centre Symmetry set.

Let M be a smooth closed m-dimensional submanifold of the affine space $\mathbb{R}^n$, with $n \leq 2m$. Let a, b be points of M.

$$\tau_{a-b} : \mathbb{R}^n \ni x \mapsto x + (a-b) \in \mathbb{R}^n$$

is the translation by the vector $(a-b)$.

Definition 2.1. A pair of points $a, b \in M$ $(a \neq b)$ is called a **weakly parallel** pair if

$$T_aM + \tau_{a-b}(T_bM) \neq \mathbb{R}^n.$$

$\mathrm{codim}(T_aM+\tau_{a-b}(T_bM))$ in $T_a\mathbb{R}^n$ is called a **codimension of a weakly parallel pair** a, b. We denote it by $\mathrm{codim}(a, b)$.

A weakly parallel pair $a, b \in M$ is called **k-parallel** if

$$\dim(T_aM \cap \tau_{b-a}(T_bM)) = k.$$

If $k = m$ the pair $a, b \in M$ is called **strongly parallel**, or just **parallel**. We also refer to k as the **degree of parallelism** of the pair (a, b) and denote it by $\deg(a, b)$. The degree of parallelism and the codimension of parallelism are related in the following way:

$$2m - \deg(a, b) = n - \operatorname{codim}(a, b). \tag{2.1}$$

Thus, *for a Lagrangian submanifold, the degree of parallelism and the codimension of a weakly parallel pair coincide.*

Definition 2.2. A **chord** passing through a pair a, b, is the line

$$l(a, b) = \{x \in \mathbb{R}^n | x = \lambda a + (1 - \lambda)b, \lambda \in \mathbb{R}\},$$

but we sometimes also refer to $l(a, b)$ as a chord *joining* a and b.

Definition 2.3. For a given λ, an **affine λ-equidistant** of M, $E_\lambda(M)$, is the set of all $x \in \mathbb{R}^n$ such that $x = \lambda a + (1-\lambda)b$, for all weakly parallel pairs $a, b \in M$. $E_\lambda(M)$ is also called a (affine) **momentary equidistant** of M. Whenever M is understood, we write E_λ for $E_\lambda(M)$.

Note that, for any λ, $E_\lambda(M) = E_{1-\lambda}(M)$ and in particular $E_0(M) = E_1(M) = M$. Thus, the case $\lambda = 1/2$ is special:

Definition 2.4. $E_{\frac{1}{2}}(M)$ is called the **Wigner caustic** of M.

Remark 2.5. This name is given for historical reasons [3][17].

The *extended affine space* is the space $\mathbb{R}^{n+1}_e = \mathbb{R} \times \mathbb{R}^n$ with coordinate $\lambda \in \mathbb{R}$ (called *affine time*) on the first factor and projection on the second factor denoted by $\pi : \mathbb{R}^{n+1}_e \ni (\lambda, x) \mapsto x \in \mathbb{R}^n$.

Definition 2.6. The **affine extended wave front** of M, $\mathbb{E}(M)$, is the union of all affine equidistants each embedded into its own slice of the extended affine space: $\mathbb{E}(M) = \bigcup_{\lambda \in \mathbb{R}} \{\lambda\} \times E_\lambda(M) \subset \mathbb{R}^{n+1}_e$.

Note that, when M is a circle on the plane, $\mathbb{E}(M)$ is the (double) cone, which is a smooth manifold with nonsingular projection π everywhere, but at its singular point, which projects to the centre of the circle. From this, we generalize the notion of centre of symmetry.

Thus, let π_r be the restriction of π to the affine extended wave front of M: $\pi_r = \pi|_{\mathbb{E}(M)}$. A point $x \in \mathbb{E}(M)$ is a **critical** point of π_r if the germ of π_r at x fails to be the germ of a regular projection of a smooth submanifold. We now introduce the main definition of this paper:

Definition 2.7. The **Global Centre Symmetry** set of M, $GCS(M)$, is the image under π of the locus of critical points of π_r.

Remark 2.8. The set $GCS(M)$ is the bifurcation set of a family of affine equidistants (family of chords of weakly parallel pairs) of M.

Remark 2.9. In general, $GCS(M)$ consists of two components: the **caustic** $\Sigma(M)$ being the projection of the singular locus of $\mathbb{E}(M)$ and the **criminant** $\Delta(M)$ being the (closure of) the image under π_r of the set of regular points of $\mathbb{E}(M)$ which are critical points of the projection π restricted to the regular part of $\mathbb{E}(M)$. $\Delta(M)$ is the envelope of the family of regular parts of momentary equidistants, while $\Sigma(M)$ contains all the singular points of momentary equidistants.

The above definition (with its following remarks) is only a very slight modification of the definition that has already been introduced and used by Giblin and Zakalyukin [10] to study centre symmetry sets of curves on the plane and surfaces in 3-space. However, in our present definition the whole manifold M is considered, as opposed to pairs of germs, as in [10], and weak parallelism is also taken into account. Of course, slightly modifying their nice definition was the easy part. On the other hand, considering the whole manifold in the definition leads to the following simple but important result:

Theorem 2.10. *The Global Centre Symmetry set of M contains the Wigner caustic of M.*

Proof. Let x be a regular point of $E_{\frac{1}{2}}(M)$. Then $x = \frac{1}{2}(a+b)$ for a weakly parallel pair $a, b \in M$. It means that x is a intersection point of the chords $l(a,b)$ and $l(b,a)$. The extended wave front $\mathbb{E}(M)$ contains the sets

$$\{(\lambda, \lambda a + (1-\lambda)b) | \lambda \in \mathbb{R}\},\ \{(\lambda, (1-\lambda)a + \lambda b) | \lambda \in \mathbb{R}\}.$$

If $(\frac{1}{2}, x)$ is a regular point of $\mathbb{E}(M)$ then the above sets are included in the tangent space to $\mathbb{E}(M)$ at $(\frac{1}{2}, x)$. It implies that a fiber $\{(\lambda, x) | \lambda \in \mathbb{R}\}$ is included in the tangent space of $\mathbb{E}(M)$. Thus if $(\frac{1}{2}, x)$ is a regular point of $\mathbb{E}(M)$ then x is in the criminant $\Delta(M)$. If $(\frac{1}{2}, x)$ is not a regular point of $\mathbb{E}(M)$ then x is in the caustic $\Sigma(M)$. □

Remark 2.11. As we shall see later (section 8), when we give a better characterization of $\Delta(M)$, apart from the cases considered in the previous remark most often $(\frac{1}{2}, x) \in \mathbb{E}(M)$ is not a regular point of $\mathbb{E}(M)$ (but the fact that $x \in \Sigma(M)$ cannot be seen by purely local considerations). In view of this fact, we divide the caustic $\Sigma(M)$ into two parts: The Wigner caustic $E_{1/2}(M)$ and the **centre symmetry caustic** $\Sigma'(M) = \Sigma(M) \setminus E_{1/2}(M)$.

As noted in the introduction, the study of Wigner caustics goes back more than 30 years and $E_{1/2}(M)$ can be described in various ways:

Let M be a smooth convex closed curve on the plane and take two nearby points on M. There is only one chord connecting these two points (with nonparallel tangent vectors), whose midpoint x lies close to M. Conversely, for such a point x, inside but close to M, there is only one chord connecting two points of M for which x is its midpoint. As the neighboring points are moved further away from each other, the midpoint of the chord connecting these points moves further inside of M. When the two points have parallel tangent vectors, $x \in E_{1/2}(M)$. As x moves inside $E_{1/2}(M)$, there are three chords connecting nonparallel pairs of points on M having x as their midpoint (when $x \in E_{1/2}(M)$ two of these three chords coalesced into one) [3].

One way to find a chord connecting points on M given a midpoint x is by reflecting M through x and looking for the intersection points of M and its reflected image $\mathcal{R}_x M$ (these are pairs of endpoints of each chord). Again, the number of intersection pairs change as x crosses the Wigner caustic and, when $x \in E_{1/2}(M)$, there is a point (or a pair of points) on M where M and $\mathcal{R}_x M$ are tangent [17].

Still another way to search for $E_{1/2}(M)$ is to look at the area of the planar region enclosed by M and a chord as a function $\bar{A}$ of the chord's midpoint $\bar{x}$ and search for the points where the hessian determinant blows up. Alternatively, a more precise and complete description is obtained by considering this area as a function A of a point x on the chord and a variable κ locating one of the endpoints of the chord on the curve. Regarding x as parameter, $A(x, \kappa)$ is a generating family for which $E_{1/2}(M)$ is its bifurcation set. Because of this description, $E_{1/2}(M)$ is also known as the "area evolute" of M [3, 13].

The first description of $E_{1/2}(M)$ can in principle be generalized to any smooth closed m-dimensional submanifold M of $\mathbb{R}^{2m}$.

The second description can be generalized to any smooth closed m-dimensional submanifold M of $\mathbb{R}^{2m}$ and it can be further generalized to any smooth closed m-dimensional submanifold M of $\mathbb{R}^n$, for $n \leq 2m$, so that, when $x \in E_{1/2}(M)$, there is a point (or pair of points) on M where M and $\mathcal{R}_x M$ are tangent in at least $2m - n + 1$ directions. Moreover, in this more general setting, there is a way to encode these reflection maps in a transformation of the space $\mathbb{R}^n \times \mathbb{R}^n$, which can be generalized for any $\lambda \neq 0, 1$ and used to characterize all sets $E_\lambda(M)$ in a simple way, as explained below in the next section.

A way to generalize the third description is to focus on the case when L is a smooth closed *Lagrangian* submanifold of $\mathbb{R}^{2m}$ with canonical affine symplectic structure. In this case, a generating family closely related to A and $\bar{A}$, above, generalize to generating families for every $E_\lambda(L)$, in each degree of parallelism, as done in section 4, below.

3. λ-chord transformations

For $\lambda = 1/2$, there is a well known procedure, sometimes known as the centre-chord change of coordinates, sometimes as the midpoint transformation, hereby also called the "$\frac{1}{2}$-chord transformation", which encodes the midpoint reflections referred to above in such a way as to facilitate the description of the Wigner caustic [20].

Consider the product affine space: $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates (x_+, x_-) and the tangent bundle to $\mathbb{R}^n$: $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with coordinate system $(x, \dot{x})$ and standard projection

$$pr : T\mathbb{R}^n \ni (x, \dot{x}) \to x \in \mathbb{R}^n.$$

Then, there exists a global linear diffeomorphism

$$\Gamma_{1/2} : \mathbb{R}^n \times \mathbb{R}^n \ni (x^+, x^-) \mapsto \left(\frac{x^+ + x^-}{2}, \ \frac{x^+ - x^-}{2} \right) = (x, \dot{x}) \in T\mathbb{R}^n,$$

with inverse

$$\Gamma_{1/2}^{-1} : T\mathbb{R}^n \ni (x, \dot{x}) \mapsto (x + \dot{x}, x - \dot{x}) = (x^+, x^-) \in \mathbb{R}^n \times \mathbb{R}^n.$$

This map $\Gamma_{1/2}$ is the $\frac{1}{2}$-chord transformation, which we now generalize.

Below, we state this generalization in the case $\mathbb{R}^n = \mathbb{R}^{2m}$, for better reference throughout the paper, but stress that this generalization and most of what follows apply to general $\mathbb{R}^n$, as done in [7].

Definition 3.1. $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$, a **$\lambda$-chord transformation**

$$\Psi_{\mu(\lambda)\rho(\lambda)} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to T\mathbb{R}^{2m} \ , \ (x^+, x^-) \mapsto (x, \dot{x})$$

is a *linear* diffeomorphism generalizing the half-chord transformation, which is defined by the *λ-point equation*:

$$x = \lambda x^+ + (1 - \lambda) x^- \ , \tag{3.1}$$

for the λ-point x, and the *general chord equation*:

$$\dot{x} - \mu(\lambda) x = \rho(\lambda)(x^+ - x^-) \ , \tag{3.2}$$

where $\rho : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ is such that $\rho(\lambda) \neq 0$, for $\lambda \neq 0, 1$, and $\rho(1/2) = 1/2$, and $\mu : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ is such that $\mu(1/2) = 0$.

If $\mu \equiv 0$, $\Psi_{0\rho(\lambda)} \equiv \Psi_{\rho(\lambda)}$ is a **faithful λ-chord transformation**. If $\mu(\lambda) \neq 0$, for $\lambda \neq 1/2$, $\Psi_{\mu(\lambda)\rho(\lambda)}$ is a **tilted λ-chord transformation**.

The inverse equations to (3.1) and (3.2) are given by

$$x^+ = \left(1 - (1-\lambda)\frac{\mu(\lambda)}{\rho(\lambda)}\right)x \ + \frac{1-\lambda}{\rho(\lambda)}\dot{x} \ , \tag{3.3}$$

$$x^- = \left(1 + \lambda\frac{\mu(\lambda)}{\rho(\lambda)}\right)x \ - \frac{\lambda}{\rho(\lambda)}\dot{x} \ . \tag{3.4}$$

Remark 3.2. If $\Psi_{\rho(\lambda)}$ is a faithful λ-chord transformation, for every affine transformation on $\mathbb{R}^{2m}$, $x^{\pm} \mapsto Ax^{\pm} + a$, with $A \in GL(2m, \mathbb{R})$ and $a \in \mathbb{R}^{2m}$, the induced affine transformation on $T\mathbb{R}^{2m}$ is λ-independent,

$$\mathcal{A} : T\mathbb{R}^{2m} \to T\mathbb{R}^{2m} \ , \ (x, \dot{x}) \mapsto (Ax + a, A\dot{x}) \ . \tag{3.5}$$

Equivalently, the image of the diagonal of $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$ by $\Psi_{\rho(\lambda)}$ is the zero section of $T\mathbb{R}^{2m}$, $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$.

If $\Psi_{\mu(\lambda)\rho(\lambda)}$ is a tilted λ-chord transformation, the image of the diagonal of $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$ by $\Psi_{\mu(\lambda)\rho(\lambda)}$ is the tilted section $\{(x, \dot{x} = \mu(\lambda)x)\}$ of $T\mathbb{R}^{2m}$ and the induced affine transformation on $T\mathbb{R}^{2m}$ is λ-dependent,

$$\mathcal{A}'_{\mu(\lambda)} : T\mathbb{R}^{2m} \to T\mathbb{R}^{2m} \ , \ (x, \dot{x}) \mapsto (Ax + a, A\dot{x} + \mu(\lambda)a) \ . \tag{3.6}$$

Note, however, that if one considers a linear ($a = 0$) transformation on $\mathbb{R}^{2m}$, the induced linear transformation on $T\mathbb{R}^{2m}$ is λ-independent.

Among the faithful λ-chord transformations, the choice $\rho(\lambda) \equiv 1/2$ is standard and, in this case, the λ-chord transformation is denoted by Γ_λ and is bijective $\forall \lambda \in \mathbb{R}$. This is the transformation used in [7].

The reason for considering tilted λ-chord transformations shall become clear in the next section. Among the tilted λ-chord transformations, the most special one, in the case of Lagrangian submanifolds, is the choice $\mu(\lambda) = 2\lambda - 1$ and $\rho(\lambda) = 2\lambda(1-\lambda)$. For this choice, the tilted λ-chord transformation shall be denoted by Φ_λ. Explicitly,

$$\Phi_\lambda : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \ni (x^+, x^-) \mapsto (x, \dot{x}) \in T\mathbb{R}^{2m}$$

is given by the λ-point equation (3.1), for x, together with

$$\dot{x} = \lambda x^+ - (1-\lambda)x^- \ , \tag{3.7}$$

so that Φ_λ^{-1} is given by:

$$x^+ = \frac{x + \dot{x}}{2\lambda} \ \ , \ \ x^- = \frac{x - \dot{x}}{2(1-\lambda)} \ . \tag{3.8}$$

Now, let L be a smooth closed Lagrangian submanifold of the affine symplectic space $(\mathbb{R}^{2m}, \omega)$ and consider the product $L \times L \subset \mathbb{R}^{2m} \times \mathbb{R}^{2m}$. Let $\mathcal{L}_{\mu(\lambda)\rho(\lambda)}$ denote the image of $L \times L$ by a λ-chord transformation,

$$\mathcal{L}_{\mu(\lambda)\rho(\lambda)} = \Psi_{\mu(\lambda)\rho(\lambda)}(L \times L) \ ,$$

which is a $2m$-dimensional smooth submanifold of $T\mathbb{R}^{2m}$.

Then we have the following general characterization:

Theorem 3.3. *The set of critical values of the standard projection $pr : T\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ restricted to $\mathcal{L}_{\mu(\lambda)\rho(\lambda)}$ is $E_\lambda(L)$.*

Proof. Let a belong to the set of critical values of $pr|_{\mathcal{L}_{\mu(\lambda)\rho(\lambda)}}$. It means that $\dim T_{(a,\dot{a})}\mathcal{L}_{\mu(\lambda)\rho(\lambda)} \cap T_{(a,\dot{a})}pr^{-1}(a)$ is positive. Let $v_1, \cdots, v_k$ be a basis of $T_{(a,\dot{a})}\mathcal{L}_{\mu(\lambda)\rho(\lambda)} \cap T_{(a,\dot{a})}pr^{-1}(a)$. Then these basis has the following form $v_j = \sum_{i=1}^{2m} \alpha_{ji} \frac{\partial}{\partial \dot{x}_i}|_{(a,\dot{a})}$ for $j = 1, \cdots, k$. By (3.3) and (3.4) we get that for $i = 1, \cdots, 2m$

$$\left(\Psi^{-1}_{\mu(\lambda)\rho(\lambda)}\right)_* \left(\frac{\partial}{\partial \dot{x}_i}\right) = \frac{1}{\rho(\lambda)} \left((1-\lambda)\frac{\partial}{\partial x_i^+} - \lambda \frac{\partial}{\partial x_i^-}\right)$$

and it follows that $(\Psi^{-1}_{\mu(\lambda)\rho(\lambda)})_*(v_j) = \frac{1}{\rho(\lambda)}(v_j^+ + v_j^-)$, where

$$v_j^+ = (1-\lambda)\sum_{i=1}^{2m} \alpha_{ji} \frac{\partial}{\partial x_i^+}|_{a^+} \in T_{a^+}L \ , \ v_j^- = -\lambda \sum_{i=1}^{n} \alpha_{ji} \frac{\partial}{\partial x_i^-}|_{a^-} \in T_{a^-}L.$$

It implies that $v_j^+ \in T_{a^+}L \cap \tau_{(a^+-a^-)}T_{a^-}L$ for $j = 1, \cdots, k$. Thus $T_{a^+}L + \tau_{(a^+-a^-)}T_{a^-}L \neq T_{a^+}\mathbb{R}^n$ and consequently a^+, a^- is a weakly parallel (k-parallel) pair. Hence $a = \lambda a^+ + (1-\lambda)a^-$ belongs to E_λ.

Now assume that a belongs to E_λ. Then $a = \lambda a^+ + (1-\lambda)a^-$ for a weakly k-parallel pair a^+, a^-. Thus there exist linearly independent vectors $v_j^+ = \sum_{i=1}^{2m} \alpha_{ji} \frac{\partial}{\partial x_i^+}|_{a^+} \in T_{a^+}L \cap \tau_{(a^+-a^-)}T_{a^-}L$ for $j = 1, \cdots, k$. Consider linearly independent vectors $v_j = (\Psi_{\mu(\lambda)\rho(\lambda)})_*((1-\lambda)v_j^+ - \lambda\tau_{(a^- - a^+)}v_j^+)$ for $j = 1, \cdots, k$. It is obvious that v_j belongs to $T_{(a,\dot{a})}\mathcal{L}_{\mu(\lambda)\rho(\lambda)}$ and $pr_*(v_j) = 0$ for $j = 1, \ldots, k$. Thus a is a critical value of $pr|_{\mathcal{L}_{\mu(\lambda)\rho(\lambda)}}$. ☐

Remark 3.4. For the characterization of $E_\lambda(L)$, the distinction between faithful and tilted λ-chord transformations is meaningless.

For local classification of singularities, we introduce the the following definition. Again, let L and $\widetilde{L}$ be smooth closed Lagrangian submanifolds of the affine symplectic space $(\mathbb{R}^{2m}, \omega)$ and let

$$\mathcal{L}_{\mu(\lambda)\rho(\lambda)} = \Psi_{\mu(\lambda)\rho(\lambda)}(L \times L) \ , \ \widetilde{\mathcal{L}}_{\mu(\lambda)\rho(\lambda)} = \Psi_{\mu(\lambda)\rho(\lambda)}(\widetilde{L} \times \widetilde{L}) \ ,$$

where $\Psi_{\mu(\lambda)\rho(\lambda)}$ is a λ-chord transformation.

Definition 3.5. $E_\lambda(L)$ and $E_\lambda(\widetilde{L})$ are $\Psi_{\mu(\lambda)\rho(\lambda)}$**-chord equivalent** if there exists a fiber-preserving diffeomorphism-germ ξ^λ of $T\mathbb{R}^{2m}$ which maps the germ of $\mathcal{L}_{\mu(\lambda)\rho(\lambda)}$ to the germ of $\widetilde{\mathcal{L}}_{\mu(\lambda)\rho(\lambda)}$, as *germs of Lagrangian submanifolds, for suitable symplectic forms on* $T\mathbb{R}^{2m}$, so that the following diagram commutes (vertical arrows indicate germs of diffeomorphisms):

$$\begin{array}{ccccc} L\times L\subset\ \mathbb{R}^{2m}\times\mathbb{R}^{2m} & \overset{\Psi_{\mu(\lambda)\rho(\lambda)}|_{L\times L}}{\longrightarrow} & T\mathbb{R}^{2m} & \overset{pr}{\longrightarrow} & \mathbb{R}^{2m} \\ \downarrow & & \downarrow \xi^{\lambda} & & \downarrow \\ \widetilde{L}\times\widetilde{L}\subset\ \mathbb{R}^{2m}\times\mathbb{R}^{2m} & \overset{\Psi_{\mu(\lambda)\rho(\lambda)}|_{\widetilde{L}\times\widetilde{L}}}{\longrightarrow} & T\mathbb{R}^{2m} & \overset{pr}{\longrightarrow} & \mathbb{R}^{2m} \end{array}$$

Whenever $\Psi_{\mu(\lambda)\rho(\lambda)}$ is subtended, $\Psi_{\mu(\lambda)\rho(\lambda)}$-chord equivalence is simply called **λ-chord equivalence**.

For global invariance considerations we also introduce the following definition, which relates invariance of the chord-classification under a group G-action on $\mathbb{R}^{2m}$ with G-equivariance of the above diagram.

Definition 3.6. Let L and L' be smooth closed Lagrangian submanifolds of the affine symplectic space $(\mathbb{R}^{2m},\omega)$ and let G be a Lie group acting properly on $\mathbb{R}^{2m}$ so that L' is the image of L by the action $\alpha_g:\mathbb{R}^{2m}\to\mathbb{R}^{2m}$ of some $g\in G$. We say that *the classification of the singularities of $E_\lambda(L)$ by $\Psi_{\mu(\lambda)\rho(\lambda)}$-chord equivalence is G-***invariant** if, $\forall g\in G$, the following diagram commutes (vertical arrows indicate global diffeomorphisms):

$$\begin{array}{ccccc} L\times L\ \subset\ \mathbb{R}^{2m}\times\mathbb{R}^{2m} & \overset{\Psi_{\mu(\lambda)\rho(\lambda)}}{\longrightarrow} & T\mathbb{R}^{2m} & \overset{pr}{\longrightarrow} & \mathbb{R}^{2m} \\ \alpha_g\times\alpha_g\downarrow & & \downarrow \theta_g^{\lambda} & & \downarrow \alpha_g \\ L'\times L'\ \subset\ \mathbb{R}^{2m}\times\mathbb{R}^{2m} & \overset{\Psi_{\mu(\lambda)\rho(\lambda)}}{\longrightarrow} & T\mathbb{R}^{2m} & \overset{pr}{\longrightarrow} & \mathbb{R}^{2m} \end{array}$$

where θ_g^{λ} is fiber-preserving.

The following statement is immediate (see remark 3.2):

Proposition 3.7. *The classification of the singularities of $E_\lambda(L)$ by λ-chord equivalence is affine symplectic invariant, that is the group G in the definition 3.6 above being the affine symplectic group.*

The above theorem, proposition, definitions and remarks set up the characterization and classification of singularities of $E_\lambda(L)$, for a smooth closed Lagrangian submanifold L of $(\mathbb{R}^{2m},\omega)$. In fact, most of what has been defined above generalizes to non Lagrangian cases [7].

However, the fact that L is a *Lagrangian* submanifold of the affine *symplectic* space $(\mathbb{R}^{2m},\omega)$ forces us to consider suitable symplectic forms on $T\mathbb{R}^{2m}$ very carefully and limits the kinds of diffeomorphism germs ξ^λ that can be used in definition 3.5, for λ-chord equivalence. When no symplectic structure is considered, the diffeomorphism-germ ξ^λ in definition 3.5 is of the general form

$$\xi^\lambda : T\mathbb{R}^{2m} \ni (x, \dot{x}) \mapsto (X(x), \dot{X}^\lambda(x, \dot{x})) \in T\mathbb{R}^{2m}, \tag{3.9}$$

but in the lagrangian case, further restrictions apply.

On the other hand, the fact that L is a *Lagrangian* submanifold of the affine *symplectic* space $(\mathbb{R}^{2m}, \omega)$ also allows us to relate this new notion of λ-chord equivalence to the well-known notion of Lagrangian equivalence and, in so doing, define very useful generating families for every $E_\lambda(L)$, in each degree of parallelism, as presented below.

4. Generating families

Let $(\mathbb{R}^{2m}, \omega)$ be an affine symplectic space with canonical Darboux coordinates p_i, q_i, so that $\omega = \sum_{i=1}^m dp_i \wedge dq_i$, and let L be a smooth closed Lagrangian submanifold of $(\mathbb{R}^{2m}, \omega)$.

The purpose of this work is to describe the singularities of $GCS(L)$. To do so, we generalize to any $\lambda \in \mathbb{R}\backslash\{0,1\}$ another construction that is well known for $\lambda = 1/2$ (for this case, see for instance [20]). This other generalization amounts to correctly weighting the symplectic form on each copy of $\mathbb{R}^{2m}$ to be consistent with λ-chord transformations.

Thus, for a fixed $\lambda \in \mathbb{R} \setminus \{0,1\}$ we consider the product affine space $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$ with the symplectic form

$$\delta_\lambda \omega = 2\lambda^2 \pi_1^* \omega - 2(1-\lambda)^2 \pi_2^* \omega \ , \tag{4.1}$$

where π_i is the projection of $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$ on i-th factor for $i = 1, 2$.

Now, let $\Psi_{\mu(\lambda)\rho(\lambda)}$ be a λ-chord transformation (3.1)(3.2). Then,

$$\left(\Psi^{-1}_{\mu(\lambda)\rho(\lambda)}\right)^* (\delta_\lambda \omega) \ = \ \Omega_{\mu(\lambda)\rho(\lambda)} =$$

$$= \ \left(\frac{2\lambda(1-\lambda)}{\rho(\lambda)}\right) \dot{\omega} \ + \ 2\left((2\lambda - 1) - \frac{2\lambda(1-\lambda)}{\rho(\lambda)}\mu(\lambda)\right) pr^* \omega \ , \tag{4.2}$$

where $pr : T\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is the standard projection and $\dot{\omega}$ is the canonical symplectic form on the tangent bundle to $(\mathbb{R}^{2m}, \omega)$, which is defined by $\dot{\omega}(x, \dot{x}) = d\{\dot{x} \lrcorner \omega\}(x)$ or, in Darboux coordinates for ω, by

$$\dot{\omega} = \sum_{i=1}^m d\dot{p}_i \wedge dq_i + dp_i \wedge d\dot{q}_i \ . \tag{4.3}$$

For the standard λ-chord transformation ($\mu(\lambda) \equiv 0$, $\rho(\lambda) \equiv 1/2$),

$$\left(\Gamma_\lambda^{-1}\right)^* (\delta_\lambda \omega) \ = \ 4\lambda(1-\lambda)\dot{\omega} \ + \ 2(2\lambda - 1) pr^* \omega \ . \tag{4.4}$$

On the other hand, if we consider the tilted λ-chord transformation defined by (3.1) and (3.7), which is given by the choices $\mu(\lambda) = 2\lambda - 1$ and $\rho(\lambda) = 2\lambda(1-\lambda)$, we obtain the λ-independent form

$$\left(\Phi_\lambda^{-1}\right)^* (\delta_\lambda \omega) \; = \; \dot{\omega} \; . \tag{4.5}$$

The pair $(T\mathbb{R}^{2m}, \dot{\omega})$ is the *canonical symplectic tangent bundle* of $(\mathbb{R}^{2m}, \omega)$ and is, thus, a $4m$-dimensional symplectic space. Similarly, for any other $\Psi_{\mu(\lambda)\rho(\lambda)}$, the pair $(T\mathbb{R}^{2m}, \Omega_{\mu(\lambda)\rho(\lambda)})$ defines a *noncanonical symplectic tangent bundle* of $(\mathbb{R}^{2m}, \omega)$, also a $4m$-dimensional symplectic space. Note that $\Omega_{\mu(\lambda)\rho(\lambda)}$ satisfies $\Omega_{\mu(1/2)\rho(1/2)} = \dot{\omega}$.

Remark 4.1. The vertical subspaces of $TT\mathbb{R}^{2m}$ are Lagrangian for $\Omega_{\mu(\lambda)\rho(\lambda)}$, as one can see from the explicit expressions (4.2) and (4.3), which means that $pr : T\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defines a **Lagrangian fiber bundle** with respect to $\Omega_{\mu(\lambda)\rho(\lambda)}$ i.e. a fiber bundle whose total space is equipped with a symplectic structure and whose fibers are Lagrangian submanifolds [2]. This follows from the *weights* in (4.1) for $\delta_\lambda\omega$.

In order to understand the ideology of this present construction, let's first focus attention on the case $\lambda = 1/2$. Consider a Lagrangian submanifold $\Lambda_{1/2} \subset (\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_{1/2}\omega)$ that is a graph onto the first factor of $(\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_{1/2}\omega)$. Then, $\Lambda_{1/2}$ is the graph of a symplectomorphism, or a *canonical transformation* $\phi : (\mathbb{R}^{2m}, \omega) \to (\mathbb{R}^{2m}, \omega)$, $\phi^*\omega = \omega$.

For the $\frac{1}{2}$-chord transformation $\Gamma_{1/2}$, if $\mathcal{L}_{1/2} = \Gamma_{1/2}\Lambda_{1/2}$ is locally a graph over the zero section of $(T\mathbb{R}^{2m}, \dot{\omega})$, then this canonical transformation $(\mathbb{R}^{2m}, \omega) \to (\mathbb{R}^{2m}, \omega)$, $x^- \mapsto x^+$, can locally be "described" by the midpoint $x = (x^+ + x^-)/2$, that is, this canonical transformation can locally be described by a *generating function* of the midpoints.[1]

This midpoint description generalizes for when $\Lambda_{1/2}$ is not a graph onto the first factor of $(\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_{1/2}\omega)$, but is still Lagrangian . In this case, $\Lambda_{1/2}$ defines a *canonical relation* on $(\mathbb{R}^{2m}, \omega)$ and, if $\mathcal{L}_{1/2} = \Gamma_{1/2}\Lambda_{1/2}$ is locally a graph over the zero section of $(T\mathbb{R}^{2m}, \dot{\omega})$, then this canonical relation $\Lambda_{1/2} = \{(x^+, x^-)\}$ can locally be "described" by the midpoints, that is, by a generating function of the midpoints, given the Lagrangian fiber bundle $pr : (T\mathbb{R}^{2m}, \dot{\omega}) \to \mathbb{R}^{2m}$ (see [20]).

Clearly, if L is Lagrangian in $(\mathbb{R}^{2m}, \omega)$, then $L \times L = \{(x^+, x^-)\}$ defines a relation on $(\mathbb{R}^{2m}, \omega)$. If we want to "describe" this relation by the midpoints, we endow the product space with the symplectic form $\delta_{1/2}\omega$ which makes $L \times L = \Lambda_{1/2}$ a canonical relation on $(\mathbb{R}^{2m}, \omega)$.

However, if we now want to "describe" the relation $\{(x^+, x^-)\}$ by another λ-point $x = \lambda x^+ + (1-\lambda)x^-$ on the chord joining the pair

[1]Such a midpoint description was first introduced by Poincaré [18].

(x^+, x^-), this relation cannot be canonical anymore. In other words, if we want to describe the relation $L \times L = \{(x^+, x^-)\}$ by a *generating function* of the λ-points $x = \lambda x^+ + (1-\lambda)x^-$, for some $\lambda \neq 1/2$, we must now *weight differently* the symplectic form ω on the two copies of $\mathbb{R}^{2m}$ in such a way as to account for the fact that we are describing the relation on $(\mathbb{R}^{2m}, \omega)$ in an *asymmetrical* way. The weights given in formula (4.1) for $\delta_\lambda \omega$ correctly account for this asymmetry.

Definition 4.2. For each $\lambda \in \mathbb{R} \setminus \{0,1\}$, a Lagrangian submanifold $\Lambda_\lambda \subset (\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_\lambda \omega)$ defines a **λ-weighted symplectic relation** on $(\mathbb{R}^{2m}, \omega)$. In particular, if L is a Lagrangian submanifold of $(\mathbb{R}^{2m}, \omega)$, then $L \times L = \Lambda_\lambda$ defines a λ-weighted symplectic relation on $(\mathbb{R}^{2m}, \omega)$.

Now, if $\Psi_{\mu(\lambda)\rho(\lambda)}$ is a λ-chord transformation, let

$$\mathcal{L}_{\mu(\lambda)\rho(\lambda)} = \Psi_{\mu(\lambda)\rho(\lambda)}(L \times L).$$

If $\mathcal{L}_{\mu(\lambda)\rho(\lambda)}$ is locally a graph over the zero section of $(T\mathbb{R}^{2m}, \Omega_{\mu(\lambda)\rho(\lambda)})$, then $\mathcal{L}_{\mu(\lambda)\rho(\lambda)}$ can locally be "described" by the λ-points $x = \lambda x^+ + (1-\lambda)x^-$, that is, by a generating function of these λ-points.

In other words, $\mathcal{L}_{\mu(\lambda)\rho(\lambda)} = \Psi_{\mu(\lambda)\rho(\lambda)}(L \times L)$ is a Lagrangian submanifold of the $4m$-dimensional symplectic tangent bundle $(T\mathbb{R}^{2m}, \Omega_{\mu(\lambda)\rho(\lambda)})$ which, with its standard projection $pr : T\mathbb{R}^{2m} \to \mathbb{R}^{2m}$, is a Lagrangian fiber bundle.

The restriction of the projection of a Lagrangian bundle to a embedding Lagrangian submanifold in the total space of this bundle is called a **Lagrangian map** [2]. So we obtain the following result.

Proposition 4.3. *$pr|_{\mathcal{L}_{\mu(\lambda)\rho(\lambda)}} : \mathcal{L}_{\mu(\lambda)\rho(\lambda)} \to \mathbb{R}^{2m}$ is a Lagrangian map.*

The set of critical values of a Lagrangian map is called a **caustic** and from Theorem 3.3 we have

Corollary 4.4. *The caustic of $pr|_{\mathcal{L}_{\mu(\lambda)\rho(\lambda)}}$ is $E_\lambda(L)$.*

Definition 4.5 ([2])**.** Two germs of Lagrangian fiber bundles are **Lagrangian equivalent** if there exists a fiber-preserving diffeomorphism-germ of the bundle spaces mapping one symplectic structure to the other. Two germs of Lagrangian maps are **Lagrangian equivalent** if there exists a Lagrangian equivalence of the corresponding germs of Lagrangian fiber bundles that sends the domain of the first map to the domain of the second.

A Lagrangian map-germ at a point is said to be **Lagrangian stable** if for every map with the given germ there is a neighbourhood in the space of Lagrangian maps (in the topology of the convergence with a finite number of derivatives on each compact set) and a neighbourhood of the original point such that each Lagrangian map belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Lagrangian equivalent to the original germ.

Remark 4.6. For $\lambda \neq 0, 1$, note that the λ-dependent affine bijection $T\mathbb{R}^{2m} \to T\mathbb{R}^{2m}$, $(x, \dot{x}) \mapsto (x, \frac{\dot{x}-(2\lambda-1)x}{4\lambda(1-\lambda)})$, relating the tilted and the (standard) faithful λ-chord transformations Φ_λ and Γ_λ, defines a *Lagrangian equivalence* between $(T\mathbb{R}^{2m}, (\Gamma_\lambda^{-1})^*(\delta_\lambda \omega))$ and $(T\mathbb{R}^{2m}, \dot{\omega})$. More generally, *the distinction between faithful and tilted λ-chord transformations looses meaning via Lagrangian equivalence.*

In view of remarks 3.4 and 4.6, *in the remaining of this paper we only use the tilted λ-chord transformation* Φ_λ defined by (3.1) and (3.7) and only consider the canonical symplectic tangent bundle $(T\mathbb{R}^{2m}, \dot{\omega})$. The only exception is the appendix, where we study curves in nonsymplectic plane and use, instead, the standard chord transformation.

So, let L and $\widetilde{L}$ be smooth closed Lagrangian submanifolds of the symplectic affine space $(\mathbb{R}^{2m}, \omega)$ and let

$$\mathcal{L}_\lambda = \Phi_\lambda(L \times L) \ , \ \widetilde{\mathcal{L}}_\lambda = \Phi_\lambda(\widetilde{L} \times \widetilde{L}) \ ,$$

be the corresponding smooth closed Lagrangian submanifolds of the canonical symplectic tangent bundle $(T\mathbb{R}^{2m}, \dot{\omega})$, where

$$\Phi_\lambda : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to T\mathbb{R}^{2m}$$

$$(x^+, x^-) \mapsto (\lambda x^+ + (1-\lambda)x^-, \lambda x^+ - (1-\lambda)x^-) \ .$$

Definition 4.7. $E_\lambda(L)$ and $E_\lambda(\widetilde{L})$ are **Lagrangian equivalent** if the Lagrangian maps $pr|_{\mathcal{L}_\lambda}$ and $pr|_{\widetilde{\mathcal{L}}_\lambda}$ are Lagrangian equivalent.

Remark 4.8. Lagrangian equivalence of affine λ-equidistants, as defined above, fulfills all the requirements for λ-chord equivalence of affine λ-equidistants, as in definition 3.5. We shall therefore use this well-known notion of Lagrangian equivalence for the classification of $E_\lambda(L)$.

It follows from above definitions and remarks and proposition 3.7:

Corollary 4.9. *The classification of $E_\lambda(L)$ by Lagrangian equivalence is affine symplectic invariant.*

Definition 4.10. From the above corollary, we also use the terms **affine-Lagrangian equivalence** and **affine-Lagrangian stability** for Lagrangian equivalence and Lagrangian stability (definition 4.5) of an affine equidistant E_λ of a Lagrangian submanifold $L \subset (\mathbb{R}^{2m}, \omega)$.

We now rely on the well known fact that any smooth Lagrangian submanifold L of a symplectic affine space can be locally described as the graph of the differential of a certain generating function.

Thus, let L^+ and L^- denote germs of L at the points a^+ and a^-.

Proposition 4.11. *If the pair a^+, a^- is k-parallel ($k = 1, \cdots, m$) then there exists canonical coordinates (p, q) on $\mathbb{R}^{2m}$ and function germs S^+ and S^- such that*

$$L^+ : p_i = \frac{\partial S^+}{\partial q_i}(q_1, \cdots, q_m), \text{ for } i = 1, \cdots, m \tag{4.6}$$

(4.7)

$$L^- : \begin{cases} p_j = \ \ \frac{\partial S^-}{\partial q_j}(q_1, \cdots, q_k, p_{k+1}, \cdots, p_m), \text{ for } j = 1, \cdots, k, \\ q_l = -\frac{\partial S^-}{\partial p_l}(q_1, \cdots, q_k, p_{k+1}, \cdots, p_m), \text{ for } l = k+1, \cdots, m \end{cases}$$

and $d^2S^+(q^+_{a,1}, \cdots, q^+_{a,m}) = 0$ and $d^2S^-(p^-_{a,1}, \cdots, p^-_{a,k}, q^-_{a,k+1}, \cdots, p^-_{a,m}) = 0$, where $a^+ = (p^+_a, q^+_a)$ and $a^- = (p^-_a, q^-_a)$.

Proof. We can find a linear symplectic change of coordinates such that the tangent (affine) spaces have the following form $T_{a^+}L^+ = \{p = p^+_a\}$, where $a^+ = (p^+_a, q^+_a)$ and $T_{a^-}L^- = \{p_1 = p^-_{a,1}, \cdots, p_k = p^-_{a,k}, q_{k+1} = q^-_{a,k+1}, \cdots, q_m = q^-_{a,m}\}$, where $a^- = (p^-_a, q^-_a)$. Since L is a smooth Lagrangian submanifold, it follows from standard considerations that it can be described locally by differentials of generating functions of the forms stated above in neighborhoods of a^+ and a^-, in which case we have that $d^2S^+|a^+ = d^2S^-|a^- = 0$. □

From the above, we state the main result of this section, which shall be used in all that follows.

Let the arguments of the function S^+ be denoted by $(q^+_1, \cdots, q^+_m)$ and the arguments of the function S^- by $(q^-_1, \cdots, q^-_k, p^-_{k+1}, \cdots, p^-_m)$. Let $q = (q_1, \cdots, q_m)$, $p = (p_1, \cdots, p_m)$, $\dot{q} = (\dot{q}_1, \cdots, \dot{q}_m)$, $\dot{p} = (\dot{p}_1, \cdots, \dot{p}_m)$.

Also, let $\beta = (\beta_1, \cdots, \beta_m)$ and, for any $k < m$, let $[k] = \{1, \cdots, k\}$, so that $\beta_{[k]} = (\beta_1, \cdots, \beta_k)$, and $\alpha_{[m]\setminus[k]} = (\alpha_{k+1}, \cdots, \alpha_m)$.

Let $L^+ \times L^-$ denote the germ of $L \times L$ at the point $(a^+, a^-) \in L \times L$ so that $\mathcal{L}_\lambda = \Phi_\lambda(L^+ \times L^-)$ is the germ at $(a, \dot{a})$, where $a = \lambda a^+ + (1-\lambda)a^-$, $\dot{a} = \lambda a^+ - (1-\lambda)a^-$, of a smooth Lagrangian submanifold of the $4m$-dimensional symplectic tangent bundle $(T\mathbb{R}^{2m}, \dot{\omega})$.

The restriction to $\mathcal{L}_\lambda$ of the projection $pr : T\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defines a germ of Lagrangian map and we have the following result:

Theorem 4.12. *If the pair a^+, a^- is k-parallel and L^+ and L^- are given by (5.3) and (5.4) then the germ of $\mathcal{L}_\lambda$ at $(a, \dot{a})$ is generated by* *the germ of the generating family F_λ which is given by*

$$F_\lambda(p,q,\alpha_{[m]\setminus[k]},\beta) = \tag{4.8}$$
$$2\lambda^2 S^+\left(\tfrac{q+\beta}{2\lambda}\right) - 2(1-\lambda)^2 S^-\left(\tfrac{q_{[k]}-\beta_{[k]}\ ,\ p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right)$$
$$-\sum_{i=1}^k p_i\beta_i + \tfrac{1}{2}\sum_{j=k+1}^m q_j\alpha_j - p_j\beta_j - \alpha_j\beta_j - p_jq_j$$

Proof. We show that

$$\mathcal{L}_\lambda = \left\{(\dot p,\dot q,p,q) : \exists(\alpha,\beta),\ \dot p = \frac{\partial F_\lambda}{\partial q},\ \dot q = -\frac{\partial F_\lambda}{\partial p},\ \frac{\partial F_\lambda}{\partial \alpha} = \frac{\partial F_\lambda}{\partial \beta} = 0\right\}. \tag{4.9}$$

We have for $i = 1,\cdots,k$ and $j = k+1,\cdots,m$

$$\dot p_i = \lambda\frac{\partial S^+}{\partial q_i^+}\left(\frac{q+\beta}{2\lambda}\right) - (1-\lambda)\frac{\partial S^-}{\partial q_i^-}\left(\frac{q_{[k]}-\beta_{[k]}, p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right), \tag{4.10}$$

$$\dot p_j = \lambda\frac{\partial S^+}{\partial q_j^+}\left(\frac{q+\beta}{2\lambda}\right) + \frac{1}{2}(\alpha_j - p_j), \tag{4.11}$$

$$\dot q_i = \beta_i, \tag{4.12}$$

$$\dot q_j = (1-\lambda)\frac{\partial S^-}{\partial p_j^-}\left(\frac{q_{[k]}-\beta_{[k]}, p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right) + \frac{1}{2}(\beta_j + q_j), \tag{4.13}$$

$$\frac{\partial F_\lambda}{\partial \alpha_j} = (1-\lambda)\frac{\partial S^-}{\partial p_j^-}\left(\frac{q_{[k]}-\beta_{[k]}, p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right) + \frac{1}{2}(q_j - \beta_j) = 0, \tag{4.14}$$

$$\frac{\partial F_\lambda}{\partial \beta_i} = \lambda\frac{\partial S^+}{\partial q_i^+}\left(\frac{q+\beta}{2\lambda}\right) + (1-\lambda)\frac{\partial S^-}{\partial q_i^-}\left(\frac{q_{[k]}-\beta_{[k]}, p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right) - p_i = 0, \tag{4.15}$$

$$\frac{\partial F_\lambda}{\partial \beta_j} = \lambda\frac{\partial S^+}{\partial q_j^+}\left(\frac{q+\beta}{2\lambda}\right) - \frac{1}{2}(\alpha_j + p_j) = 0. \tag{4.16}$$

By (4.12) we get $\beta_i = \dot q_i$ for $i = 1,\cdots,k$. (4.13) and (4.14) imply that $\beta_j = \dot q_j$ for $j = k+1,\cdots,m$. By (4.11) and (4.16) we have $\alpha_j = \dot p_j$ for $j = k+1,\cdots,m$. Thus we eliminate $(\alpha_{[m]\setminus[k]},\beta)$.

Then (4.10) implies that for $i = 1,\cdots,k$

$$\dot p_i = \lambda\frac{\partial S^+}{\partial q_i^+}\left(\frac{q+\dot q}{2\lambda}\right) - (1-\lambda)\frac{\partial S^-}{\partial q_i^-}\left(\frac{q_{[k]}-\dot q_{[k]}, p_{[m]\setminus[k]}-\dot p_{[m]\setminus[k]}}{2(1-\lambda)}\right). \tag{4.17}$$

(4.15) implies that for $i = 1, \cdots, k$
(4.18)
$$p_i = \lambda \frac{\partial S^+}{\partial q_i^+}\left(\frac{q+\dot{q}}{2\lambda}\right) + (1-\lambda)\frac{\partial S^-}{\partial q_i^-}\left(\frac{q_{[k]} - \dot{q}_{[k]}, p_{[m]\setminus[k]} - \dot{p}_{[m]\setminus[k]}}{2(1-\lambda)}\right).$$

By (4.11) and (4.16) we have for $j = k+1, \cdots, m$

$$\frac{1}{2\lambda}(\dot{p}_j + p_j) = \frac{\partial S^+}{\partial q_j^+}\left(\frac{q+\dot{q}}{2\lambda}\right). \tag{4.19}$$

(4.13) and (4.14) imply that for $j = k+1, \cdots, m$
(4.20)
$$\frac{1}{2(1-\lambda)}(q_j - \dot{q}_j) = -(1-\lambda)\frac{\partial S^-}{\partial p_j^-}\left(\frac{q_{[k]} - \dot{q}_{[k]}, p_{[m]\setminus[k]} - \dot{p}_{[m]\setminus[k]}}{2(1-\lambda)}\right).$$

If (p^+, q^-) (p^-, q^-) are points in L^+ and L^- described by (5.3) and (5.4) respectively then (4.17)-(4.20) describe $\mathcal{L}_\lambda$ in coordinates given by (3.1)-(9.1). □

Remark 4.13. It is clear from the form of the generating family, given by (4.8), that *the degree of parallelism is the corank of the singularity* i. e. the corank of the Hessian of the function

$$\mathbb{R}^{2m-k} \ni (\alpha_{[m]\setminus[k]}, \beta) \mapsto F_\lambda(p_a, q_a, \alpha_{[m]\setminus[k]}, \beta) \in \mathbb{R}$$

Now, let $x = (p, q)$. We recall that two germs of generating families $F = F(x, \kappa)$ and $G(x, \kappa)$ are **R^+-equivalent** if there exists a fiber-preserving diffeomorphism-germ $Z(x, \kappa) = (\phi(x), \zeta(x, \kappa))$ and a function-germ g such that $F(x, \kappa) = G(Z(x, \kappa)) + g(x)$.

The families F and $\tilde{F}$ with common parameters x but in general with different spaces of arguments κ and $\tilde{\kappa}$ are **stably R^+-equivalent** if there exist nondegenerate quadratic forms Q and $\tilde{Q}$ (in the new arguments) such that families $F + Q$ and $\tilde{F} + \tilde{Q}$ are R^+-equivalent.

Theorem 4.14 ([2])**.** *Two germs of Lagrangian maps are Lagrangian equivalent if and only if the germs of their generating families are stably R^+-equivalent.*

Corollary 4.15. *Let L and $\tilde{L}$ be smooth closed Lagrangian submanifolds of the symplectic affine space $(\mathbb{R}^{2m}, \omega)$. Germs $E_\lambda(L)$ and $E_\lambda(\tilde{L})$ are Lagrangian equivalent if and only if the corresponding germs of generating families for $\mathcal{L}_\lambda$ and $\tilde{\mathcal{L}}_\lambda$ are stably R^+-equivalent.*

5. Singularities of equidistants of Lagrangian submanifolds

In this section we study the singularities of momentary equidistants of closed Lagrangian submanifolds up to Lagrangian equivalence. Remind that, for $E_\lambda(L)$, Lagrangian stability is affine-Lagrangian stability (corollary 4.9 and definition 4.10). We have the following results:

Theorem 5.1. *Any caustic of stable Lagrangian singularity in the $4m$-dimensional symplectic tangent bundle $(T\mathbb{R}^{2m}, \dot{\omega})$ is realizable as $E_\lambda(L)$, for some smooth closed Lagrangian submanifold L in $(\mathbb{R}^{2m}, \omega)$.*

Corollary 5.2. *For a smooth Lagrangian curve L, generic singularities of $E_\lambda(L)$ are cusps. In the neighborhood of its regular points, $E_\lambda(L)$ is a smooth curve in $(\mathbb{R}^2, \omega)$.*

Corollary 5.3. *For a smooth Lagrangian surface L, generic singularities of $E_\lambda(L)$ can be cusps A_3, swallowtails A_4, butterflies A_5, hyperbolic umbilics D_4^+, elliptic umbilics D_4^-, or parabolic umbilics D_5. In the neighborhood of its regular points, $E_\lambda(L)$ is a 3-dimensional smooth submanifold of $(\mathbb{R}^4, \omega)$.*

The proof of Theorem 5.1 is based on the following description of the stable Lagrangian singularities.

Theorem 5.4 ([2])**.** *The germ at (x_0, κ_0) of Lagrangian map $(x, \kappa) \mapsto x$ given by a Lagrangian submanifold $\mathcal{L}^* \subset (T^*\mathbb{R}^n, \omega_{can})$,*

$$\mathcal{L}^* = \left\{ (x^*, x) \mid \exists \kappa \ \ \frac{\partial F}{\partial \kappa} = 0, \ x^* = \frac{\partial F}{\partial x} \right\},$$

where

$$rank_{(x_0,\kappa_0)} \left[\frac{\partial^2 F}{\partial \kappa^2}, \ \frac{\partial^2 F}{\partial \kappa \partial x} \right]$$

is equal to the dimension of κ-space, is Lagrangian stable if and only if

$$\mathcal{E}_\kappa \Big/ \left\langle \frac{\partial f}{\partial \kappa} \right\rangle \ = \ span_{\mathbb{R}} \left\{ 1, \frac{\partial F}{\partial x}(x_0, \kappa) \right\}, \tag{5.1}$$

where $\mathcal{E}_\kappa$ is the ring of germs at κ_0 of functions in κ, $f(\kappa) = F(x_0, \kappa)$ and $\langle \partial f / \partial \kappa \rangle$ denotes the ideal in $\mathcal{E}_\kappa$ generated by $\partial f / \partial \kappa_i$ for $i = 1, \cdots, 2m - k$.

Remark 5.5. (5.1) means that $F(x, \kappa) + x_0$ is a $\mathcal{R}$-versal deformation of $f(\kappa)$ ([2]).

First, we calculate the formula appearing in Theorem 5.4 for the special case of the generating family F_λ and the Lagrangian submanifold $\mathcal{L}_\lambda \subset (T\mathbb{R}^{2m}, \dot{\omega})$, given by theorem 4.12. For a fixed λ, let $x = (p, q)$ and $\kappa = (\alpha, \beta)$. From (4.8) we easily see that

$$\mathrm{rank}_{(a,\dot{a})}\left[\frac{\partial^2 F_\lambda}{\partial \kappa^2},\ \frac{\partial^2 F_\lambda}{\partial \kappa \partial x}\right] = 2m - k,$$

hence is equal to the dimension of κ-space. The caustic of $\mathcal{L}_\lambda$ generated by $F_\lambda(x, \kappa)$ is given by

$$E_\lambda = \left\{x \in \mathbb{R}^{2m} \mid \exists \kappa \ \ \frac{\partial F_\lambda}{\partial \kappa} = 0,\ \det\left[\frac{\partial^2 F_\lambda}{\partial \kappa_i \partial \kappa_j}\right] = 0\right\}. \tag{5.2}$$

By Proposition 4.11 we obtain that

$$S^+(q^+) = \sum_{i=1}^{m} p^+_{a,i}(q^+_i - q^+_{a,i}) + S^+_3(q^+ - q^+_a) \tag{5.3}$$

$$\begin{aligned} S^-(q^-_{[k]}, p^-_{[m]\setminus[k]}) &= \sum_{i=1}^{k} p^-_{a,i}(q^-_i - q^-_{a,i}) - \sum_{i=k+1}^{m} q^-_{a,i}(p^-_i - p^-_{a,i}) + \\ &+ S^-_3(q^-_{[k]} - q^-_{a,[k]}, p^-_{[m]\setminus[k]} - p^-_{a,[m]\setminus[k]}), \end{aligned}$$

where$S^\pm_3 \in \mathfrak{m}^3$ ($\mathfrak{m}$ is the maximal ideal of the ring of smooth function-germs on $\mathbb{R}^n$ at 0).

We write the generating families in coordinates $\tilde{p} = p - p_a$, $\tilde{q} = q - q_a$, $s = \alpha - \dot{p}_a$, $t = \beta - \dot{q}_a$, where $a = (p_a, q_a)$, $\dot{a} = (\dot{p}_a, \dot{q}_a)$. Then by Theorem 4.12 we obtain

$$\begin{gathered} F_\lambda(\tilde{p}, \tilde{q}, s, t) = \\ 2\lambda^2 S^+_3\left(\tfrac{\tilde{q}+t}{2\lambda}\right) - 2(1-\lambda)^2 S^-_3\left(\tfrac{\tilde{q}_{[k]} - t_{[k]}\ ,\ \tilde{p}_{[m]\setminus[k]} - s_{[m]\setminus[k]}}{2(1-\lambda)}\right) \\ - \textstyle\sum_{i=1}^{k} \tilde{p}_i t_i + \frac{1}{2}\sum_{j=k+1}^{m} \tilde{q}_j s_j - \tilde{p}_j t_j - s_j t_j - \tilde{p}_j \tilde{q}_j + \sum_{l=1}^{m} \dot{p}_{a,l}\tilde{q}_l - \dot{q}_{a,l}\tilde{p}_l \end{gathered} \tag{5.4}$$

$$\begin{gathered} f_\lambda(s, t) = F_\lambda(0, 0, s, t) = \\ 2\lambda^2 S^+_3\left(\tfrac{t}{2\lambda}\right) - 2(1-\lambda)^2 S^-_3\left(\tfrac{-t_{[k]}, -s_{[m]\setminus[k]}}{2(1-\lambda)}\right) - \tfrac{1}{2}\textstyle\sum_{j=k+1}^{m} s_j t_j \end{gathered} \tag{5.5}$$

The ideal $\left\langle \frac{\partial f_\lambda}{\partial \kappa} \right\rangle$ is generated by the function germs (we let the indices $i = 1, \cdots, k$; $j = k+1, \cdots, m$)

$$\frac{\partial f_\lambda}{\partial t_i} = \lambda \frac{\partial S^+_3}{\partial q^+_i}\left(\frac{t}{2\lambda}\right) + (1-\lambda)\frac{\partial S^-_3}{\partial q^-_i}\left(\frac{-t_{[k]}, -s_{[m]\setminus[k]}}{2(1-\lambda)}\right) \tag{5.6}$$

$$\frac{\partial f_\lambda}{\partial t_j} = -\frac{1}{2}s_j + \lambda \frac{\partial S^+_3}{\partial q^+_j}\left(\frac{t}{2\lambda}\right) \tag{5.7}$$

$$\frac{\partial f_\lambda}{\partial s_j} = -\frac{1}{2}t_j + (1-\lambda)\frac{\partial S_3^-}{\partial p_j^-}\left(\frac{-t_{[k]}, -s_{[m]\setminus[k]}}{2(1-\lambda)}\right) \tag{5.8}$$

and partial derivatives with respect to the parameters at $\tilde{p} = \tilde{q} = 0$ are

$$\frac{\partial F_\lambda}{\partial \tilde{p}_i}(0,0,s,t) = -\dot{q}_{ai} - t_i \tag{5.9}$$

$$\frac{\partial F_\lambda}{\partial \tilde{p}_j}(0,0,s,t) = -\dot{q}_{aj} - \frac{1}{2}t_j - (1-\lambda)\frac{\partial S_3^-}{\partial p_j^-}\left(\frac{-t_{[k]}, -s_{[m]\setminus[k]}}{2(1-\lambda)}\right) \tag{5.10}$$

(5.11)

$$\frac{\partial F_\lambda}{\partial \tilde{q}_i}(0,0,s,t) = \dot{p}_{ai} + \lambda\frac{\partial S_3^+}{\partial q_i^+}\left(\frac{t}{2\lambda}\right) - (1-\lambda)\frac{\partial S_3^-}{\partial q_i^-}\left(\frac{-t_{[k]}, -s_{[m]\setminus[k]}}{2(1-\lambda)}\right)$$

$$\frac{\partial F_\lambda}{\partial \tilde{q}_j}(0,0,s,t) = \dot{p}_{aj} + \frac{1}{2}s_j + \lambda\frac{\partial S_3^+}{\partial q_j^+}\left(\frac{t}{2\lambda}\right) \tag{5.12}$$

In order to prove Theorem 5.1, we analyze separately the cases of 1-parallelism and 2-parallelism, in every dimension.

5.1. Singularities of E_λ for 1-parallelism.

Proposition 5.6. *Any A_k singularity can be realizable as $E_\lambda(L)$, for 1-parallelism and $k \leq 2m+1$.*

Proof. We use the generating family of the form (5.4) for $k=1$. To realize A_{2l} singularity take the following function-germs

$$S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + (\tilde{q}_1^+)^{2l+1} + \sum_{i=2}^{l} \tilde{q}_i^+ (\tilde{q}_1^+)^{2i-1},$$

$$S_3^-(\tilde{q}_1^-, \tilde{p}_2^-, \cdots, \tilde{p}_m^-) = -(1-\lambda)(\tilde{q}_1^-)^3 + \sum_{i=2}^{l-1} \tilde{p}_i^- (\tilde{q}_1^-)^{2(l-i+1)}.$$

A_{2l+1} singularity is realizable by the following function-germs

$$S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + (\tilde{q}_1^+)^{2l+2} + \sum_{i=2}^{l} \tilde{q}_i^+ (\tilde{q}_1^+)^{2i-1},$$

$$S_3^-(\tilde{q}_1^-, \tilde{p}_2^-, \cdots, \tilde{p}_m^-) = -(1-\lambda)(\tilde{q}_1^-)^3 + \sum_{i=2}^{l} \tilde{p}_i^- (\tilde{q}_1^-)^{2(l-i+2)}.$$

By long but straightforward calculations using (5.6)-(5.12) one can check that (5.1) is satisfied. Theorem 5.4 completes the proof. □

5.2. **Singularities of E_λ for 2-parallelism.**

Proposition 5.7. *Any D_k ($k \geq 4$) or E_k ($k = 6, 7, 8$) singularity can be realizable as $E_\lambda(L)$, for 2-parallelism and $k \leq 2m+1$.*

Proof. In case of 2-parallelism we use the generating family of the form (5.4) with $k = 2$. The following singularities are realizable by the following generating functions:

$$D_{2l}:$$

$$S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + \tilde{q}_2^+(\tilde{q}_1^+)^2 \pm (\tilde{q}_2^+)^{2l-1} + \lambda(\tilde{q}_2^+)^3 + \sum_{i=2}^{l-1} \tilde{q}_{i+1}^+(\tilde{q}_2^+)^{2i-1},$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)(\tilde{q}_1^-)^3 - (1-\lambda)(\tilde{q}_2^-)^3 + \sum_{i=2}^{l-2} \tilde{p}_{i+1}^-(\tilde{q}_2^-)^{2(l-i)}.$$

$$D_{2l+1}:$$

$$S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + \tilde{q}_2^+(\tilde{q}_1^+)^2 \pm (\tilde{q}_2^+)^{2l} + \lambda(\tilde{q}_2^+)^3 + \sum_{i=2}^{l-1} \tilde{q}_{i+1}^+(\tilde{q}_2^+)^{2i-1},$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)(\tilde{q}_1^-)^3 - (1-\lambda)(\tilde{q}_2^-)^3 + \sum_{i=2}^{l-1} \tilde{p}_{i+1}^-(\tilde{q}_2^-)^{2(l-i+1)}.$$

$$E_6:$$

$$S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 \pm (\tilde{q}_2^+)^4 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2\tilde{q}_3^+,$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3.$$

$$E_7:$$

$$S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + (\tilde{q}_2^+)^3\tilde{q}_3^+,$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3 + (\tilde{q}_2^-)^4\tilde{p}_3^-.$$

$$E_8:$$

$$S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 + (\tilde{q}_2^+)^5 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2\tilde{q}_3^+ + \tilde{q}_1^+(\tilde{q}_2^+)^3\tilde{q}_4^+,$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3 + (\tilde{q}_2^-)^3\tilde{p}_3^-.$$

We use the method described in the proof of Proposition 5.6. By long but straightforward calculations we obtain the result. □

6. The GCS of a Lagrangian submanifold: the criminant

We now begin the study of singularities of the global centre symmetry set of a smooth closed Lagrangian submanifold $L \subset (\mathbb{R}^{2m}, \omega)$.

Remind that, in terms of the projection

$$\pi : \mathbb{R} \times \mathbb{R}^{2m} \ni (\lambda, x) \mapsto x \in \mathbb{R}^{2m} \ , \tag{6.1}$$

definition 2.7 states that $GCS(L)$ is the locus of critical points of $\pi|_{\mathbb{E}(L)}$, where

$$\mathbb{E}(L) = \bigcup_{\lambda \in \mathbb{R}} \{\lambda\} \times E_\lambda(L) \subset \mathbb{R} \times \mathbb{R}^{2m} \ . \tag{6.2}$$

From remarks 2.9 and 2.11, $GCS(L)$ consists of two parts which can be further refined to comprise three parts:

(i) the **Wigner caustic** $E_{1/2}(L)$.

(ii) the **centre symmetry caustic** $\Sigma'(L)$, consisting of the λ-family of π-projections of singularities of $\mathbb{E}(L)$, excluding the Wigner caustic.

(iii) the **criminant** $\Delta(L)$, being the π-projection of smooth parts of the extended wave front $\mathbb{E}(L)$ that are tangent to the fibers of π.

The classification of the Wigner caustic of a Lagrangian submanifold L has been mostly carried out in the last section, since the Wigner caustic is the $\lambda = 1/2$ affine equidistant. In a subsequent paper [8], we study $E_{1/2}(L)$ in a neighborhood L, considered in a broader sense, that is, considering pairs of points of the type $(a, a) \in L \times L$ as strongly parallel pairs. Then, in a neighborhood of L, we look for singularities of the Wigner caustic that have maximal co-rank m, that is, that are singularities of strong parallel pairs, for pairs of type (a, a).

In terms of the generating families of section 4, these must now have the special (simplest) form

$$F_{1/2}(p, q, \beta) = \frac{1}{2} S(q + \beta) - \frac{1}{2} S(q - \beta) - \sum_{i=1}^{m} p_i \beta_i \ , \tag{6.3}$$

where S is the local generating function of a germ of the Lagrangian submanifold $L \subset (\mathbb{R}^{2m}, \omega)$. It follows immediately from (6.3) that

$$F_{1/2}(p, q, -\beta) = -F_{1/2}(p, q, \beta) \tag{6.4}$$

and therefore only the generating families for singularities of co-rank m which are *odd* functions of β should be considered, in this case.

This point had already been made in [17], but, in order to classify such singularities, we must consider the condition of versality of unfoldings *in the category of odd functions* [8]. Condition (6.4) for the generating families implies $\mathbb{Z}_2$-symmetric singularities for the Wigner caustic on-shell. A first study of such symmetric singularities, for the case of surfaces in nonsymplectic $\mathbb{R}^4$, is presented in [14].

In order to study the centre symmetry caustic $\Sigma'(L)$ and the criminant $\Delta(L)$, the whole λ-family must be considered together.

Due to the Lagrangian condition, we resort to a classification via generating families, as was done in sections 4 and 5 for the λ-equidistants. From results of the previous sections we know that $E_\lambda(L)$ is the caustic of the Lagrangian submanifold $\mathcal{L}_\lambda = \Phi_\lambda(L\times L)$ in the Lagrangian fiber bundle $(T\mathbb{R}^{2m}, \dot\omega) \to \mathbb{R}^{2m}$, where Φ_λ be the *tilted* chord transformation given by equations (3.1) and (3.7), that is

$$\Phi_\lambda : \mathbb{R}^{2m}\times\mathbb{R}^{2m} \to T\mathbb{R}^{2m} ,$$

$$\Phi_\lambda : (x^+, x^-) \mapsto (x, \dot x) = (\lambda x^+ + (1-\lambda)x^-, \lambda x^+ - (1-\lambda)x^-) .$$

By Theorem 4.12 the generating family for $\mathcal{L}_\lambda$ is given by $F_\lambda(p, q, \alpha, \beta)$ of the form (4.8). Then the germ of $E_\lambda(L)$ is described as in equation (5.2), that is (for $\kappa = (\alpha, \beta)$),

$$E_\lambda(L) = \left\{ (p, q) \in \mathbb{R}^{2m} \ \ | \ \exists\kappa \ \ \frac{\partial F_\lambda}{\partial\kappa} = 0, \ \det\left[\frac{\partial^2 F_\lambda}{\partial\kappa_i\partial\kappa_j}\right] = 0 \right\}.$$

Since $\mathbb{E}(L)$ is the union of $\{\lambda\}\times E_\lambda$ we obtain that the germ of $\mathbb{E}(L)$ is described in the following way.

Proposition 6.1. $\mathbb{E}(L) = \left\{ (\lambda, p, q) : \exists\kappa \ \frac{\partial F_\lambda}{\partial\kappa} = 0, \ \det\left[\frac{\partial^2 F_\lambda}{\partial\kappa_i\partial\kappa_j}\right] = 0 \right\}$.

We now find a Lagrangian fiber bundle and the germ of a Lagrangian submanifold $\mathcal{L}$ in this bundle such that $\mathbb{E}(L)$ is the caustic of $\mathcal{L}$.

Lets us consider the fiber bundle

$$Pr : T^*\mathbb{R}\times T\mathbb{R}^{2m} \ni ((\lambda^*, \lambda), (\dot p, \dot q, p, q)) \mapsto (\lambda, (p, q)) \in \mathbb{R}\times\mathbb{R}^m. \tag{6.5}$$

The above bundle with the canonical symplectic structure

$$d\lambda^*\wedge d\lambda + \dot\omega$$

is a Lagrangian fiber bundle. For F_λ given by (4.8) in theorem 4.12, let

$$F(\lambda, p, q, \alpha, \beta) = F_\lambda(p, q, \alpha, \beta).$$

Then, for $\kappa = (\alpha, \beta) = (\alpha_{[m]\setminus[k]}, \beta) = (\kappa_1, \cdots, \kappa_{2m-k})$, we have the following immediate result:

Proposition 6.2. *The germ of $\mathbb{E}(L)$ is the caustic of the germ of a Lagrangian submanifold $\mathcal{L}$ of the Lagrangian fiber bundle $(T^*\mathbb{R}\times T\mathbb{R}^{2m}, d\lambda^*\wedge d\lambda + \dot\omega)$ generated by the family F in the following way*

$$\mathcal{L} = \left\{ ((\lambda^*, \lambda), (\dot p, \dot q, p, q)) : \exists\kappa \ \ \lambda^* = \frac{\partial F}{\partial\lambda}, \ \dot p = \frac{\partial F}{\partial q}, \ \dot q = -\frac{\partial F}{\partial p}, \ \frac{\partial F}{\partial\kappa} = 0 \right\}. \tag{6.6}$$

6.1. **Geometric characterization of the criminant of the GCS of a Lagrangian submanifold.** Let $L \subset (\mathbb{R}^{2m}, \omega)$ be a smooth closed Lagrangian submanifold. Remind that the criminant $\Delta(L)$ is the (closure of) the image under π_r of the set of regular points of $\mathbb{E}(L)$ which are critical points of the projection π restricted to the regular part of $\mathbb{E}(L)$. That is, the criminant $\Delta(L)$ is the envelope of the family of regular parts of momentary equidistants. We find the condition for the tangency to the fibers of the projection $\pi : (\lambda, p, q) \mapsto (p, q)$.

The results stated in this section are also valid for the criminant of the GCS of arbitrary smooth submanifolds [7], which generalize results in [10]-[12] for hypersurfaces, but here we present the results for Lagrangian submanifolds and their proofs in terms of generating families.

Proposition 6.3. *If (λ, a) is a regular point of $\mathbb{E}(L)$ then there exists a 1-parallel pair a^+, a^- such that $a = \lambda a^+ + (1-\lambda)a^-$.*

Proof. (λ_a, p_a, q_a) is a regular point of $\mathbb{E}(L)$ then the rank of the map

$$\kappa \mapsto \left(\frac{\partial F}{\partial \kappa}(\lambda_a, p_a, q_a, \kappa),\ \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa)\right]\right)$$

is maximal $2m-k$. It implies that corank $\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right]$ is 1. By Remark 4.13 we obtain that a^+, a^- is a 1-parallel pair. □

Proposition 6.4. *Let $(\lambda_a, a) = (\lambda_a, p_a, q_a)$ be a regular point of $\mathbb{E}(L)$. Then the fiber of π_r is tangent to $\mathbb{E}(L)$ at (λ_a, p_a, q_a) if and only if*

$$\text{rank}\left[\frac{\partial^2 F}{\partial \lambda \partial \kappa_j}, \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}\right] = \text{rank}\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}\right] = 2m-2 \tag{6.7}$$

at $(\lambda_a, p_a, q_a, \kappa_a)$ such that

$$\frac{\partial F}{\partial \kappa}(\lambda_a, p_a, q_a, \kappa_a) = 0,\ \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right] = 0.$$

Proof. By Proposition 6.3 if (λ_a, p_a, q_a) is a regular point of $\mathbb{E}(L)$ then the rank of the map

$$\kappa \mapsto \left(\frac{\partial F}{\partial \kappa}(\lambda_a, p_a, q_a, \kappa),\ \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa)\right]\right)$$

is maximal $2m-1$. We also have that rank $\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right]$ is $2m-2$ which implies that one of the columns of this matrix is linearly dependent on the others. For simplicity we assume that this is the first column. Thus a rank of the map

$$\kappa \mapsto \left(\frac{\partial F}{\partial \kappa_{[2m-1]\setminus[1]}}(\lambda_a, p_a, q_a, \kappa),\ \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa)\right]\right)$$

is maximal $2m-1$. By the implicit function theorem there exists a smooth map germ $\mathcal{K}:\mathbb{R}^{2m+1}_e\to\mathbb{R}^{2m-1}$ at(λ_a,p_a,q_a), such that $\kappa=\mathcal{K}(\lambda,p,q)$ if and only if

$$\frac{\partial F}{\partial \kappa_{[2m-1]\setminus[1]}}(\lambda,p,q,\kappa)=0,\ \det\left[\frac{\partial^2 F}{\partial\kappa_i\partial\kappa_j}(\lambda,p,q,\kappa)\right]=0.$$

Then the germ of $\mathbb{E}(L)$ at(λ_a,p_a,q_a) has the following form:

$$\mathbb{E}(L)=\left\{(\lambda,p,q):\frac{\partial F}{\partial\kappa_1}(\lambda,p,q,\mathcal{K}(\lambda,p,q))=0\right\}.$$

The fiber of π_r is tangent to $\mathbb{E}(L)$ at (λ_a,p_a,q_a) if and only if

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial F}{\partial\kappa_1}(\lambda,p,q,\mathcal{K}(\lambda,p,q))\right)(\lambda_a,p_a,q_a)=0,$$

which can be rewritten as
(6.8)
$$\frac{\partial^2 F}{\partial\lambda\partial\kappa_1}(\lambda_a,p_a,q_a,\kappa_a)+\sum_{j=1}^{2m-1}\frac{\partial^2 F}{\partial\kappa_j\partial\kappa_1}(\lambda_a,p_a,q_a,\kappa_a)\frac{\partial\mathcal{K}_j}{\partial\lambda}(\lambda_a,p_a,q_a)=0.$$

On the other hand, differentiating $\frac{\partial F}{\partial\kappa_{[2m-1]\setminus[1]}}(\lambda,p,q,\mathcal{K}(\lambda,p,q))=0$ with respect to λ we obtain for $i=2,\cdots,2m-1$
(6.9)
$$\frac{\partial^2 F}{\partial\lambda\partial\kappa_i}(\lambda_a,p_a,q_a,\kappa_a)+\sum_{j=1}^{2m-1}\frac{\partial^2 F}{\partial\kappa_j\partial\kappa_i}(\lambda_a,p_a,q_a,\kappa_a)\frac{\partial\mathcal{K}_j}{\partial\lambda}(\lambda_a,p_a,q_a)=0.$$

Thus (6.8)-(6.9) imply (6.7). On the other hand (6.9) and (6.7) imply (6.8). □

Theorem 6.5. *The point $a=\lambda a^++(1-\lambda)a^-$ belongs to the criminant $\Delta(L)$ of the Global Centre Symmetry set of L if and only if there exists a bitangent hyperplane to L at points a^+ and a^-.*

Proof. First assume that (λ,a) is a regular point of $\mathbb{E}(L)$. By Propositions 6.3-6.4 $a+,a^-$ is a 1-parallel pair and $a=(p,q)$ is in the criminant if and only if (λ, a) satisfies (6.7). Thus $\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}\right]$ has the following form

$$\frac{1}{2}\begin{bmatrix} \frac{\partial^2 S^+}{(\partial q_1^+)^2} - \frac{\partial^2 S^-}{(\partial q_1^-)^2} & \frac{\partial^2 S^+}{\partial q_1^+ \partial q_2^+} & \cdots & \frac{\partial^2 S^+}{\partial q_1^+ \partial q_m^+} & -\frac{\partial^2 S^-}{\partial q_1^- \partial p_2^-} & \cdots & -\frac{\partial^2 S^-}{\partial q_1^- \partial p_m^-} \\ \frac{\partial^2 S^+}{\partial q_1^+ \partial q_2^+} & \frac{\partial^2 S^+}{(\partial q_2^+)^2} & \cdots & \frac{\partial^2 S^+}{\partial q_2^+ \partial q_m^+} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 S^+}{\partial q_1^+ \partial q_m^+} & \frac{\partial^2 S^+}{\partial q_2^+ \partial q_m^+} & \cdots & \frac{\partial^2 S^+}{(\partial q_m^+)^2} & 0 & \cdots & -1 \\ -\frac{\partial^2 S^-}{\partial q_1^- \partial p_2^-} & -1 & \cdots & 0 & -\frac{\partial^2 S^-}{(\partial p_2^-)^2} & \cdots & \frac{\partial^2 S^-}{\partial p_2^- \partial p_m^-} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial^2 S^-}{\partial q_1^- \partial p_m^-} & 0 & \cdots & -1 & \frac{\partial^2 S^-}{\partial p_2^- \partial p_m^-} & \cdots & -\frac{\partial^2 S^-}{(\partial p_m^-)^2} \end{bmatrix}$$

On the other hand

$$\frac{\partial^2 F}{\partial \lambda \partial \beta_1} = p_1^+ - p_1^- - \sum_{j=1}^{n} q_j^+ \frac{\partial^2 S^+}{\partial q_1^+ \partial q_j^+} + q_1^- \frac{\partial^2 S^-}{(\partial q_1^-)^2} + \sum_{j=2}^{n} p_j^- \frac{\partial^2 S^-}{\partial q_1^- \partial p_j^-},$$

$$\frac{\partial^2 F}{\partial \lambda \partial \beta_i} = p_i^+ - \sum_{i=1}^{n} q_j^+ \frac{\partial^2 S^+}{\partial q_i^+ \partial q_j^+}, \text{ for } i = 2, \cdots, m,$$

$$\frac{\partial^2 F}{\partial \lambda \partial \alpha_i} = q_i^- + q_1^- \frac{\partial^2 S^+}{\partial p_i^- \partial q_1^-} + \sum_{j=2}^{n} p_j^- \frac{\partial^2 S^+}{\partial p_i^- \partial p_j^-}, \text{ for } i = 2, \cdots, m,$$

where $q^+ = \frac{q+\beta}{2\lambda}, p^+ = \frac{\partial S^+}{\partial q^+}$ are coordinates of $a^+ \in L^+$ and $q_1^- = \frac{q_1 - \beta_1}{2(1-\lambda)}$, $p^-_{[m]\setminus[2]} = \frac{p_{[m]\setminus[2]} - \alpha_{[m]\setminus[2]}}{2(1-\lambda)}$, $p_1^- = \frac{\partial S^-}{\partial q_1^-}$, $q^-_{[m]\setminus[2]} = -\frac{\partial S^-}{\partial p^-_{[m]\setminus[2]}}$ are coordinates of $a^- \in L^-$.

Then (6.7) is equivalent to

$$(a^+ - a^-) \in T_{a^+}L^+ + T_{a^-}L^-, \tag{6.10}$$

since $T_{a^+}L^+$ is spanned by vectors $\sum_{j=1}^m \frac{\partial^2 S^+}{\partial q_i^+ \partial q_j^+}\frac{\partial}{\partial p_j} + \frac{\partial}{\partial q_i}$ for $i = 1, \cdots, m$ and $T_{a^-}L^-$ is spanned by vectors $\frac{\partial^2 S^-}{(\partial q_1^-)^2}\frac{\partial}{\partial p_1} - \sum_{j=2}^m \frac{\partial^2 S^-}{\partial q_1^- \partial p_j^-}\frac{\partial}{\partial q_j} + \frac{\partial}{\partial q_1}$ and $\frac{\partial^2 S^-}{\partial p_i^- \partial q_1^-}\frac{\partial}{\partial p_1} - \sum_{j=2}^m \frac{\partial^2 S^-}{\partial p_i^- \partial p_j^-}\frac{\partial}{\partial q_j} + \frac{\partial}{\partial p_i}$ for $i = 2, \cdots, m$.

a^+, a^- is 1-parallel then (6.10) exactly means that there exists a bitangent hyperplane to L^+ at a^+ and to L^- at a^-. By continuity, a point in the closure of the set of points which satisfy (6.10) also satisfies this condition. □

Corollary 6.6. *If, for some λ, the point $a = \lambda a^+ + (1-\lambda)a^-$ belongs to the criminant $\Delta(L) \subset GCS(L)$, then the whole chord $l(a^+, a^-)$ belongs to $GCS(L)$. Equivalently, if there exists a bitangent hyperplane to L at points a^+ and a^-, then the chord $l(a^+, a^-)$ belongs to $GCS(L)$.*

In view of these results, we now generalize the notion of convexity of a curve on the plane.

Definition 6.7. A smooth closed Lagrangian submanifold L of the affine symplectic space $(\mathbb{R}^{2m}, \omega)$ is **weakly convex** if there is no bi-tangent hyperplane to L.

Corollary 6.8. *If L is a weakly convex closed Lagrangian submanifold of $(\mathbb{R}^{2m}, \omega)$ then the criminant $\Delta(L)$ of $GCS(L)$ is empty.*

7. Affine-Lagrangian stable singularities of the GCS of Lagrangian submanifolds

We now turn to the definition of an equivalence relation to be used for the classification of the singularities of $GCS(L)$. Due to the Lagrangian condition, we look for an equivalence of generating families.

Remind that, for the classification of $\mathbb{E}(\lambda)$ and $GCS(L)$, because λ is no longer fixed it has become an extra parameter that unfolds the generating families F. The naive approach is to consider the extended parameter space $\mathbb{R} \times \mathbb{R}^{2m} \ni (\lambda, x)$ for unfolding the generating families and then classify their stable unfoldings in the usual way.

This approach, which treats $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^{2m}$ on an equal footing,

$$(\lambda, (p, q)) = (\lambda, x) = y \in \mathbb{R}^{1+2m} \ ,$$

becomes clearer if we change from tangent to cotangent bundle to $\mathbb{R}^{2m}$,

$$(T\mathbb{R}^{2m}, \dot{\omega}) \ni (\dot{p}, \dot{q}, p, q) \mapsto (p^*, q^*, p, q) \in (T^*\mathbb{R}^{2m}, \omega_{can})$$

$$p^* = -\dot{q} \ \ , \ \ q^* = \dot{p} \ , \tag{7.1}$$

where ω_{can} is the canonical symplectic form on $T^*\mathbb{R}^n$, $\forall n$, which is given in terms of coordinates $(y^*, y) \in T^*\mathbb{R}^n$ by $\omega_{can} = \sum_{i=1}^{n} dy_i^* \wedge dy_i$. Then, (7.1) induces the symplectomorphism

$$(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega}) \to (T^*\mathbb{R}^{1+2m}, \omega_{can}) \ ,$$

so that, for $F(\lambda, p, q, \alpha, \beta) = F(y, \kappa)$, $y = (\lambda, x)$, $\kappa = (\alpha, \beta)$, we have the analogous of proposition 6.2, namely, that the germ of $\mathbb{E}(L)$ is the caustic of the germ of a Lagrangian submanifold $\mathcal{L}^*$ of the Lagrangian fiber bundle $(T^*\mathbb{R}^{1+2m}, \omega_{can})$, which is generated by the family F in the canonical way. In this setting, Lagrangian equivalence of $\mathbb{E}(L)$ and $\mathbb{E}(\widetilde{L})$ is defined in terms of Lagrangian equivalence of $\mathcal{L}^*$ and $\widetilde{\mathcal{L}^*}$ in the usual way, which means that their generating families must be stably R^+-equivalent (theorem 4.14). Because $GCS(L)$ is obtained from $\mathbb{E}(L)$

using projection (6.1), we could classify singularities of $GCS(L)$ using the above equivalence relation for classifying $\mathbb{E}(L)$.

However, such a classification of $GCS(L)$ would not take into account the projection (6.1) in a proper way, because it is not possible to introduce the notion of affine symplectic invariance for such a classification of $GCS(L)$, since the above Lagrangian equivalence of $\mathbb{E}(L)$ does not distinguish the affine time $\lambda \in \mathbb{R}$ from $x \in \mathbb{R}^{2m}$.

Now, if $\mathcal{A} = (A, a)$ is an element of the affine symplectic group $iSp_{\mathbb{R}}^{2m} = Sp(2m, \mathbb{R}) \ltimes \mathbb{R}^{2m}$, with $A \in Sp(2m, \mathbb{R})$, $a \in \mathbb{R}^{2m}$, then

$$\mathcal{A} : (\mathbb{R}^{2m}, \omega) \supset L \to L' \subset (\mathbb{R}^{2m}, \omega) \ ,$$
$$\mathcal{A} : x \mapsto \mathcal{A}x = Ax + a \ .$$

From this, we define the natural action

$$id_{T^*\mathbb{R}} \times \mathcal{A} \times \mathcal{A} : T^*\mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to T^*\mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^{2m} \ ,$$
$$(\lambda, \lambda^*, x^+, x^-) \mapsto (\lambda, \lambda^*, \mathcal{A}x^+, \mathcal{A}x^-) \ ,$$

which, via the chord transformation Φ_λ, induces an action

$$iSp_{\mathbb{R}}^{2m} \ \ni \ id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi \ : \ T^*\mathbb{R} \times T\mathbb{R}^{2m} \supset \mathcal{L} \ \to \ \mathcal{L}' \subset T^*\mathbb{R} \times T\mathbb{R}^{2m},$$
$$id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi : (\lambda, \lambda^*, \Phi_\lambda(x^+, x^-)) \mapsto (\lambda, \lambda^*, \Phi_\lambda(\mathcal{A}x^+, \mathcal{A}x^-)),$$
$$id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi : (\lambda, \lambda^*, x, \dot{x}) \mapsto (\lambda, \lambda^*, Ax + a, A\dot{x} + (2\lambda - 1)a),$$

that commutes with projection $id_{T^*\mathbb{R}} \times pr : T^*\mathbb{R} \times T\mathbb{R}^{2m} \to T^*\mathbb{R} \times \mathbb{R}^{2m}$, that is, defining the obvious action $id_{\mathbb{R}} \times \mathcal{A}$ on $\mathbb{R} \times \mathbb{R}^{2m}$, we have

$$(id_{\mathbb{R}} \times \mathcal{A}) \circ (id_{T^*\mathbb{R}} \times pr) = (id_{T^*\mathbb{R}} \times pr) \circ (id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi). \tag{7.2}$$

In view of the above and proposition 6.2, we now define a modified Lagrangian equivalence which takes into account projection (6.1).

Definition 7.1. Germs of Lagrangian submanifolds $\mathcal{L}$, $\widetilde{\mathcal{L}}$ of the Lagrangian fiber bundle $(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega})$ are **(1,2m)-Lagrangian equivalent** if there exists a symplectomorphism-germ Υ of $T^*\mathbb{R} \times T\mathbb{R}^{2m}$ such that $\Upsilon(\mathcal{L}) = \widetilde{\mathcal{L}}$ and the following diagram commutes, where the vertical arrows indicate diffeomorphism-germs

$$\begin{array}{ccccccc} \mathcal{L} & \hookrightarrow & T^*\mathbb{R} \times T\mathbb{R}^{2m} & \stackrel{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} & \stackrel{\pi}{\rightarrow} & \mathbb{R}^{2m} \\ & & \downarrow \Upsilon & & \downarrow & & \downarrow \\ \widetilde{\mathcal{L}} & \hookrightarrow & T^*\mathbb{R} \times T\mathbb{R}^{2m} & \stackrel{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} & \stackrel{\pi}{\rightarrow} & \mathbb{R}^{2m} \end{array}$$

The first two vertical diffeomorphism-germs (from right to left) read:

$$x \mapsto X(x)$$

$$(\lambda, x) \mapsto (\Lambda(\lambda, x), X(x)).$$

Moreover, germs $\mathcal{L}$, $\widetilde{\mathcal{L}}$ at $(\frac{1}{2}, a, \dot{a})$ are **(1,2m)-Lagrangian equivalent for** $\lambda = \frac{1}{2}$ if, in addition, for every $x \in \mathbb{R}^{2m}$

$$\Lambda(\frac{1}{2}, x) = \frac{1}{2}. \tag{7.3}$$

Remark 7.2. Condition (7.3) is introduced for the classification of the Wigner caustic $E_{\frac{1}{2}}(L)$ as a part of $GCS(L)$. If (7.3) is satisfied then the diffeomorphism (Λ, X) preserves the Wigner caustic.

Remark 7.3. $(1, 2m)$-Lagrangian equivalence of germs of Lagrangian submanifolds of the Lagrangian fiber bundle $(T^*\mathbb{R}\times T\mathbb{R}^{2m}, d\lambda^*\wedge d\lambda+\dot{\omega})$ is the equivalence of bifurcations of Lagrangian maps (Section 10.1 in [2]), that is, diagrams of maps of the form:

$$D(\mathcal{L}): \ \mathcal{L} \hookrightarrow T^*\mathbb{R} \times T\mathbb{R}^{2m} \stackrel{Pr}{\longrightarrow} \mathbb{R} \times \mathbb{R}^{2m} \stackrel{\pi}{\rightarrow} \mathbb{R}^{2m}$$

A Lagrangian submanifold $\mathcal{L}$ is **(1,2m)-Lagrangian stable** if the diagram of maps $D(\mathcal{L})$ is stable i.e. every Lagrangian submanifold $\widetilde{\mathcal{L}}$ with nearby diagram $D(\widetilde{\mathcal{L}})$ is $(1, 2m)$-Lagrangian equivalent to $\mathcal{L}$.

Definition 7.4. If L is a smooth closed Lagrangian submanifold of $(\mathbb{R}^{2m}, \omega)$ and $\mathcal{L}$ is a lagrangian submanifold of $(T^*\mathbb{R}\times T\mathbb{R}^{2m}, d\lambda^*\wedge d\lambda+\dot{\omega})$, with $\mathbb{E}(L) = Pr(\mathcal{L})$, we say that the classification of $GCS(L)$ by $(1, 2m)$-Lagrangian equivalence of $\mathcal{L}$ is **affine symplectic invariant** because, $\forall \mathcal{A} \in iSp^{2m}_{\mathbb{R}}$, the following diagram commutes (see (7.2)):

$$\begin{array}{ccccc}
T^*\mathbb{R} \times T\mathbb{R}^{2m} & \stackrel{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} & \stackrel{\pi}{\rightarrow} & \mathbb{R}^{2m} \\
\downarrow id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi & & \downarrow id_{\mathbb{R}} \times \mathcal{A} & & \downarrow \mathcal{A} \\
T^*\mathbb{R} \times T\mathbb{R}^{2m} & \stackrel{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} & \stackrel{\pi}{\rightarrow} & \mathbb{R}^{2m}
\end{array}$$

so that, if $L' = \mathcal{A}(L)$, then $\mathcal{L}' = (id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi)(\mathcal{L})$ and therefore $\mathbb{E}(L) = (id_{\mathbb{R}} \times \mathcal{A})\mathbb{E}(L)$ and $GCS(L') = \mathcal{A}(GCS(L))$.

Thus, for an affine symplectic invariant classification of $GCS(L)$, the generating families for $\mathcal{L}$ cannot be unfolded by the parameters $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^{2m}$ as if they were on an equal footing.

However, in a natural way, the $(1, 2m)$-Lagrangian equivalence of Lagrangian submanifolds of $T^*\mathbb{R} \times T\mathbb{R}^{2m}$ leads to the following equivalence of their generating families.

Definition 7.5. The function-germs $F, \widetilde{F} : \mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^k \to \mathbb{R}$ are **(1,2m)-$\mathcal{R}^+$-equivalent** if there exists a diffeomorphism-germ

$$(\lambda, x, \kappa) \mapsto (\Lambda(\lambda, x), X(x), K(\lambda, x, \kappa))$$

and a smooth function-germ $g : \mathbb{R} \times \mathbb{R}^{2m} \to \mathbb{R}$ such that

$$\widetilde{F}(\lambda, x, \kappa) = F(\Lambda(\lambda, x), X(x), K(\lambda, x, \kappa)) + g(\lambda, x).$$

F and $\widetilde{F}$ are **stably (1,2m)-$\mathcal{R}^+$-equivalent** if there are quadratic forms Q and $\widetilde{Q}$ such that $F+Q$ and $\widetilde{F}+\widetilde{Q}$ are $(1,2m)$-$\mathcal{R}^+$-equivalent. Germs F and $\widetilde{F}$ at $(\frac{1}{2}, a, \kappa_a)$ are **(stably) (1,2m)-$\mathcal{R}^+$-equivalent for** $\lambda = \frac{1}{2}$ if, in addition, for every $x \in \mathbb{R}^m$ condition (7.3) is satisfied (the role of condition (7.3) is explained in remark 7.2.

Remark 7.6. $(1,2m)$-$\mathcal{R}^+$-equivalence is a special case of Wassermann's $(1,2m)$-equivalence studied in [22]. See also Section 10.1 in [2], where relations between (r,s)-classification of families of functions ([22]), classification of bifurcations of caustics ([1] and [23]) and classification of bifurcations of Lagrangian maps (see Remark 7.3) were discussed.

We have the following result, whose proof is a minor modification for $(1,2m)$-Lagrangian equivalence of the proof of Theorem 4.14 in [2].

Proposition 7.7. *Germs of Lagrangian submanifolds $\mathcal{L}$, $\widetilde{\mathcal{L}}$ of the Lagrangian fiber bundle $(T^*\mathbb{R}\times T\mathbb{R}^{2m}, d\lambda^*\wedge d\lambda+\dot{\omega})$ are $(1,2m)$-Lagrangian equivalent if and only if the corresponding germs of generating families F and $\widetilde{F}$ are stably $(1,2m)$-$\mathcal{R}^+$-equivalent.*

Definition 7.8. $GCS(L)$ and $GCS(\widetilde{L})$ are **(1,2m)-Lagrangian equivalent** if the generating families F and $\widetilde{F}$ for $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are stably $(1,2m)$-$\mathcal{R}^+$-equivalent.

Definition 7.9. The function-germ F at z is **(1,2m)-$\mathcal{R}^+$-stable** if for any neighborhood U of z in $\mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^k$ and representative function F' of the germ F defined on U, there exists a neighborhood V of F' in $C^\infty(U, \mathbb{R})$ (with the weak C^∞-topology) such that for any function $G' \in V$ there exists a point $z' \in U$ such that the germ of G' at z' is $(1,2m)$-$\mathcal{R}^+$-equivalent to F. F is $(1,2m)$-$\mathcal{R}^+$-stable iff the corresponding germs of $\mathcal{L}$ and $GCS(L)$ are $(1,2m)$-Lagrangian stable, whenever realizable. In view of definition 7.4, we also use the term **affine-Lagrangian stability** for $(1,2m)$-Lagrangian stability of $\mathcal{L}$ and $GCS(L)$.

Definitions 7.1-7.9 are the ones we were looking for. The following theorems show that the only affine-Lagrangian stable singularities of GCS are singularities of the criminant, the smooth part of the Wigner caustic and the "tangent" union of them.

Theorem 7.10. *Let $\lambda_a \neq \frac{1}{2}$. If F is the germ at (λ_a, a, κ_a) of a $(1,2m)$-$\mathcal{R}^+$-stable unfolding of $f \in \mathfrak{m}^2$ then F is stably $(1,2m)$-$\mathcal{R}^+$-equivalent to the germ of the trivial unfolding (if f has A_1 singularity) or to one of the following germs at $(0,0,0)$ of unfoldings of $f(t) = t^3$*

$$A_2^{A_k^{\pm}} \ : \ F(\lambda, x, t) = t^3 + t\left(\sum_{i=1}^{k} x_i \lambda^{i-1} \pm \lambda^{k+1}\right), \tag{7.4}$$

for $k = 0, 1, 2, \cdots, 2m$

Proof. If f has A_1 singularity than it is obvious that F is stably $(1,2m)$-$\mathcal{R}^+$-equivalent to the trivial unfolding. Now we assume that f has A_2 singularity. Since F is stable than F is stable $(1,2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda, x, t) = t^3 + tg(\lambda, x)$, where g is a smooth function-germ vanishing at 0. If g is a versal unfolding of the function-germ $\lambda \mapsto g(\lambda, 0)$ with A_k singularity we can reduce F to the form (7.4) by a diffeomorphism-germ of the form $(\lambda, x, t) \mapsto (\Lambda(\lambda, x), X(x), t)$.

We show that these are the only $(1,2m)$-$\mathcal{R}^+$-stable unfoldings. The proof is based on the following lemma. □

Lemma 7.11. *Unfoldings of $A_3^{\pm}$ singularity are not $(1,2m)$-$\mathcal{R}^+$-stable.*

Proof. If f has A_3 singularity then F is stable $(1,2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda, x, t) = \pm t^4 + t^2 g_2(\lambda, x) + t g_1(\lambda, x)$, where g_1, g_2 are smooth function-germs vanishing at 0. Now we use the standard arguments of the singularity theory that stability implies infinitesimal stability. In the case of $(1,2m)$-$\mathcal{R}^+$-equivalence the infinitesimal stability implies the following condition:

$$\mathcal{E}_2 = \mathcal{E}_2\left\langle \frac{\partial F}{\partial t}|_{\mathbb{R}^2}\right\rangle + \mathcal{E}_1\left\langle 1, \frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2}\right\rangle + \mathbb{R}\left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^2}, \cdots, \frac{\partial F}{\partial x_{2m}}|_{\mathbb{R}^2}\right\rangle + \mathfrak{m}_2^{2m+4}, \tag{7.5}$$

where $\mathbb{R}^2$ denotes the t, λ-plane $\{x = 0\}$, $\mathcal{E}_2$ is the ring of smooth function-germs in λ and t, $\mathfrak{m}_2$ is the maximal ideal in $\mathcal{E}_2$ and $\mathcal{E}_1$ is the ring of smooth function-germs in λ. Now we use the method in [22].

Let $V = \mathcal{E}_2 \big/ \left(\mathcal{E}_2\left\langle \frac{\partial F}{\partial t}|_{\mathbb{R}^2}\right\rangle + \mathfrak{m}_2^{2m+4}\right)$ and let $\pi : \mathcal{E}_2 \to V$ be the projection. We have $\pi(t^3) = \pi(\mp 1/2 t g_2|_{\mathbb{R}^2} \mp 1/4 g_1|_{\mathbb{R}^2})$ in V. Thus elements $\pi(t^i \lambda^j)$ for $i = 0, 1, 2$ and $j < 2m + 4 - i$ form a basis of V over $\mathbb{R}$. It implies that $\dim_{\mathbb{R}} V = 6m + 9$. Moreover $\frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2} = t\left(t\frac{\partial g_2}{\partial \lambda}|_{\mathbb{R}^2} + \frac{\partial g_2}{\partial \lambda}|_{\mathbb{R}^2}\right)$. Then

$$\dim_{\mathbb{R}} \pi\left(\mathcal{E}_1\left\langle 1, \frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2}\right\rangle\right) \leq 4m + 7$$

and

$$\dim_{\mathbb{R}} \pi\left(\mathbb{R}\left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^2}, \cdots, \frac{\partial F}{\partial x_{2m}}|_{\mathbb{R}^2}\right\rangle\right) \leq 2m.$$

So if (7.5) held we would have $\dim_{\mathbb{R}} V \leq 6m+7 < 6m+9$, which is impossible. Therefore F is not $(1,2m)$-$\mathcal{R}^+$-stable and A_3 singularity has no $(1,2m)$-$\mathcal{R}^+$-stable unfoldings. □

To study the Wigner caustic in the GCS set we consider the germ of F at $(1/2, a, \kappa_a)$.

Theorem 7.12. *If F is the germ at $(\frac{1}{2}, a, \kappa_a)$ of a $(1,2m)$-$\mathcal{R}^+$-stable unfolding of $f \in \mathfrak{m}^2$ then F is stably $(1,2m)$-$\mathcal{R}^+$-equivalent (for $\lambda = 1/2$) to the germ of the trivial unfolding (if f has A_1 singularity) or to one of the following germs at $(\frac{1}{2}, 0, 0)$ of unfoldings of $f(t) = t^3$*
(7.6)

$$A_2^{A_k^{\pm}}(1/2) \;:\; F(\lambda, x, t) = t^3 + t\left(\sum_{i=0}^{k} x_{i+1}\left(\lambda - \frac{1}{2}\right)^i \pm \left(\lambda - \frac{1}{2}\right)^{k+1}\right),$$

for $k = 0, 1, 2, \cdots, 2m-1$

Proof. If f has A_1 singularity than it is obvious that F is stably $(1,2m)$-$\mathcal{R}^+$-equivalent to the trivial unfolding. Now we assume that f has A_2 singularity. Since F is stable than F is stable $(1,2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda, x, t) = t^3 + tg(\lambda, x)$, where g is a smooth function-germ vanishing at $(1/2, 0)$. If g is a versal unfolding of the function-germ $\lambda \mapsto g(\lambda, 0)$ with $A_k^{\pm}$ singularity on a manifold (λ-space) with the boundary ($\lambda = \frac{1}{2}$) (see [1]) then we can reduce F to the form (7.6) by a diffeomorphism-germ of the form $(\lambda, x, t) \mapsto (1/2 + (\lambda - 1/2)\Lambda(\lambda, x), X(x), t)$. □

Theorem 7.13. *If the generating family F for $\mathcal{L}$ has $A_2^{A_k^{\pm}}$ singularity, for $k = 0, 1, 2, \cdots, 2m$, then $\mathbb{E}(L)$ is a germ of a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^{2m}$.*

If F has $A_2^{A_0}$ singularity at (λ_a, a, κ_a) then $\mathbb{E}(L)$ is transversal at (λ_a, a) to the fibers of projection π.

If F has $A_2^{A_k^{\pm}}$ singularity at (λ_a, a, κ_a) then $\mathbb{E}(L)$ is k-tangent at (λ_a, a) to the fibers of projection π, a belongs to the criminant $\Delta(L)$ of $GSC(L)$ and the germ of $\Delta(L)$ at a is the caustic of $A_k^{\pm}$ singularity.

Proof. By Proposition 6.1 and the normal form of F for $A_2^{A_k^{\pm}}$ singularity we obtain that

$$\mathbb{E}(L) = \{(\lambda, x) \in \mathbb{R} \times \mathbb{R}^{2m} : \sum_{i=1}^{k} x_i \lambda^{i-1} \pm \lambda^{k+1} = 0\}.$$

It is easy to see that $\mathbb{E}(L)$ is the germ at $(0,0)$ of a smooth hypersurface and $\mathbb{E}(L)$ is transversal at $(0,0)$ to $\{\lambda = 0\}$ for $k = 0$ and $\mathbb{E}(L)$ is k-tangent to $\{\lambda = 0\}$ at $(0,0)$ for $k = 1, 2, \cdots, 2m$. The germ of the criminat $\Delta(L)$ at 0 is described in the following way

$$\{x\in\mathbb{R}^{2m}: \exists\lambda\ \sum_{i=1}^{k} x_i\lambda^{i-1}\pm\lambda^{k+1}=0,\ \sum_{i=2}^{k}(i-1)x_i\lambda^{i-2}\pm(k+1)\lambda^k=0\}.$$

So $\Delta(L)$ is a caustic of $A_k^{\pm}$ singularity. □

Theorem 7.14. *If the germ at $(\frac{1}{2},a,\kappa_a)$ of a generating family F for $\mathcal{L}$ has $A_2^{A_k^{\pm}}(1/2)$ singularity, for $k=0,1,2,\cdots,2m-1$, then $\mathbb{E}(L)$ is a germ of a smooth hypersurface in $\mathbb{R}\times\mathbb{R}^{2m}$.*

If F has $A_2^{A_0}(1/2)$ singularity at $(\frac{1}{2},a,\kappa_a)$ then $\mathbb{E}(L)$ is transversal at $(\frac{1}{2},a)$ to the fibers of projection π. The germ of $GCS(L)$ at a is the germ of a smooth hypersurface of $\mathbb{R}^{2m}$ - the Wigner caustic $E_{\frac{1}{2}}(L)$.

If F has $A_2^{A_k^{\pm}}(1/2)$ singularity at $(\frac{1}{2},a,\kappa_a)$ then $\mathbb{E}(L)$ is k-tangent at $(1/2,a,t)$ to the fibers of projection π. The germ of $GCS(L)$ at a consists of two tangent components: the germ of a smooth hypersurface - the Wigner caustic $E_{\frac{1}{2}}(L)$ and the germ of the caustic of $A_k^{\pm}$ singularity - the criminant $\Delta(L)$.

Proof. By Proposition 6.1 and the normal form of F for $A_2^{A_k^{\pm}}(1/2)$ singularity we obtain that

$$\mathbb{E}(L)=\{(\lambda,x)\in\mathbb{R}\times\mathbb{R}^{2m}: \sum_{i=0}^{k}x_{i+1}(\lambda-1/2)^i\pm(\lambda-1/2)^{k+1}=0\}.$$

It is easy to see that $\mathbb{E}(L)$ is the germ at $(1/2,0)$ of a smooth hypersurface and $\mathbb{E}(L)$ is transversal at $(1/2,0)$ to $\{\lambda=1/2\}$ for $k=0$ and $\mathbb{E}(L)$ is k-tangent to $\{\lambda=1/2\}$ at $(1/2,0)$ for $k=1,2,\cdots,2m-1$.

The Wigner caustic

$$E_{1/2}(L)=\{x\in\mathbb{R}^{2m}: x_1=0\}$$

is the germ of a smooth hypersurface. The germ of the criminat $\Delta(L)$ at 0 is described in the following way

$$\{x\in\mathbb{R}^{2m}: \exists\tau\ \sum_{i=0}^{k}x_{i+1}\tau^i\pm\tau^{k+1}=0,\ \sum_{i=1}^{k}ix_{i+1}\tau^{i-1}\pm(k+1)\tau^k=0\}.$$

So $\Delta(L)$ is a caustic of $A_k^{\pm}$ singularity and $E_{1/2}(L)$ is tangent to $\Delta(L)$ at 0. □

Remark 7.15. Not all $(1,2m)$-$\mathcal{R}^+$-stable singularities can be realizable as singularities of generating families F for $\mathcal{L}$ which are of the special form given in Theorem 4.12. In the next section, in Theorem 8.7, we prove that the $A_2^{A_2}$ singularity is not realizable for Lagrangian curves.

8. Classifications of the GCS of Lagrangian curves

In this section, using the equivalence of $GCS(L)$ introduced in section 6, we classify the singularities of the Global Centre Symmetry set of a Lagrangian curve L, that is, a curve $L \subset (\mathbb{R}^2, \omega)$.

To set the stage, we first state the results for the GCS of a curve on the affine plane $\mathbb{R}^2$, when no symplectic structure on $\mathbb{R}^2$ is considered.

The results for this non-Lagrangian case, summarized in theorem 8.1 below, were obtained in [3], [16] and [10]-[11] by various methods. In the appendix, this theorem is proved using the affine-invariant method of chord equivalence, which is the analogous of $(1, 2m)$-Lagrangian equivalence when no symplectic structure is considered.

Theorem 8.2 presents global results for the GCS of a convex curve, some of which have not been stated before.

Theorem 8.1. *Affine stable GCS of a smooth convex closed curve $M \subset \mathbb{R}^2$ (no symplectic structure) consists of three components: i) The CSS, a smooth curve with (possible) self intersections and cusps singularities, ii) the Wigner caustic, a smooth curve with (possible) self intersections and cusps singularities lying on the smooth part of the CSS, and iii) the middle axes, which are smooth half-lines starting at the the cusp points of the CSS.*

In theorem 8.1, the CSS and the middle axes form, together, the centre symmetry caustic $\Sigma'(M)$.

Theorem 8.2. *Let M be a generic smooth convex closed curve in $\mathbb{R}^2$. The number of cusps of the Wigner caustic of M is odd and not smaller than 3. The number of cusps of the CSS of M is odd and not smaller than 3. The number of cusps of the Wigner caustic of M is not greater than the number of cusps of the CSS of M.*

Proof. The first statement, on the number of cusps of Wigner caustics, was first proven by Berry [3] and the second statement, on the number of cusps of CSS, was first proven by Giblin and Holtom [9]. The last inequality follows immediately from the characterization in [9] of cusps of $E_{1/2}(M)$ by the curvature ratio being 1 and cusps of CSS of M by the derivative of the curvature ratio being 0, and from Rolle's theorem. □

Figures of $GCS(M)$ where the number of cusps of the CSS and of the Wigner caustic are equal to three and neither curve is self intersecting can be found in [9]. We picture below a case when the number of cusps of the Wigner caustic is three and the CSS is self intersecting and the number of its cusps is five, and another case when both the Wigner caustic and the CSS are self intersecting and each one has five cusps.

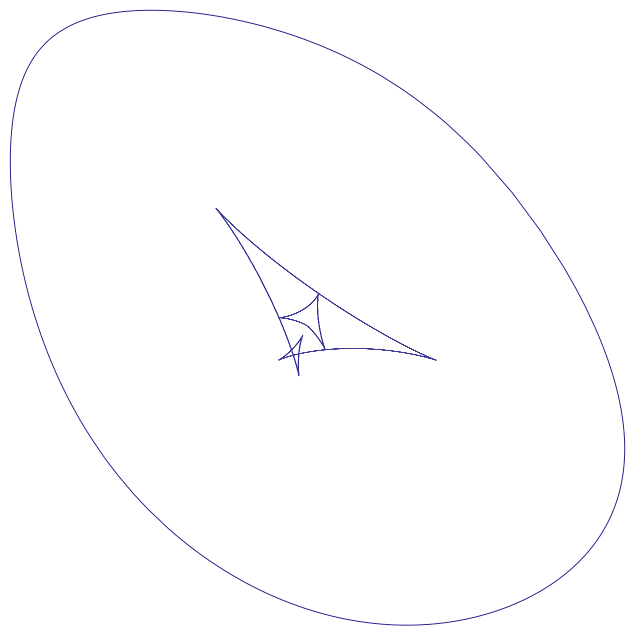

Figure 1. GCS of an oval in nonsymplectic plane: CSS with 5 cusps and Wigner caustic with 3 cusps (the middle axes are not shown here).

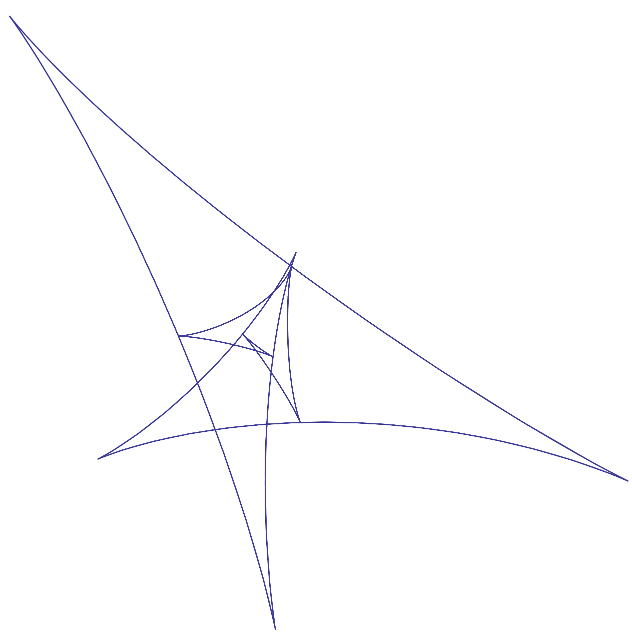

Figure 2. Both the CSS and the Wigner caustic with five cusps.

8.1. **Affine symplectic invariant classification of GCS of Lagrangian curves.** Let L be a smooth closed (Lagrangian) curve in the symplectic affine space $(\mathbb{R}^2, \omega = dp \wedge dq)$. Using the $(1,2)$-Lagrangian equivalence introduced in the previous section (definition 7.8), we classify the singularities of $GCS(L)$.

Let $a^+ = (p_a^+, q_a^+), a^- = (p_a^-, q_a^-) \in L$ be a parallel pair on L and $a_\lambda = \lambda a^+ + (1-\lambda)a^-$, $\dot{q}_\lambda = \lambda q_a^+ - (1-\lambda)q_a^-$. Let $S^\pm$ be germs of generating functions of L at $a^\pm$ satisfying the conditions in Proposition 4.11. Then the germ of generating family of $\mathcal{L}$ has the following form

$$F(\lambda, p, q, t) = 2\lambda^2 S^+(\frac{q+t}{2\lambda}) - 2(1-\lambda)^2 S^+(\frac{q-t}{2(1-\lambda)}) - pt.$$

The big front is described in the following way

$$\mathbb{E}(L) = \left\{ (\lambda, p, q) \in \mathbb{R} \times \mathbb{R}^2 : \exists t \ \frac{\partial F}{\partial t}(\lambda, p, q, t) = \frac{\partial^2 F}{\partial t^2}(\lambda, p, q, t) = 0 \right\}.$$

In the following propositions we present descriptions of different positions of $\mathbb{E}(L)$ with respect to the fiber bundle π in terms of the generating family F, generating functions S^+ and S^- and their geometric interpretations.

Proposition 8.3. *The following conditions are equivalent*

(i) (λ, a_λ) *belongs the regular part of* $\mathbb{E}(L)$,
(ii) $\exists t \ \frac{\partial^3 F}{\partial t^3}(\lambda, a_\lambda, t) \neq 0$, $\frac{\partial F}{\partial t}(\lambda, a_\lambda, t) = \frac{\partial^2 F}{\partial t^2}(\lambda, a_\lambda, t) = 0$,
(iii) $\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0$,
(iv) $\frac{1}{\lambda}\kappa(a^+) + \frac{1}{1-\lambda}\kappa(a^-) \neq 0$, *where* $\kappa(x)$ *is the curvature of* L *at* x.

Proof. Equivalence of (i) and (ii) follows from the definition of the regular part of $\mathbb{E}(L)$. Equivalence of (ii) and (iii) is obtained by direct calculations. (iv) is obvious since $\kappa(a^\pm) = \frac{\partial^3 S^\pm}{\partial (q^\pm)^3}(q_a^\pm)$. □

Proposition 8.4. *The following conditions are equivalent*

(v) *the regular part of* $\mathbb{E}(L)$ *is tangent to the fiber of* π *at* (λ, a_λ),
(vi) $\exists t$ *(ii) is satisfied and* $\frac{\partial^2 F}{\partial \lambda \partial t}(\lambda, a_\lambda, t) = 0$.
(vii) *(iii) is satisfied and* $p_a^+ = \frac{\partial S^+}{\partial q^+}(q_a^+) = \frac{\partial S^-}{\partial q^-}(q_a^-) = p_a^-$.
(viii) *(iv) is satisfied and* $l(a^+, a^-)$ *is bitangent to* a^+, a^- *to* L.

Proof. All statements follow from Proposition 6.4 and Theorem 6.5. □

Proposition 8.5. *The following conditions are equivalent*

(ix) *the regular part of* $\mathbb{E}(L)$ *is* 1*-tangent to the fiber of* π *at* (λ, a_λ),

(x) $\exists t$ *(vi) is satisfied and*

$$\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}(\lambda,a_\lambda,t)\right)^2-\frac{\partial^3 F}{\partial t^3}(\lambda,a_\lambda,t)\frac{\partial^3 F}{\partial\lambda^2\partial t}(\lambda,a_\lambda,t)\neq 0. \tag{8.1}$$

(xi) *(vii) is satisfied and* $\frac{\partial^3 S^+}{\partial(q^+)^2}(q_a^+)\frac{\partial^3 S^-}{\partial(q^-)^3}(q_a^-)\neq 0.$

(xii) *(iv) is satisfied and* $l(a^+,a^-)$ *is* 1*-tangent to* L *at* a^+ *and* a^-

Proof. (λ,a_λ) is a regular point of $\mathbb{E}(L)=\left\{(\lambda,p,q):\exists t\ \frac{\partial F}{\partial t}=\frac{\partial^2 F}{\partial t^2}=0\right\}$. By Proposition 8.3 it means that $\frac{\partial^3 F}{\partial t^3}(\lambda,a_\lambda,t)\neq 0$. It implies that there exists a smooth function-germ T on $\mathbb{R}^3$ such that $\frac{\partial^2 F}{\partial t^2}(\lambda,p,q,t)=0$ iff $t=T(\lambda,p,q)$. Then $\mathbb{E}(L)=\left\{(\lambda,p,q):\frac{\partial F}{\partial t}(\lambda,p,q,T(\lambda,p,q))=0\right\}$. Then (ix) is equivalent to

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial F}{\partial t}(\lambda,p,q,T(\lambda,p,q))\right)\Big|_{(\lambda,a_\lambda)}=0 \tag{8.2}$$

$$\frac{\partial^2}{\partial\lambda^2}\left(\frac{\partial F}{\partial t}(\lambda,p,q,T(\lambda,p,q))\right)\Big|_{(\lambda,a_\lambda)}\neq 0. \tag{8.3}$$

Using the formulae

$$\frac{\partial T}{\partial\lambda}(\lambda,p,q)=-\left(\frac{\partial^2 F}{\partial t^3}(\lambda,p,q,T(\lambda,p,q)\right)^{-1}\frac{\partial^2 F}{\partial\lambda\partial t^2}(\lambda,p,q,T(\lambda,p,q)) \tag{8.4}$$

it is easy to check that (8.2)-(8.3) are equivalent to (x). Equivalence of (x) and (xi) is obtained by direct calculation and the last equivalence is obvious. □

Proposition 8.6. *The following conditions are equivalent*

(xiii) *the regular part of* $\mathbb{E}(L)$ *is* 2*-tangent to the fiber of* π *at* (λ,a_λ)*,*

(xiv) $\exists t$ *(vi) is satisfied, (8.1) is not satisfied and*

$$\Big(\frac{\partial^4 F}{\partial\lambda^3\partial t}\left(\frac{\partial^3 F}{\partial t^3}\right)^3-3\frac{\partial^4 F}{\partial\lambda^2\partial t^2}\left(\frac{\partial^3 F}{\partial t^3}\right)^2\frac{\partial^3 F}{\partial\lambda\partial t^2}+$$
$$+3\frac{\partial^4 F}{\partial\lambda\partial t^3}\frac{\partial^3 F}{\partial t^3}\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}\right)^2-\frac{\partial^4 F}{\partial t^4}\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}\right)^3\Big)(\lambda,a_\lambda,t)\neq 0.$$

(xv) *(vii) is satisfied,*

$$\frac{\partial^3 S^+}{\partial(q^+)^3}(q_a^+)=0\ \wedge\ \frac{\partial^4 S^+}{\partial(q^+)^4}(q_a^+)\neq 0$$

or

$$\frac{\partial^3 S^-}{\partial(q^-)^3}(q_a^-)=0\ \wedge\ \frac{\partial^4 S^-}{\partial(q^-)^4}(q_a^-)\neq 0.$$

(xvi) *(iv) is satisfied and $l(a^+, a^-)$ is 1-tangent to L at one of points a^+, a^- and 2-tangent to L at the other.*

Proof. We use the same notation as in the proof of Proposition 8.5. (xiii) means that (8.2) is satisfied, (8.3) is not satisfied and

$$\frac{\partial^3}{\partial\lambda^3}\left(\frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q))\right)|_{(\lambda, a_\lambda)} \neq 0. \tag{8.5}$$

Using (8.4) it is easy to check that these conditions are equivalent to (xiv). By direct calculation one can obtain that (xiv) is equivalent to (xv) and (xvi) is obvious geometric description of (xv). □

Theorem 8.7. *Let $\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0$ (for statements (1)-(2) below, $\lambda = 1/2$).*

(1) *If the chord $l(a^+, a^-)$ is not bitangent to L at a^+, a^- then the germ of F at $(1/2, a_{1/2}, \dot{q}_{1/2})$ has $A_2^{A_0}(1/2)$ singularity and the germ of GCS at $a_{1/2}$ is a smooth curve (the smooth part of the Wigner caustic).*
(2) *If the chord $l(a^+, a^-)$ is 1-tangent to L at a^+ and at a^- then the germ of F at $(1/2, a_{1/2}, \dot{q}_{1/2})$ has $A_2^{A_1}(1/2)$ singularity and the germ of GCS at $a_{1/2}$ is a union of two 1-tangent smooth curves (the smooth part of the Wigner caustic and the smooth part of the criminant).*
(3) *If the chord $l(a^+, a^-)$ is 1-tangent to L at a^+ and at a^- then the germ of F at $(\lambda, a_\lambda, \dot{q}_\lambda)$ for $\lambda \neq 1/2$ has $A_2^{A_1}$ singularity and the germ of GCS at a_λ is a smooth curve (the smooth part of the criminant).*
(4) *If the chord $l(a^+, a^-)$ is 1-tangent to L at one of the points a^+, a^- and 2-tangent to L at the other point then the germ of F at $(\lambda, a_\lambda, \dot{q}_\lambda)$ for $\lambda \neq 1/2$ is not $(1,2)$-$\mathcal{R}^+$-stable. $A_2^{A_2}$ is not realizable as a singularity of GCS of a Lagrangian curve.*

Proof. By Proposition 8.3 if

$$\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0 \tag{8.6}$$

then the germ of a generating family F of $\mathcal{L}$ is a unfolding of the function-germ with A_2 singularity. Therefore we can reduce F to the following form $F'(\lambda, p, q, t) = t^3 + g(\lambda, p, q)t$, where g is a smooth function-germ vanishing at $(\lambda_a, 0)$ (for $\lambda_a = 0$ or $\lambda_a = 1/2$).

By Proposition 8.4 if the chord $l(a^+, a^-)$ is not bitangent to L at a^+, a^- then $\frac{\partial F'}{\partial t \partial \lambda}(1/2, 0, 0) \neq 0$ and this implies that $\frac{\partial g}{\partial \lambda}(1/2, 0) \neq 0$. By Theorems 7.12 and 7.14 we obtain (1).

If the chord $l(a^+,a^-)$ is tangent to L at a^+,a^- then by Proposition 8.4 we get that $p_a^+=p_a^-$ and $\frac{\partial F'}{\partial t\partial\lambda}(\lambda_a,0,0)=0$ and this implies that $\frac{\partial g}{\partial\lambda}(\lambda_a,0)=0$. But $dg|_{(\lambda_a,0)}\neq 0$ since $\frac{\partial F}{\partial t\partial p}(\lambda_a,a,\dot{q}_a)\neq 0$.

By Proposition 8.5 if $l(a^+,a^-)$ is 1-tangent to L at a^+,a^- then

$$\left(\frac{\partial^3 F'}{\partial\lambda\partial t^2}(\lambda_a,0,0)\right)^2-\frac{\partial^3 F'}{\partial t^3}(\lambda_a,0,0)\frac{\partial^3 F'}{\partial\lambda^2\partial t}(\lambda_a,0,0)\neq 0. \tag{8.7}$$

But this implies that $\frac{\partial^2 g}{\partial\lambda^2}(\lambda_a,0,)\neq 0$. Thus if $\lambda_a=1/2$ by Theorems 7.12 and 7.14 we obtain (2) and otherwise by Theorems 7.10 and 7.13 we obtain (3).

Finally, let us assume that the chord $l(a^+,a^-)$ is 1-tangent to L at a^+ and 2-tangent at a^-. By Proposition 8.6 we get $\frac{\partial^2 g}{\partial\lambda^2}(\lambda_a,0,)=0$ and

$$\Big(\frac{\partial^4 F}{\partial\lambda^3\partial t}\left(\frac{\partial^3 F}{\partial t^3}\right)^3-3\frac{\partial^4 F}{\partial\lambda^2\partial t^2}\left(\frac{\partial^3 F}{\partial t^3}\right)^2\frac{\partial^3 F}{\partial\lambda\partial t^2}+$$
$$+3\frac{\partial^4 F}{\partial\lambda\partial t^3}\frac{\partial^3 F}{\partial t^3}\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}\right)^2-\frac{\partial^4 F}{\partial t^4}\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}\right)^3\Big)(\lambda_a,0,0)\neq 0.$$

Thus, $\frac{\partial^3 g}{\partial\lambda^3}(\lambda_a,0,)\neq 0$. We know that $\frac{\partial g}{\partial p}(\lambda_a,0,)\neq 0$ since $\frac{\partial^2 F}{\partial t\partial p}(\lambda_a,a,\dot{q}_a)\neq 0$. It is easy to see that $\frac{\partial^2 F}{\partial t\partial q}(\lambda_a,a,\dot{q}_a)=0$. Thus F has $A_2^{A_2}$ singularity at $(\lambda_a,a,\dot{q}_a)$ iff the following condition is satisfied

$$\frac{\partial^3 F}{\partial\lambda\partial q\partial t}(\lambda_a,a,\dot{q}_a)\frac{\partial^3 F}{\partial t^3}(\lambda_a,a,\dot{q}_a)-\frac{\partial^3 F}{\partial\lambda\partial t^2}(\lambda_a,a,\dot{q}_a)\frac{\partial^3 F}{\partial q\partial t^2}(\lambda_a,a,\dot{q}_a)\neq 0$$

By direct calculation it is easy to see that this is equivalent to

$$\frac{(q_a^+-q_a^-)}{\lambda_a(1-\lambda_a)}\frac{\partial^3 S^+}{\partial(q^+)^3}(q_a^+)\frac{\partial^3 S^-}{\partial(q^-)^3}(q_a^-)\neq 0,$$

which is not satisfied, since $l(a^+,a^-)$ is 2-tangent to L at a^-. □

Corollary 8.8. *Let L be a smooth closed convex curve in $(\mathbb{R}^2,\omega)$. The middle axes and the whole CSS are not $(1,2)$-Lagrangian stable. The smooth part of the Wigner caustic is $(1,2)$-Lagrangian stable, but the cusp singularities of the Wigner caustic, seen as part of the $GCS(L)$, are not $(1,2)$-Lagrangian stable.*

Remark 8.9. A comparison of theorem 8.1 and corollary 8.8 shows that, for the case of convex curves in $\mathbb{R}^2$, various singularities which are affine stable are not affine-Lagrangian stable. In other words, there is a breakdown of stability of various singularities due to the presence of a symplectic form in $\mathbb{R}^2$ to be accounted for. Other examples of breakdown of stability due to a symplectic form can be found in [4]-[6].

Remark 8.10. Although the cusp singularities of the Wigner caustic are affine-Lagrangian stable when the Wigner caustic is considered by itself (corollary 5.2), they are not affine-Lagrangian stable when the Wigner caustic is considered as part of the GCS. That is, the meeting of the Wigner caustic and the CSS is not affine-Lagrangian stable.

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9. Appendix

Here we prove theorem 8.1. To do so, first we define affine stability. Remind that, $\forall\lambda \in \mathbb{R}$, the standard chord transformation

$$\Gamma_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n \ , \ (x^+, x^-) \mapsto (x, \dot{x}) \ ,$$

is the chord transformation defined by the choices $\mu \equiv 0$ and $\rho \equiv 1/2$. Explicitly, x is given by the λ-point equation (3.1) and $\dot{x}$ is given by the standard chord equation

$$\dot{x} = \frac{1}{2}(x^+ - x^-) \ . \tag{9.1}$$

One distinguishing feature of the standard chord transformation is that it is bijective $\forall\lambda \in \mathbb{R}$. Explicitly, its inverse is given by

$$\Gamma_\lambda^{-1} : T\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \ , \ (x, \dot{x}) \mapsto (x^+, x^-) \ ,$$

$$x^+ = x + 2(1-\lambda)\dot{x} \ , \ x^- = x - 2\lambda\dot{x} \ . \tag{9.2}$$

It follows that the standard extended chord transformation

$$\Gamma : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \times T\mathbb{R}^n \ ,$$

$$(\lambda, x^+, x^-) \mapsto (\lambda, \Gamma_\lambda(x^+, x^-)) \ ,$$

with Γ_λ given by equations (3.1), (9.1), is also bijective. For this reason, it is preferable to use the standard chord transformation Γ to the tilted chord transformation Φ when no symplectic structure has to be accounted for (as is also done in [7]).

Now, let M and $\widetilde{M}$ be germs of m-dimensional smooth submanifolds of $\mathbb{R}^n$, $n \le 2m$, and let $\mathbb{M}$ and $\widetilde{\mathbb{M}}$ be the chord transformed cylinders

$$\mathbb{M} = \Gamma(\mathbb{R} \times M \times M) \ , \ \widetilde{\mathbb{M}} = \Gamma(\mathbb{R} \times \widetilde{M} \times \widetilde{M}) \ .$$

Definition 9.1. Germs of $GCS(M)$ and $GCS(\widetilde{M})$ are **chord equivalent** if there exists a diffeomorphism-germ Θ of $\mathbb{R} \times T\mathbb{R}^n$ such that $\widetilde{\mathbb{M}} = \Theta(\mathbb{M})$ and the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{R} \times T\mathbb{R}^n & \overset{id_{\mathbb{R}} \times pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^n & \overset{\pi}{\longrightarrow} & \mathbb{R}^n \\ \downarrow \Theta & & \downarrow & & \downarrow \\ \mathbb{R} \times T\mathbb{R}^n & \overset{id_{\mathbb{R}} \times pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^n & \overset{\pi}{\longrightarrow} & \mathbb{R}^n \end{array}$$

where *vertical* arrows indicate diffeomorphism-germs, as follows:

$$\Theta : \mathbb{R} \times T\mathbb{R}^n \ni (\lambda, x, \dot{x}) \mapsto (\Lambda(\lambda, x), X(x), \dot{X}(\lambda, x, \dot{x})) \in \mathbb{R} \times T\mathbb{R}^n,$$

$$\mathbb{R} \times \mathbb{R}^n \ni (\lambda, x) \mapsto (\Lambda(\lambda, x), X(x)) \in \mathbb{R} \times \mathbb{R}^n,$$

$$\mathbb{R}^n \ni x \mapsto X(x) \in \mathbb{R}^n.$$

Remark 9.2. The chord equivalence is a special case of the equivalence of cascades of projection defined in [15].

Now, let $\mathcal{B} = (B, b) \in iGL^n_{\mathbb{R}} = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ act standardly on $\mathbb{R}^n$

$$x \mapsto \mathcal{B}x = Bx + b \ , \tag{9.3}$$

and define its induced action on $\mathbb{R} \times T\mathbb{R}^n$ as

$$id_{\mathbb{R}} \times \mathcal{B}_\Gamma : (\lambda, \Gamma_\lambda(x^+, x^-)) \mapsto (\lambda, \Gamma_\lambda(\mathcal{B}x^+, \mathcal{B}x^-)) \ ,$$

$$id_{\mathbb{R}} \times \mathcal{B}_\Gamma : (\lambda, x, \dot{x}) \mapsto (\lambda, Bx + b, B\dot{x}) \ , \tag{9.4}$$

which clearly satisfies, on $\mathbb{R} \times T\mathbb{R}^n$,

$$(id_{\mathbb{R}} \times \mathcal{B}) \circ (id_{\mathbb{R}} \times pr) = (id_{\mathbb{R}} \times pr) \circ (id_{\mathbb{R}} \times \mathcal{B}_\Gamma) \ . \tag{9.5}$$

Also, let $\mathbb{B} = (\beta, \mathcal{B}) \in \mathbb{R} \times iGL^n_{\mathbb{R}}$ and define its action on $\mathbb{R} \times T\mathbb{R}^n$ as

$$\mathbb{B}_\Gamma : (\lambda, \Gamma_\lambda(x^+, x^-)) \mapsto (\lambda + \beta, \Gamma_{\lambda+\beta}(\mathcal{B}x^+, \mathcal{B}x^-)) \ ,$$

$$\mathbb{B}_\Gamma : (\lambda, x, \dot{x}) \mapsto (\lambda + \beta, Bx + b + \beta B\dot{x}, B\dot{x}) \ . \tag{9.6}$$

Then, $\mathbb{B}_\Gamma \neq id_{\mathbb{R}} \times \mathcal{B}_\Gamma$, but, if $\mathbb{R}^n \supset M \ni x \mapsto x' = \mathcal{B}x \in M' \subset \mathbb{R}^n$,

$$\mathbb{M} = \Gamma(\mathbb{R} \times M \times M) \ , \ \mathbb{M}' = \Gamma(\mathbb{R} \times M' \times M') \ ,$$

we have that, as sets,

$$\forall \beta \in \mathbb{R} \ , \ \mathbb{B}_\Gamma(\mathbb{M}) = (id_{\mathbb{R}} \times \mathcal{B}_\Gamma)(\mathbb{M}) = \mathbb{M}' \ . \tag{9.7}$$

Furthermore, if $\mathbb{E}(M) = (id_{\mathbb{R}} \times pr)(\mathbb{M})$, $\mathbb{E}(M)' := (id_{\mathbb{R}} \times pr)(\mathbb{M}')$, with $\mathbb{M}$ and $\mathbb{M}'$ related by equation (9.7), then, as sets, we have that

$$\mathbb{E}(M)' = \mathbb{E}(M') = \mathbb{E}(\mathcal{B}(M)) = (id_{\mathbb{R}} \times \mathcal{B})(\mathbb{E}(M)) \ , \tag{9.8}$$

$$GCS(M)' = GCS(M') = GCS(\mathcal{B}(M)) = \mathcal{B}(GCS(M)) \ . \tag{9.9}$$

Definition 9.3. Because equations (9.5) and (9.7)-(9.9) are satisfied for the actions (9.3), (9.4) and (9.6) of the affine group $iGL^n_{\mathbb{R}}$ and its trivial $\mathbb{R}$-extension, we say that the classification of singularities of $GCS(M)$ by chord equivalence is **strongly affine invariant**.

Definition 9.4. A singularity of $GCS(M)$ is **affine stable** if it is a stable singularity under its classification by chord equivalence.

9.1. **Proof of theorem 8.1.** Let M be a smooth closed convex curve in $\mathbb{R}^2$. Let $a^+ = (a_1^+, a_2^+)$, $a^- = (a_1^-, a_2^-)$ be a pair of parallel point of M. Then M is locally around a^+ and a^- described as follows:

$$M^+ : x_2^+ = f^+(x_1^+), \quad M^- : x_2^- = f^-(x_1^-), \tag{9.10}$$

where $f^+,\ f^-$ are smooth function-germs on $\mathbb{R}$ such that $a_2^+ = f^+(a_1^+)$, $a_2^- = f^-(a_1^-)$, $\frac{df^+}{dx^+}(a_1^+) = \frac{df^-}{dx^-}(a_1^-) = 0$.

By a affine transformation we can get that $a_1^+ = a_1^- = 0$.

Then $\mathbb{M} = \Gamma(\mathbb{R} \times M \times M)$ is locally around point

$$(a, \dot{a}) = (\lambda_0 a^+ + (1-\lambda_0)a^-, 1/2(a^+ - a^-)) = (a_1, a_2, 0, \dot{a_2})$$

described as in the following way:

$$x_2 = \lambda f^+ (x_1 + 2(1-\lambda)\dot{x}_1) + (1-\lambda) f^- (x_1 - 2\lambda \dot{x}_1)\,, \tag{9.11}$$

$$\dot{x_2} = \frac{1}{2}\left(f^+ (x_1 + 2(1-\lambda)\dot{x}_1) - f^- (x_1 - 2\lambda \dot{x}_1)\right). \tag{9.12}$$

Using a diffeomorphism-germ of $\mathbb{R} \times T\mathbb{R}^2$ of the form

$$(\lambda, x, \dot{x}) \mapsto \left(\lambda, x, \dot{x}_1, \dot{x}_2 - \frac{1}{2}\left(f^+ (x_1 + 2(1-\lambda)\dot{x}_1) - f^- (x_1 - 2\lambda \dot{x}_1)\right)\right)$$

we show that (9.11)-(9.12) is chord equivalent to

$$x_2 = \lambda f^+ (x_1 + 2(1-\lambda)\dot{x}_1) + (1-\lambda) f^- (x_1 - 2\lambda \dot{x}_1)\,,\ \dot{x}_2 = 0. \tag{9.13}$$

Let f denote the following function-germ on $\mathbb{R} \times T\mathbb{R}^2$ at 0

$$\begin{aligned} f(\lambda, x, \dot{x}_1) &= \lambda f^+ (x_1 + 2(1-\lambda)\dot{x}_1) + \\ &+ (1-\lambda) f^- (x_1 - 2\lambda \dot{x}_1) - x_2. \end{aligned}$$

By definition of f we obtain that

$$f(\lambda_a, a, 0) = \frac{\partial f}{\partial \dot{x}_1}(\lambda_a, a, 0) = 0. \tag{9.14}$$

Now consider the following conditions for $a_1^+ = a_1^- = 0$:

$$\left((1-\lambda_a)\frac{d^2 f^+}{d(x_1^+)^2}(a_1^+) = -\lambda_a \frac{d^2 f^-}{d(x_1^-)^2}(a_1^-)\right) \Leftrightarrow \frac{\partial^2 f}{\partial \dot{x}_1^2}(\lambda_a, a, 0) = 0, \tag{9.15}$$

$$\left((1-\lambda_a)^2\frac{d^3f^+}{d(x_1^+)^3}(a_1^+)=\lambda_a^2\frac{d^3f^-}{d(x_1^-)^3}(a_1^-)\right)\Leftrightarrow\frac{\partial^3 f}{\partial\dot{x}_1^3}(\lambda_a,a,0)=0, \tag{9.16}$$

$$\left((1-\lambda_a)^3\frac{d^4f^+}{d(x_1^+)^4}(a_1^+)\neq-\lambda_a^3\frac{d^4f^-}{d(x_1^-)^4}(a_1^-)\right)\Leftrightarrow\frac{\partial^4 f}{\partial\dot{x}_1^4}(\lambda_a,a,0)\neq 0. \tag{9.17}$$

The curve M is convex. It implies that there is no bitangent line to M at points a^+,a^-. Therefore $a_2^+\neq a_2^-$ and it implies that $\frac{\partial f}{\partial\lambda}(\lambda_a,a,0)\neq 0$. Then by the implicit function theorem the equation $f(\lambda,x,\dot{x}_1)=0$ may be solved in the neighborhood of $(\lambda_a,a,0)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}$ with respect to λ. Thus we obtain

$$\lambda=\lambda_a+g(x,\dot{x}_1),\ \ \dot{x}_2=0, \tag{9.18}$$

where g is a function-germ such that $g(a,0)=\frac{\partial g}{\partial\dot{x}_1}(a,0)=0$. This implies that that at all smooth points, $\mathbb{E}(M)$ is transversal to the fibers of π. So the criminant $\Delta(M)$ is empty for a convex curve M.

If (9.15) is not satisfied then the function-germ $\dot{x}_1\mapsto g(a,\dot{x}_1)$ is locally equivalent to $\dot{x}_1\mapsto\pm\dot{x}_1^2$. Since $g(x,\dot{x}_1)$ is a deformation of $g(a,\dot{x}_1)$ then by a diffeomorphism-germ of $\mathbb{R}\times T\mathbb{R}^2$ the form $(\lambda,x,\dot{x})\mapsto(\lambda,X(x),\dot{X}_1(x,\dot{x}_1),\dot{x}_2)$ we reduce (9.18) to the following form

$$\lambda=\lambda_a\pm\dot{x}_1^2+g_0(x),\quad \dot{x}_2=0,$$

where g_0 is a function-germ vanishing at a. By a diffeomorphism-germ

$$(\lambda,x,\dot{x})\mapsto(\lambda-g_0(x),x,\dot{x}) \tag{9.19}$$

we obtain

$$\lambda=\lambda_a\pm\dot{x}_1^2,\quad \dot{x}_2=0.$$

Then $\mathbb{E}(M)=\{(\lambda,x):\lambda=\lambda_a\}$ is a smooth surface-germ transversal to fibers of π. So $GCS(M)$ is empty in this case.

If condition (9.15) is satisfied and (9.16) is not satisfied then $\dot{x}_1\mapsto g(a,\dot{x}_1)$ is locally diffeomorphic to $\dot{x}_1\mapsto\pm\dot{x}_1^3$. The function-germ $(\lambda,x,\dot{x}_1)\mapsto g(x,\dot{x}_1)+\lambda_0-\lambda$ at $(\lambda_a,a,0)$ is a deformation of $\dot{x}_1\mapsto g(a,\dot{x}_1)$. By (9.15) and the implicit function theorem we have that

$$\frac{\partial^2 g}{\partial\dot{x}_1\partial x_1}(a,0)\neq 0 \tag{9.20}$$

if the following condition at $a_1^+=a_1^-=0$ is not satisfied

$$\frac{d^2f^+}{d(x_1^+)^2}(a_1^+)=\frac{d^2f^-}{d(x_1^-)^2}(a_1^-)=0. \tag{9.21}$$

We may assume that $\frac{d^2f^+}{d(x_1^+)^2}(a_1^+)\frac{d^2f^-}{d(x_1^-)^2}(a_1^-)\neq 0$ since M is a generic convex curve. It is easy to see that (9.20) implies that $(\lambda,x,\dot{x}_1)\mapsto$

$g(x,\dot{x}_1)+\lambda_0-\lambda$ is the versal deformation. By a diffeomorphism-germ of $\mathbb{R}\times T\mathbb{R}^2$ of the form $(\lambda,x,\dot{x})\mapsto(\lambda-g_0(x),X(x),\dot{X}_1(x,\dot{x}_1),\dot{x}_2)$ we reduce (9.18) to the following form

$$\lambda=\lambda_a+\dot{x}_1^3+x_1\dot{x}_1,\quad \dot{x}_2=0.$$

Then $\mathbb{E}(M)$ is the cusp singularity $(\times\mathbb{R})$ in $\mathbb{R}^3$ and $GCS(M)$ is a singular set of $\mathbb{E}(M)$. So $GCS(M)$ is a germ of a smooth curve.

Conditions (9.15)-(9.17) imply that $\dot{x}_1\mapsto g(a,\dot{x}_1)$ is locally equivalent to $\dot{x}_1\mapsto\pm\dot{x}_1^4$. By (9.15)-(9.16) and the implicit function theorem it is easy to show that

$$\frac{\partial^2 g}{\partial\dot{x}_1\partial x_2}(a,0)=0,\ \frac{\partial^3 g}{\partial\dot{x}_1^2\partial x_2}(a,0)\neq 0. \tag{9.22}$$

Together with (9.20) it imply that a function-germ $(\lambda,x,\dot{x}_1)\mapsto g(x,\dot{x}_1)+\lambda_0-\lambda$ at $(\lambda_a,a,0)$ is the versal deformation of $\dot{x}_1\mapsto g(a,\dot{x}_1)$. Then by a diffeomorphism-germ of $\mathbb{R}\times T\mathbb{R}^2$ of the form $(\lambda,x,\dot{x})\mapsto(\lambda-g_0(x),X(x),\dot{X}_1(x,\dot{x}_1),\dot{x}_2)$ we reduce (9.18) to the following form

$$\lambda=\lambda_a\pm\dot{x}_1^4+x_2\dot{x}_1^2+x_1\dot{x}_1,\ \dot{x}_2=0.$$

Then $\mathbb{E}(M)$ is the swallow tail singularity and $GCS(M)$ is a singular set of $\mathbb{E}(M)$. So $GCS(M)$ is composed of the smooth curve with the cusp singularity (CSS) and of the smooth half-line of the self-intersections starting at the cusp point and lying inside the cusp. This half line is a part of the middle axis of M.

By Theorem 2.10 $GCS(M)$ contains also the Wigner caustic $E_{\frac{1}{2}}(M)$. For the classification of the Wigner caustic in $GCS(M)$ we use the Γ-chord equivalence with the extra assumption

$$\Lambda(\lambda_0,x)=\lambda_0\ ,\ \lambda_0=1/2\ . \tag{9.23}$$

We use the same arguments as in the first part of the proof. In this case (9.18) has the following form

$$\lambda=1/2+g(x,\dot{x}_1),\ \dot{x}_2=0, \tag{9.24}$$

where g is a function-germ such that $g(a,0)=\frac{\partial g}{\partial\dot{x}_1}(a,0)=0$.

Since $\partial f/\partial x_2(1/2,a,0)=-1\neq 0$ we obtain that

$$\frac{\partial g}{\partial x_2}(a,0)\neq 0. \tag{9.25}$$

It implies that if (9.15) is not satisfied then $g(x,\dot{x}_1)$ is the versal deformation of $g(a,\dot{x}_1)$. Thus we reduce (9.24) to

$$\lambda=1/2\pm\dot{x}_1^2+x_1,\quad \dot{x}_2=0.$$

In this case the Wigner caustic is a smooth curve $x_1=0$.

If (9.15) is satisfied but (9.16) is not satisfied (for $\lambda_a = 1/2$) we get that (9.20) is satisfied. Together with (9.25) it implies that we can reduce (9.24) to the form

$$\lambda = 1/2 + \dot{x}_1^3 + x_2\dot{x}_1 + x_1, \quad \dot{x}_2 = 0.$$

In this case the germ $GCS(M)$ consists of the Wigner caustic which is the cusp curve $27x_1^2 + 4x_2^3 = 0$ and a germ of $CSS(M)$ which is the smooth curve $x_2 = 0$.

Remark 9.5. From the proof of this theorem we get conditions (9.15)-(9.17) which distinguished various singularities of the GCS set of a convex smooth curve. They were first presented in [9] in terms of (derivatives of) the ratio of the curvatures of the curve M at points a^+ and a^-. Note that $\partial(f^\pm)^2/\partial(x^\pm)^2(a^\pm)$ is the curvature of M at $a^\pm$.

Remark 9.6. Although the possibility of self intersections of both the CSS and the Wigner caustic have been illustrated in section 8 and stated in theorem 8.1, no such possibility is found in its proof. This is because the proof only concerns a local classification of singularities and these self intersections are of global, or multilocal nature.

Remark 9.7. We used the standard extended chord transformation Γ to define affine-stability and work out the proof of theorem 8.1 because it is geometrically simpler than the tilted chord transformation Φ and is the natural choice for non-Lagrangian cases, as studied in [7]. Γ allows for the action (9.6) of $\mathbb{R} \times iGL^n_{\mathbb{R}}$ to be globally defined on $\mathbb{R} \times T\mathbb{R}^n$ and Γ also has the property of affine rigidity, meaning that (9.4) defines an action $iGL^n_{\mathbb{R}} : T\mathbb{R}^n \to T\mathbb{R}^n$. By comparison, Φ is only linearly rigid (see remark 3.2) and, via Φ, the similar action of $\mathbb{R} \times iGL^n_{\mathbb{R}}$ is only defined on a subset of $\mathbb{R} \times T\mathbb{R}^n$ (pinched at $\lambda = 0$ and $\lambda = 1$). However, theorem 8.1 can be similarly stated and proved using Φ instead of Γ.

Remark 9.8. Via Φ and the similar action of $\mathbb{R} \times iSp^{2m}_{\mathbb{R}}$ on the proper subset of $\mathbb{R} \times T\mathbb{R}^{2m}$, it is possible to introduce the notion of strong affine symplectic invariance for an equivalence of GCS of Lagrangian submanifolds by imposing, on the diagram of definition 7.1, commutativity also with respect to the projection $T^*\mathbb{R} \times T\mathbb{R}^{2m} \to \mathbb{R} \times T\mathbb{R}^{2m}$. However, this stronger equivalence relation is so rigid in the Lagrangian case that not even the singularities of the criminant are stable.

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