Semiclassical Evolution of Dissipative Markovian Systems

(submitted in July 2007)

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Abstract.

A semiclassical approximation for an evolving density operator, driven by a "closed" hamiltonian operator and "open" markovian Lindblad operators, is obtained. The theory is based on the chord function, i.e. the Fourier transform of the Wigner function. It reduces to an exact solution of the Lindblad master equation in the case where the hamiltonian operator is a quadratic function and the Lindblad operators are linear functions of positions momenta.

Initially, the semiclassical formulae for the case of hermitian Lindblad operators are reinterpreted in terms of a (real) double phase space, generated by an appropriate classical double Hamiltonian. Then, the general case of dissipative markovian evolution, that results from non-hermitian Lindblad operators, is incorporated semiclassically through an extra "open" term that is added to the double Hamiltonian. The particular case of generic hamiltonian operators, but linear dissipative Lindblad operators, is studied in more detail. A Liouville-type equivariance still holds for the corresponding classical evolution in double phase, but the centre subspace, which supports the Wigner function, is compressed, along with expansion of its conjugate subspace, which supports the chord function.

Decoherence narrows the relevant region of double phase space to the neighborhood of a caustic for both the Wigner function and the chord function. This difficulty is avoided by a propagator in a mixed representation, so that a further "small-chord" approximation leads to a simple generalization of the quadratic theory for evolving Wigner functions.

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1. Introduction

The Lindblad master equation describes the general evolution for markovian open systems under the weakest possible constraints [1] (see also e.g. [2, 3]). Given the internal Hamiltonian, $\hat{H}$, and the Lindblad operators, $\hat{L_k}$, which account for the action of the random environment, the evolution of the density operator may be reduced to the canonical form,

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \frac{1}{\hbar}\sum_k (\hat{L_k}\hat{\rho}\hat{L_k}^\dagger - \frac{1}{2}\hat{L_k}^\dagger\hat{L_k}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L_k}^\dagger\hat{L_k}), \tag{1.1}$$

so that, in the absence of the environment ($\hat{L_k} = 0$), the motion is governed by the Liouville-Von Neumann equation appropriate for unitary evolution.

A typical example is based on the Jaynes-Cummings model, which describes the interaction of a two-level atom with a single mode of the optical field in a cavity. The statistically independent arrival of atoms leads to the *damped harmonic oscillator equation* for the photon field,

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} = &-\frac{i}{\hbar}[\hat{a}^\dagger\hat{a},\hat{\rho}] + \frac{A}{\hbar}(\nu+1)(\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{a}^\dagger\hat{a}) \\ &+ \frac{A}{\hbar}\nu(\hat{a}^\dagger\hat{\rho}\hat{a} - \frac{1}{2}\hat{a}\hat{a}^\dagger\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{a}\hat{a}^\dagger),\end{aligned} \tag{1.2}$$

where we identify the pair of Lindblad operators as proportional to the anihilation operator $\hat{a} = (\widehat{q} + i\widehat{p})/\sqrt{2}$ and the creation operator $\hat{a}^\dagger = (\widehat{q} - i\widehat{p})/\sqrt{2}$ for photons in the field mode (see e.g. [4, 5] and references therein).

The case where the Lindblad operators are all self-adjoint has deserved special attention. It is known that the corresponding Lindblad equation describes decoherence, or dephasing, as well as diffusion, but no dissipation [3]. Since the latter is usually a much slower process, it is often useful to simplify the evolution by considering only the self-adjoint part of the $\hat{L_j}$'s when studying the decoherence process (as in the semiclassical theory proposed in [6]). However, most physical processes for an open system such as (1.2) are dissipative. It is therefore desirable to develop a semiclassical theory for the evolution of the density operator that combines the description of both the initial decoherence process and the more classical development of diffusion and dissipation.

In this paper, we develop a formalism for treating the semiclassical limit for the evolution of density operators subject to equation (1.1), in cases where the Lindblad operators are not self-adjoint. By semiclassical, we mean generalized WKB expansions (see e.g. [7]), as opposed to simple power expansions in $\hbar$. The present theory expands on our phase-space treatment for the semiclassical evolution of closed systems [8] and of non-dissipative open markovian systems [6]. To this purpose, we will adapt the theory developed in these papers (particularly [6]) in two respects.

Recalling that $\mathbf{R^{2N}}$ stands for a $(2N)$-dimensional phase space, which is a symplectic vector space, $\{\mathbf{x} = (\mathbf{p},\mathbf{q})\}$, first we switch from the Weyl representation, where $\hat{\rho}$ is represented by the Wigner function $W(\mathbf{x})$, to its Fourier transformed representation, the chord representation, where $\hat{\rho}$ is represented by the chord function, $\chi(\boldsymbol{\xi})$, also known as the quantum characteristic function, given by

$$\chi(\boldsymbol{\xi}) = \frac{1}{(2\pi\hbar)^N}\int d\mathbf{x}\; W(\mathbf{x})\;\exp\{\frac{i}{\hbar}(\boldsymbol{\xi}\wedge\mathbf{x})\}\;, \tag{1.3}$$

where we have used the skew product,

$$\mathbf{x}\wedge\mathbf{x}' = \sum_{l=1}^{N}(p_l q_l' - q_l p_l') = \mathbf{J}\,\mathbf{x}\cdot\mathbf{x}', \tag{1.4}$$

which also defines the skew symplectic matrix $\mathbf{J}$. The *chord*, $\boldsymbol{\xi} = (\xi_p, \xi_q)$, is the Fourier conjugate variable of the *centre* $\mathbf{x}$ and stands for a tangent vector in phase space, as in the scheme for a Legendre transform. In contrast to the Wigner function, the chord function is not necessarily real, but its semiclassical expression is often similar to that of the Wigner function, as discussed in [9].

In the case where the initial Hamiltonian operator is at most quadratic in the momentum and position operators, $\widehat{\mathbf{x}} = (\hat{p}, \hat{q})$, and the Lindblad operators are linear in $\widehat{\mathbf{x}}$, as in example (1.2), the Lindblad equation reduces to a Fokker-Planck equation in the chord representation, which can be solved exactly [10]. Keeping to linear (but not self-adjoint) Lindblad operators, we here obtain an appropriate semiclassical generalization to the evolution of the chord function, $\chi(\boldsymbol{\xi}, t)$, for generic Hamiltonian operators, given an initial pure state, $\chi(\boldsymbol{\xi}, 0)$. This is similar to the theory for the evolution of the Wigner function in [6], in which the Lindblad operators were assumed to be self-adjoint (no dissipation). But the present treatment has the immediate advantage of being exact in the quadratic case.

In fact, the inverse Fourier transform of the semiclassical evolution for the chord function, evaluated in the stationary phase approximation, produces the semiclassical evolution for the Wigner function, as presented in [6]. However, this can now be seen to be a poorer approximation than that for the semiclassical chord function. Indeed, the direct Weyl treatment does not describe diffusion, which should progressively coarse-grain the Wigner function, as is clear from the analysis of the quadratic case [10].

The second modification to the WKB semiclassical theory, which is required for treating markovian dissipation, is more profound: We find that it is necessary to work in *double phase space*, $(\mathbf{x}, \boldsymbol{\xi}) \in \mathbf{R^{2N}} \times \mathbf{R^{2N}}$. This is a natural setting for the corresponding description of the semiclassical evolution of the density operator [11, 12], or indeed, for the representation of general operators acting on the Hilbert space of quantum states, considered as superpositions of $|ket\rangle\langle bra|$ elements. Just as an evolving quantum state, $|\psi\rangle$, corresponds to an evolving submanifold in simple phase space, $\mathbf{x} \in \mathbf{R^{2N}}$, the unitary evolution of a pure state density operator in a closed system, that is, a projector, $|\psi\rangle\langle\psi|$, corresponds to the evolution of a submanifold in double phase space. (In both cases, the respective submanifold satisfies an appropriate lagrangian property, to be specified). We thus obtain a formal generalization of the WKB framework, where an approximate oscillating solution of the Schrödinger equation is built from a classically evolving lagrangian submanifold [7, 13, 14]. However, the restriction to a closed system and hence unitary quantum evolution strongly restricts the allowed form of the corresponding classical double Hamiltonian [15].

The crucial point is that, an additional term in the double Hamiltonian, that accounts for dissipation, is produced naturally by the semiclassical treatment of the open terms in the master equation. This new term depends exclusively on the Lindblad operators and cancels in the special case where these are self-adjoint. As it should be, the resulting description of the full semiclassical evolution again coincides with the exact solution of the master equation in the quadratic case.

Our use of classical double phase is limited to the semiclassical approximation. No attempt has here been made to define a generalized quantum mechanics that would classically correspond to double phase space. This could lead to a fully quantum path integral for markovian systems, as an alternative to the one developed by Strunz [16]. His approach relies on the position representation, with the obervables defined in the Weyl represntation. Several of the ingredients in our theory already appear in Strunz's path integral, though its semiclassical limit is expressed in terms of complex orbits, whereas we deal only with real phase space propagation.

This paper is divided in three parts. In the initial sections 2-5, we review basic material and reformulate the semiclassical theory for closed evolution [8] and nondissipative open markovian evolution [6], within the chord representation, so as to perfectly fit the exact quadratic results in [10].

In the second part, the ingredients in the basic result, equation (5.2), are reinterpreted within the double phase space scenario. This leads to the identification of the dissipative hamiltonian, in section 6, and the consequent semiclassical treatment of dissipative markovian dynamics, in section 7 (for linear Lindblad operators).

It turns out that the *classical region* of double phase space, to which decoherence drives the evolution, projects singularly as a caustic onto the subspaces where either the Wigner function, or the chord function are defined. For this reason, in the final part of this paper, section 8, our semiclassical theory is adapted to the evolving *centre-chord propagator* [17], which takes an initial density operator, expressed as a Wigner function, into a final chord function, thus avoiding caustics for a finite time. This, composed with a *small chord approximation* for the evolution of the Wigner function itself, provides a generalization of the known behavior, in the case of quadratic Hamiltonians, for non-quadratic cases.

2. Review of the semiclassical theory for density operators

The chord representation of an operator $\hat{A}$ on the Hilbert space $L^2(\mathbf{R^N})$ is defined via the decomposition of $\hat{A}$ as a linear (continuous) superposition of *translation operators*,

$$\hat{T}_{\boldsymbol{\xi}} = \exp\{\frac{i}{\hbar}(\boldsymbol{\xi} \wedge \hat{\mathbf{x}})\}. \tag{2.1}$$

Each of these corresponds classically to a uniform translation of phase space points $\mathbf{x}_0 \in \mathbf{R^{2N}}$ by the vector $\boldsymbol{\xi} \in \mathbf{R^{2N}}$, that is: $\mathbf{x}_0 \mapsto \mathbf{x}_0 + \boldsymbol{\xi}$. In this way,

$$\hat{A} = \frac{1}{(2\pi\hbar)^N} \int d\boldsymbol{\xi}\ \tilde{A}(\boldsymbol{\xi})\ \hat{T}_{\boldsymbol{\xi}} \tag{2.2}$$

and the expansion coefficient, a function on $\mathbf{R^{2N}}$, is the *chord symbol* of the operator $\hat{A}$:

$$\tilde{A}(\boldsymbol{\xi}) = \mathrm{tr}\ (\hat{T}_{-\boldsymbol{\xi}}\ \hat{A}). \tag{2.3}$$

The Fourier transform of the translation operators defines the *reflection operators*,

$$2^N\ \hat{R}_{\mathbf{x}} = \frac{1}{(2\pi\hbar)^N} \int d\boldsymbol{\xi}\ \exp\{\frac{i}{\hbar}(\mathbf{x} \wedge \boldsymbol{\xi})\}\ \hat{T}_{\boldsymbol{\xi}}, \tag{2.4}$$

such that each of these corresponds classically to a reflection of phase space $\mathbf{R^{2N}}$ through the point $\mathbf{x}$, that is $\mathbf{x}_0 \mapsto 2\mathbf{x} - \mathbf{x}_0$. The same operator $\hat{A}$ can then be decomposed into a linear superposition of reflection operators

$$\hat{A} = 2^N \int \frac{d\mathbf{x}}{(2\pi\hbar)^N}\ A(\mathbf{x})\ \hat{R}_{\mathbf{x}}, \tag{2.5}$$

thus defining the *centre symbol or Weyl symbol* of operator $\hat{A}$,

$$A(\mathbf{x}) = 2^N \mathrm{tr}\ (\hat{R}_{\mathbf{x}}\ \hat{A}). \tag{2.6}$$

It follows that the centre and chord symbols are always related by Fourier transform:

$$\tilde{A}(\boldsymbol{\xi}) = \frac{1}{(2\pi\hbar)^N} \int d\mathbf{x}\ A(\mathbf{x}) \exp\{\frac{i}{\hbar}(\mathbf{x} \wedge \boldsymbol{\xi})\}\ , \tag{2.7}$$

$$A(\mathbf{x}) = \frac{1}{(2\pi\hbar)^N} \int d\boldsymbol{\xi}\ \tilde{A}(\boldsymbol{\xi}) \exp\{\frac{i}{\hbar}(\boldsymbol{\xi} \wedge \mathbf{x})\}\ . \tag{2.8}$$

In the case of the density operator, $\hat{\rho}$, it is convenient to normalize its chord symbol, so that we define the *chord function* as

$$\chi(\boldsymbol{\xi}) = \frac{1}{(2\pi\hbar)^N}\mathrm{tr}\ (\hat{T}_{-\boldsymbol{\xi}}\ \hat{\rho}) = \frac{\tilde{\rho}(\boldsymbol{\xi})}{(2\pi\hbar)^N}\ , \tag{2.9}$$

whose Fourier transform is the *Wigner function*,

$$W(\mathbf{x}) = \frac{1}{(2\pi\hbar)^N}\int d\boldsymbol{\xi}\exp\{\frac{i}{\hbar}(\boldsymbol{\xi}\wedge\mathbf{x})\}\ \chi(\boldsymbol{\xi}), \tag{2.10}$$

or alternatively [18]

$$W(\mathbf{x}) = \frac{1}{(\pi\hbar)^N}\mathrm{tr}\ (\hat{R}_{\mathbf{x}}\ \hat{\rho}). \tag{2.11}$$

The expectation value of any operator $\hat{A}$, defined as

$$\langle\hat{A}\rangle = \mathrm{tr}\ (\hat{\rho}\ \hat{A}), \tag{2.12}$$

can then be written, according to (2.5),

$$\langle\hat{A}\rangle = 2^N\int\frac{d\mathbf{x}}{(2\pi\hbar)^N}\ A(\mathbf{x})\ \mathrm{tr}\ (\hat{\rho}\ \hat{R}_{\mathbf{x}})\ = \int d\mathbf{x}\ A(\mathbf{x})\ W(\mathbf{x})\ , \tag{2.13}$$

which justifies the Wigner function being dubbed a "quasi-probability", even though it can be negative. The normalization condition reads

$$1 = \mathrm{tr}\ \hat{\rho} = \int d\mathbf{x}\ W(\mathbf{x}) = (2\pi\hbar)^N\chi(\mathbf{0})\ . \tag{2.14}$$

The Weyl representation and its Fourier transform have a long history. References [19, 20, 21, 22, 23, 18, 24, 25] develop many of its aspects, with unavoidable variations in notation and interpretation. Our presentation is largely based on the review [26].

A strictly quantum mechanical treatment requires us to choose between representations based on conjugate variables. This is just as true for the centre and chord symbols, related by (2.7) and (2.8), as for the more familiar position and momentum representations. However, the WKB semiclassical treatment links the $\mathbf{x}$ variable and the $\boldsymbol{\xi}$ variable through the stationary phase approximation. Indeed, starting from the integral expressions of (2.11) or (2.9), this stationary phase method replaces respectively an integral over $\mathbf{x}$, or $\boldsymbol{\xi}$, by its integrand, evaluated at one or several points $\mathbf{x}_i$, or $\boldsymbol{\xi}_j$. Because of the Fourier relation (2.10) between the pair of representations, each chord $\boldsymbol{\xi}$ is then associated with a discrete set of "centres" $\mathbf{x}$ - this denomination will become clear in the following - while each "centre" $\mathbf{x}$ specifies a discrete set of chords $\boldsymbol{\xi}$.

This correspondence is geometrically clear in the case of a pure state, $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$, classically associated with a (quantized) lagrangian submanifold, $\mathcal{L}_\psi$, in the simple phase space $\mathbf{x}\in\mathbf{R^{2N}}$, that is, an N-dimensional submanifold $\mathcal{L}_\psi$ with the property that

$$\oint_\gamma \boldsymbol{p}\cdot\mathrm{d}\boldsymbol{q} = 0, \tag{2.15}$$

for any reducible circuit γ lying in $\mathcal{L}_\psi$ (see, e.g. [27, 28, 29], for more on symplectic manifolds and their lagrangian submanifolds). Then, for every point $\mathbf{x}$, one can draw a discrete set of chords $\boldsymbol{\xi}_j$ of the submanifold $\mathcal{L}_\psi$, such that $\boldsymbol{\xi}_j = \mathbf{x}_j^+ - \mathbf{x}_j^-$ and $\mathbf{x}$ is the midpoint of $[\mathbf{x}_j^-, \mathbf{x}_j^+]$. Reciprocally, every vector $\boldsymbol{\xi}$ coincides with a discrete set of chords to $\mathcal{L}_\psi$ with midpoints $\mathbf{x}_j$. These are the basic elements for the construction of a WKB semiclassical theory of density operators using this pair of conjugate representations, as was first noticed by Berry [24].

More explicitly, the construction of chords from centres, or vice versa, is realized as follows: To determine the set of centres that are conjugate to a given chord, $\boldsymbol{\xi}$, for each $\hat{\rho}$, first translate the whole lagrangian submanifold, $\mathcal{L}$, by the vector $-\boldsymbol{\xi}$, then pick the set $\{\mathbf{x}^-{}_j\}$ of all points of intersection between $\mathcal{L}$ and the translated submanifold $\mathcal{L}_{-\boldsymbol{\xi}}$. The midpoint of each straight line, between $\mathbf{x}^+{}_j = \mathbf{x}^-{}_j + \boldsymbol{\xi}$ and $\mathbf{x}^-{}_j$, defines $\mathbf{x}_j(\boldsymbol{\xi}) = \mathbf{x}^-{}_j + \boldsymbol{\xi}/2$, the centre associated to $\boldsymbol{\xi}$ [9, 12]. To determine set of chords associated to each centre $\mathbf{x}$, first reflect $\mathcal{L}$ through $\mathbf{x}$ and pick the set $\{\mathbf{x}^\pm{}_j\}$ of all points of intersection between $\mathcal{L}$ and the reflected submanifold $\mathcal{L}_{\mathbf{x}}$. Then, each reflected pair of intersections defines a chord associated to $\mathbf{x}$ [30, 12], i.e. $\boldsymbol{\xi}_j(\mathbf{x}) = \mathbf{x}^+{}_j - \mathbf{x}^-{}_j$.

Given $\hat{\rho}$ and the corresponding $\mathcal{L}$, the simplest semiclassical approximation for the chord function $\chi(\boldsymbol{\xi})$ relates an amplitude $\alpha_j(\boldsymbol{\xi})$ and a phase $\sigma_j(\boldsymbol{\xi})$ to each of the centres $\mathbf{x}_j(\boldsymbol{\xi})$ above, so that [9]

$$\chi(\boldsymbol{\xi}) = \sum_j \alpha_j(\boldsymbol{\xi})\, \mathrm{e}^{i\sigma_j(\boldsymbol{\xi})/\hbar} = \sum_j \chi_j(\boldsymbol{\xi}) \ , \tag{2.16}$$

in such a way that

$$\mathbf{x}_j(\boldsymbol{\xi}) = \mathbf{J}\frac{\partial \sigma_j}{\partial \boldsymbol{\xi}} \ . \tag{2.17}$$

Similarly, the simplest WKB semiclassical approximation for the Wigner function [24]

$$W(\mathbf{x}) = \sum_j a_j(\mathbf{x})\, \mathrm{e}^{iS_j(\mathbf{x})/\hbar} = \sum_j W_j(\mathbf{x}) \ , \tag{2.18}$$

relates an amplitude $a_j(\mathbf{x})$ and a phase $S_j(\mathbf{x})$ to each of the chords $\xi_j(\mathbf{x})$ above, in such a way that

$$\boldsymbol{\xi}_j(\mathbf{x}) = -\mathbf{J}\frac{\partial S_j}{\partial \mathbf{x}} \ . \tag{2.19}$$

The phases $\sigma_j(\xi)$ (or $S_j(\mathbf{x})$) are also specified geometrically, as half the action (or symplectic area) of a circuit taken along the original submanifold $\mathcal{L}$ and closed along the translated submanifold $\mathcal{L}_{-\boldsymbol{\xi}}$ (or the reflected submanifold $\mathcal{L}_{\mathbf{x}}$). § The fact that the possible chords associated to a given centre always come in pairs $(\pm\boldsymbol{\xi}_j)$ guarantees that the semiclassical Wigner function is real, as it should be. There is no such restriction for the chord function, unless the manifold itself has a special symmetry [9].

This simplest semiclassical approximation for the chord and Wigner functions is valid far from caustics, which arise for arguments of the chord function whose associated centres coalesce, or for arguments of the Wigner function whose associated chords coalesce, respectively. Hence, caustics are related to points of tangency between $\mathcal{L}$ and $\mathcal{L}_{-\boldsymbol{\xi}}$, or between $\mathcal{L}$ and $\mathcal{L}_{\mathbf{x}}$, respectively [24, 30, 9]. For the Wigner function, this occurs whenever $\mathbf{x}$ approaches $\mathcal{L}$, in which case every pair of associated chords coalesce at a null chord (however, $\mathcal{L}$ is not the only region of centre caustics, generically). The null chord caustic is more severe for the chord function, because in this case $\mathcal{L}$ and $\mathcal{L}_{-\boldsymbol{\xi}}$ coincide. Thus, all the points in $\mathcal{L}$ are associated centres to the null chord.

The amplitude of each term in the above semiclassical approximation depends on N variables that are constant along $\mathcal{L}$. Defining the initial quantum state as an eigenstate of N commuting quantum operators, the corresponding lagrangian surface, $\mathcal{L}$ (an N-dimensional

§ Further *Maslov corrections* [7] should be included in the phase of the WKB semiclassical Wigner functions [24] and chord functions [9]. These are semiclassically small and do not alter the geometric relations (2.19) and (2.17).

torus, if it is compact) will be defined by N action variables $\mathcal{I}_n(\mathbf{x})$ in involution, i.e. all the Poisson brackets $\{\mathcal{I}_n, \mathcal{I}_{n'}\} = 0$. Let us now define the transported action variables,

$$\mathcal{I}_n^{\pm} = \mathcal{I}_n(\mathbf{x}^{\pm}) = \mathcal{I}_n(\mathbf{x} \pm \boldsymbol{\xi}/2), \tag{2.20}$$

which may be considered either as functions of $\mathbf{x}$, for fixed $\boldsymbol{\xi}$, or vice versa. Then, generally, $\{\mathcal{I}_n^+, \mathcal{I}_{n'}^-\} \neq 0$ and it is found that the amplitudes are

$$a(\mathbf{x}) = |\det\{\mathcal{I}_n^+, \mathcal{I}_{n'}^-\}|^{-1/2} = \alpha(\boldsymbol{\xi}), \tag{2.21}$$

within an overall normalization constant. This determinant can be reexpressed in terms of the Jacobian between the centre or chord variable and the $2N$ variables $(\mathcal{I}_n^+, \mathcal{I}_{n'}^-)$ [30]:

$$|\det \frac{\partial(\mathcal{I}_n^+, \mathcal{I}_{n'}^-)}{\partial \mathbf{x}}| = |\det\{\mathcal{I}_n^+, \mathcal{I}_{n'}^-\}| = |\det \frac{\partial(\mathcal{I}_n^+, \mathcal{I}_{n'}^-)}{\partial \boldsymbol{\xi}}|. \tag{2.22}$$

Clearly, the amplitudes, $\alpha_j(\boldsymbol{\xi})$ (or $a_j(\mathbf{x})$), depend on the degree of transversality of the intersection between $\mathcal{L}$ and $\mathcal{L}_{\boldsymbol{\xi}}$ (or $\mathcal{L}$ and $\mathcal{L}_{\mathbf{x}}$) and so they diverge at caustics [24, 30, 9].

It should be noted that the equality between the amplitudes in both representations, equations (2.21) and (2.22), holds for a specific pair of points $(\mathbf{x}^-, \mathbf{x}^+)$ on the torus and hence for a specific centre-chord pair. In the centre representation, the Poisson brackets are considered as functions of $\mathbf{x}$ and we define $\mathbf{x}^{\pm}(\mathbf{x})$. For the chord representation, these same endpoints are a function of $\boldsymbol{\xi}$ and so are the above Poisson brackets. The index, j, for the branch of the chord function or the Wigner function has been ommited from (2.21), because a specific centre-chord pair $(\mathbf{x},\ \xi)$ will be a particular member of a set $\{(\mathbf{x},\ \boldsymbol{\xi}_j(\mathbf{x}))\}$ for the Wigner function and, generically, a member of another set $\{(\mathbf{x}_{j'}(\boldsymbol{\xi}),\ \boldsymbol{\xi})\}$ for the chord function.

3. Review of the semiclassical limit for unitary evolution

A theory for the semiclassical limit of unitary evolution, appropriate to density operators or unitary operators in closed systems, has been established in both Weyl and chord representations [8, 15, 17]. It is worthwhile to adapt the deduction of phase space propagators in [17] for the needs of the foregoing theory. The starting point is the product formula for any pair of operators, $\widehat{B}\widehat{A}$, in the chord representation:

$$(\widehat{B}\widehat{A})(\boldsymbol{\xi}) = \frac{1}{(2\pi\hbar)^N} \int d\xi'\ \tilde{A}(\boldsymbol{\xi}')\ \tilde{B}(\boldsymbol{\xi} - \boldsymbol{\xi}')\ e^{\frac{i}{2\hbar}(\boldsymbol{\xi}\wedge\boldsymbol{\xi}')} \tag{3.1}$$

(see e.g. [26]). Here, when dealing with products of operators, we abuse the notation and use $(\widehat{B}\widehat{A})(\boldsymbol{\xi})$ to denote the chord symbol $\tilde{C}(\boldsymbol{\xi})$ of the operator $\widehat{C} = \widehat{B}\widehat{A}$ and, similarly, $(\widehat{B}\widehat{A})(\mathbf{x})$ stands for the Weyl symbol $C(\mathbf{x})$ ‖.

The problem is that we will work with the chord representation of $\widehat{\rho}$, though the Hamiltonian should be specified in the Weyl representation. This latter is indeed a smooth function, $H(\mathbf{x})$, exactly classical, or at least close to it within the order of $\hbar^2$, whereas its Fourier transform, $\tilde{H}(\boldsymbol{\xi})$, is highly singular. By defining the translation of an operator as

$$\hat{A}_{\boldsymbol{\eta}} := \hat{T}_{\boldsymbol{\eta}}\hat{A}\hat{T}_{-}\boldsymbol{\eta}, \tag{3.2}$$

whose chord representation is given by

$$\tilde{A}_{\boldsymbol{\eta}}(\boldsymbol{\xi}) = e^{\frac{i}{\hbar}\boldsymbol{\eta}\wedge\boldsymbol{\xi}}\ \tilde{A}(\boldsymbol{\xi}), \tag{3.3}$$

‖ Sometimes, the Weyl symbol of $\widehat{B}\widehat{A}$ is denoted by the *star product* $B\star A$, when B is the Weyl symbol of $\widehat{B}$ and A is the Weyl symbol of $\widehat{A}$.

while its Weyl representation reads

$$A_{\boldsymbol{\eta}}(\mathbf{x}) = A(\mathbf{x}+\boldsymbol{\eta}), \tag{3.4}$$

the phase factor in (3.1) can be incorporated into an integral involving both representations. Then, using (2.4), (3.3) and (3.4), we rewrite (3.1) as

$$(\widehat{B}\widehat{A})(\boldsymbol{\xi}) = \frac{1}{(2\pi\hbar)^{2N}} \int d\boldsymbol{\xi}' d\mathbf{x}' \; A(\mathbf{x}'-\boldsymbol{\xi}/2)\; \tilde{B}(\boldsymbol{\xi}')\; e^{\frac{i}{\hbar}(\boldsymbol{\xi}-\boldsymbol{\xi}')\wedge\mathbf{x}'}. \tag{3.5}$$

In this way, we obtain the chord representation of the commutator between $\hat{H}$ and the evolving density operator, $\hat{\rho}(t)$, as the *mixed* integral,

$$(\hat{H}\hat{\rho}-\hat{\rho}\hat{H})(\boldsymbol{\xi}) = \int \frac{d\boldsymbol{\xi}' d\mathbf{x}'}{(2\pi\hbar)^N}\; [H(\mathbf{x}'-\boldsymbol{\xi}/2) - H(\mathbf{x}'+\boldsymbol{\xi}/2)]\; \chi(\boldsymbol{\xi}')\; e^{\frac{i}{\hbar}(\boldsymbol{\xi}-\boldsymbol{\xi}')\wedge\mathbf{x}'}. \tag{3.6}$$

Here, we emphasize, $H(\mathbf{x})$ is the Weyl representation of $\widehat{H}$, which is a smooth function, so that (3.6) can be integrated in the stationary phase approximation. In the special case where $H(\mathbf{x})$ is a polynomial, we can perform the integrals in (3.6) exactly. For a quadratic Hamiltonian we thus re-derive [10]

$$(2\pi\hbar)^{-N}\; (\hat{H}\hat{\rho}-\hat{\rho}\hat{H})(\boldsymbol{\xi}) = i\hbar\,\Big\{H(\boldsymbol{\xi}),\; \chi(\boldsymbol{\xi})\Big\}, \tag{3.7}$$

which emulates the familiar result that the the Wigner function evolves classically, when the Hamiltonian is quadratic [22].

For general Hamiltonians, we now insert the semiclassical approximation (2.16) for $\chi(\boldsymbol{\xi},t)$ in (3.6). Because of the linearity of the evolution equation for the density operator, it can be decomposed into branches $\widehat{\rho}_j(t)$, each evolving separately, as represented by one of the semiclassical components, $\chi_j(\boldsymbol{\xi},t)$ in (2.16). Then (3.6) can be integrated by stationary phase, to yield the lowest order semiclassical approximation:

$$\begin{aligned}
&(2\pi\hbar)^{-N}\; (\hat{H}\hat{\rho}-\hat{\rho}\hat{H})_{SC}(\boldsymbol{\xi}) \\
&= \sum_j \alpha_j(\boldsymbol{\xi})\Big(H(\mathbf{J}\frac{\partial\sigma_j}{\partial\boldsymbol{\xi}} - \frac{\boldsymbol{\xi}}{2}) - H(\mathbf{J}\frac{\partial\sigma_j}{\partial\boldsymbol{\xi}} + \frac{\boldsymbol{\xi}}{2})\Big)\; e^{i\sigma_j(\boldsymbol{\xi})/\hbar} &(3.8)\\
&= \sum_j \Big(H(\mathbf{x}_j(\boldsymbol{\xi})-\boldsymbol{\xi}/2) - H(\mathbf{x}_j(\boldsymbol{\xi})+\boldsymbol{\xi}/2)\Big)\; \chi_j(\boldsymbol{\xi})\;. &(3.9)
\end{aligned}$$

Thus, by comparing with the unitary part of the master equation (1.1), we find that the classical chord action $\sigma_j(\boldsymbol{\xi},t)$ evolves according to the Hamilton-Jacobi equation [17]:

$$\frac{\partial\sigma_j}{\partial t}(\boldsymbol{\xi},t) = H(\mathbf{J}\frac{\partial\sigma_j}{\partial\boldsymbol{\xi}} + \frac{\boldsymbol{\xi}}{2}) - H(\mathbf{J}\frac{\partial\sigma_j}{\partial\boldsymbol{\xi}} - \frac{\boldsymbol{\xi}}{2}), \tag{3.10}$$

similarly to the evolution for the centre action [15].

It must be remembered that in (3.10), as well as in (3.6) through (3.9), the function $H(\mathbf{x})$ is the Weyl representation of the quantum hamiltonian operator, which will be either identical, or semiclassically close to the classical hamiltonian function. In the general case where this hamiltonian function is nonlinear, the resulting evolution of the chord action is a consequence of the classical motion $\mathbf{x}_j^{\pm}(\boldsymbol{\xi},t)$ of both the chord tips, $\mathbf{x}_j^{\pm}(\boldsymbol{\xi},0) = \mathbf{x}_j^{\pm}(\boldsymbol{\xi}) = \mathbf{x}_j(\boldsymbol{\xi}) \pm \boldsymbol{\xi}/2$, whereas neither the chord, $\boldsymbol{\xi}$ itself, nor the corresponding centres, $\mathbf{x}_j(\boldsymbol{\xi})$, will generally follow their respective hamiltonian phase space trajectories [8]. By working directly on the double phase space, as discussed in section 6, a new Hamiltonian function can be defined on this doubled space to take account of the motion of both chord tips in a single trajectory [15].

The above approximation for the unitary evolution of the chord function does not include the evolution of the amplitudes, $\alpha_j(\boldsymbol{\xi})$ in (2.16), which can be obtained by including the next

order in $\hbar$ in the theory. Alternatively, we note that, up to the leading order, the evolution can be portrayed as resulting from the full classical motion, i.e. all the trajectories generated by the hamiltonian, $H(\mathbf{x})$, which transports the entire lagrangian submanifold, $\mathcal{L}(t)$, and its neighborhood. Thus, each pair of points, $\mathbf{x}_j^{\pm}(\boldsymbol{\xi},t)$ on $\mathcal{L}(t)$ defines an evolving chord,

$$\overline{\boldsymbol{\xi}}_j(t) = \mathbf{x}_j^{+}(\boldsymbol{\xi},t) - \mathbf{x}_j^{-}(\boldsymbol{\xi},t) \tag{3.11}$$

and an evolving centre,

$$\overline{\overline{\mathbf{x}}}_j(\boldsymbol{\xi},t) = \Big(\mathbf{x}_j^{+}(\boldsymbol{\xi},t) + \mathbf{x}_j^{-}(\boldsymbol{\xi},t)\Big)/2\ . \tag{3.12}$$

One should note that, here, $\mathbf{x}_j^{\pm}(\boldsymbol{\xi},t)$ denotes the hamiltonian trajectories of $\mathbf{x}_j^{\pm}(\boldsymbol{\xi},0) = \mathbf{x}_j^{\pm}(\boldsymbol{\xi}) = \mathbf{x}_j(\boldsymbol{\xi}) \pm \boldsymbol{\xi}/2$. Therefore, $\overline{\boldsymbol{\xi}}_j(t)$ is generally different from the hamiltonian trajectory $\boldsymbol{\xi}_j(t)$ of the initial chord, $\boldsymbol{\xi}_j(0) = \overline{\boldsymbol{\xi}}_j(0) = \boldsymbol{\xi}$, unless the hamiltonian is not quadratic. Similarly, $\overline{\overline{\mathbf{x}}}_j(\boldsymbol{\xi},t)$ is generally different from the hamiltonian trajectory $\mathbf{x}_j(\boldsymbol{\xi},t)$ of $\mathbf{x}_j(\boldsymbol{\xi},0) = \overline{\overline{\mathbf{x}}}_j(\boldsymbol{\xi},0) = \mathbf{x}_j(\boldsymbol{\xi})$ [8].

By reconstructing the chord function according to the semiclassical prescription (2.16) at each instant, the same phase evolution is obtained as from the Hamilton-Jacobi equation (3.10), but now the evolution of the amplitudes will also be included, as long as we also allow the action variables $\mathcal{I}_n(\mathbf{x}^{\pm})$ in (2.21) to evolve according to $\mathcal{I}_n(\mathbf{x}^{\pm},t) = \mathcal{I}_n(\mathbf{x}^{\pm}(t))$, where $\mathbf{x}^{\pm}(t)$ is the hamiltonian trajectory of $\mathbf{x}^{\pm}(0)$. Again, it must be stressed that this semiclassical evolution of the chord function (or the Wigner function), obtained via global classical motion together with the geometric reconstruction of the representation at each instant, can only be identified with Liouville evolution, i.e. the evolution obtained from the hamiltonian trajectory of the argument of the the chord function (or the Wigner function), if the Hamiltonian is quadratic [8].

4. Chord representation of the open interaction term

We now address various integral representations of the chord symbol for the open interaction term. The starting point is the product rule in the chord representation [26],

$$(\widehat{A}\widehat{B}\widehat{C})(\boldsymbol{\xi}) = \int \frac{d\boldsymbol{\xi}' d\boldsymbol{\xi}'' d\boldsymbol{\xi}'''}{(2\pi\hbar)^{2N}} \tilde{A}(\boldsymbol{\xi}')\tilde{B}(\boldsymbol{\xi}'')\tilde{C}(\boldsymbol{\xi}''')\ \delta(\boldsymbol{\xi}-\boldsymbol{\xi}'-\boldsymbol{\xi}''-\boldsymbol{\xi}''')\ \exp\left[\frac{-i}{2\hbar}(\boldsymbol{\xi}\wedge\boldsymbol{\xi}' + \boldsymbol{\xi}''\wedge\boldsymbol{\xi}''')\right], \tag{4.1}$$

where, again, we abuse the notation and write $(\widehat{A}\widehat{B}\widehat{C})(\boldsymbol{\xi})$ for the chord symbol of $\widehat{A}\widehat{B}\widehat{C}$. The exponent in the integrand is here one of the many different expressions for the symplectic area of the quadrilateral with sides: $\boldsymbol{\xi}', \boldsymbol{\xi}'', \boldsymbol{\xi}''', -\boldsymbol{\xi}$. Incorporating the phase factor for translation into the chord representation, as in (3.3), leads to the compact expression,

$$(\widehat{A}\widehat{B}\widehat{C})(\boldsymbol{\xi}) = \int \frac{d\boldsymbol{\xi}' d\boldsymbol{\xi}''}{(2\pi\hbar)^{2N}}\ \tilde{A}_{\boldsymbol{\xi}/2}(\boldsymbol{\xi}')\ \tilde{B}(\boldsymbol{\xi}'')\ \tilde{C}_{\boldsymbol{\xi}''/2}(\boldsymbol{\xi}-\boldsymbol{\xi}'-\boldsymbol{\xi}''). \tag{4.2}$$

Even if the Lindblad operators $\hat{L}$ are not observables, as in the optical example (2), their Weyl representation are smooth functions on phase space, $L(\mathbf{x})$, whereas their chord representation, $\tilde{L}(\boldsymbol{\xi})$, are quite singular. Therefore, we again need a mixed product rule, where a pair of operators, $\widehat{A}$ and $\widehat{C}$ are expressed in the Weyl representation:

$$(\widehat{A}\widehat{B}\widehat{C})(\boldsymbol{\xi}) = \int \frac{d\mathbf{x}' d\boldsymbol{\xi}''}{(2\pi\hbar)^{2N}}\ A_{\boldsymbol{\xi}/2}(\mathbf{x}')\ \tilde{B}(\boldsymbol{\xi}'')\ C_{\boldsymbol{\xi}''/2}(\mathbf{x}')\ \exp\left[\frac{i}{\hbar}\mathbf{x}'\wedge(\boldsymbol{\xi}-\boldsymbol{\xi}'')\right]. \tag{4.3}$$

To obtain the desired expression for the nonunitary term of the Lindblad equation (1.1), the order of the operators is permuted, which leads to sign changes for translated operators

(3.2), given by $A_{\boldsymbol{\eta}}(\mathbf{x}) = A(\mathbf{x}+\boldsymbol{\eta})$ in the Weyl representation, so that

$$(\hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^\dagger\hat{L})(\boldsymbol{\xi}) = \int \frac{d\boldsymbol{\xi}' d\mathbf{x}'}{(2\pi\hbar)^N}\ \ \chi(\boldsymbol{\xi}')\ \exp\Big[\frac{i}{\hbar}\mathbf{x}'\wedge(\boldsymbol{\xi}-\boldsymbol{\xi}')\Big]$$
$$\Big\{L(\mathbf{x}'+\frac{\boldsymbol{\xi}}{2})L^*(\mathbf{x}'-\frac{\boldsymbol{\xi}'}{2}) - \frac{1}{2}[L(\mathbf{x}'+\frac{\boldsymbol{\xi}'}{2})L^*(\mathbf{x}'+\frac{\boldsymbol{\xi}}{2}) + L(\mathbf{x}'-\frac{\boldsymbol{\xi}}{2})L^*(\mathbf{x}'-\frac{\boldsymbol{\xi}'}{2})]\Big\}. \tag{4.4}$$

Note that $L^*(\mathbf{x}'+\boldsymbol{\xi}/2)$ is the Weyl symbol of the operator $\hat{L}^\dagger$ translated by $\boldsymbol{\xi}/2$, which is not equal to the adjoint of $\hat{L}_{\boldsymbol{\xi}/2}$.

The exact formula (4.4) is at a par with the representation of the commutator (3.6). It is interesting that, although (4.4) represents products of three operators, the dimension of the integral is the same as in (3.6). Thus, including the presence of an internal Hamiltonian and a single Lindblad operator, the exact equation of motion for the chord function is given by

$$\hbar\frac{\partial\chi}{\partial t}(\boldsymbol{\xi},t) = \int d\boldsymbol{\xi}' d\mathbf{x}'\ \ \chi(\boldsymbol{\xi}',t)\ \exp\Big[\frac{i}{\hbar}[\mathbf{x}'\wedge(\boldsymbol{\xi}-\boldsymbol{\xi}')]\Big]\ \ \Big\{-\mathrm{i}[H(\mathbf{x}'-\frac{\boldsymbol{\xi}}{2}) - H(\mathbf{x}'+\frac{\boldsymbol{\xi}}{2})]$$
$$+\Big[L(\mathbf{x}'+\frac{\boldsymbol{\xi}}{2})L^*(\mathbf{x}'-\frac{\boldsymbol{\xi}'}{2}) - \frac{1}{2}[L(\mathbf{x}'+\frac{\boldsymbol{\xi}'}{2})L^*(\mathbf{x}'+\frac{\boldsymbol{\xi}}{2}) + L(\mathbf{x}'-\frac{\boldsymbol{\xi}}{2})L^*(\mathbf{x}'-\frac{\boldsymbol{\xi}'}{2})]\Big]\Big\}. \tag{4.5}$$

If there are more Lindblad operators in the master equation (1.1), then one must sum over these in the integrand on the right hand side of (4.5). We have not included this obvious extension, so as not to confuse this sum with the further sum over semiclassical branches in the following formulae.

Far from caustics, one can evaluate (4.4) approximately, by stationary phase, if $L(\mathbf{x})$ is assumed to be a smooth function, by inserting the semiclassical approximation for each separate branch of the chord function (2.16) as in the previous section. The stationary phase condition singles out $\boldsymbol{\xi}' = \boldsymbol{\xi}$ and $\mathbf{x}' = \mathbf{x}_j(\boldsymbol{\xi})$, one of the centres associated to a geometrical chord $\boldsymbol{\xi}$ of the classical submanifold $\mathcal{L}$. The full semiclassical approximation is simply

$$(2\pi\hbar)^{-N}(\hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^\dagger\hat{L})_{SC}(\boldsymbol{\xi}) = \sum_j \Big\{L(\mathbf{x}_j(\boldsymbol{\xi})+\boldsymbol{\xi}/2)L^*(\mathbf{x}_j(\boldsymbol{\xi})-\boldsymbol{\xi}/2)$$
$$-\frac{1}{2}\{|L(\mathbf{x}_j(\boldsymbol{\xi})+\boldsymbol{\xi}/2)|^2 + |L(\mathbf{x}_j(\boldsymbol{\xi})-\boldsymbol{\xi}/2)|^2\}\Big\}\ \alpha_j(\boldsymbol{\xi})\ \ e^{i\sigma_j(\boldsymbol{\xi})/\hbar}. \tag{4.6}$$

In terms of the chord tips, $\mathbf{x}_j^{\pm}(\boldsymbol{\xi}) = \mathbf{x}_j(\boldsymbol{\xi}) \pm \boldsymbol{\xi}/2$, the semiclassical approximation to the chord representation of the open interaction term can be rewritten as

$$(2\pi\hbar)^{-N}(\hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^\dagger\hat{L})_{SC}(\boldsymbol{\xi}) =$$
$$-\frac{1}{2}\sum_j \Big\{|L(\mathbf{x}_j^+(\boldsymbol{\xi})) - L(\mathbf{x}_j^-(\boldsymbol{\xi}))|^2 - \mathrm{i}\,\mathcal{I}m\{L(\mathbf{x}_j^+(\boldsymbol{\xi}))L^*(\mathbf{x}_j^-(\boldsymbol{\xi}))\}\Big\}\ \chi_j(\boldsymbol{\xi}), \tag{4.7}$$

where $\mathcal{I}m$ denotes the imaginary part and $\chi_j(\boldsymbol{\xi})$ is a branch of the semiclassical chord function given by (2.16). In the case of a linear function,

$$L(\mathbf{x}) = \boldsymbol{l}\cdot\mathbf{x} = \boldsymbol{l}'\cdot\mathbf{x} + i\,\boldsymbol{l}''\cdot\mathbf{x}\ , \tag{4.8}$$

as in the optical example (1.2), the semiclassical approximation for the open interaction term simplifies to

$$(2\pi\hbar)^{-N}(\hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{L}\hat{L}^\dagger\hat{\rho})_{SC}(\boldsymbol{\xi}) =$$
$$\sum_j \Big(-i(\boldsymbol{l}'\wedge\boldsymbol{l}'')\mathbf{x}_j(\boldsymbol{\xi})\wedge\boldsymbol{\xi} - \frac{1}{2}[(l'\cdot\boldsymbol{\xi})^2 + (\boldsymbol{l}''\cdot\boldsymbol{\xi})^2]\Big)\ \chi_j(\boldsymbol{\xi}). \tag{4.9}$$

On the other hand, (4.4) can be integrated exactly, for a linear Lindblad operator (or even if it is a polynomial). Then (4.4) becomes

$$(2\pi\hbar)^{-N}(\hat{L}\hat{\rho}\hat{L}^{\dagger} - \frac{1}{2}\hat{L}^{\dagger}\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^{\dagger}\hat{L})(\boldsymbol{\xi}) = \hbar(\boldsymbol{l}' \wedge \boldsymbol{l}'')\ \boldsymbol{\xi}\cdot\frac{\partial\chi}{\partial\boldsymbol{\xi}} - \frac{1}{2}\left[(\boldsymbol{l}'\cdot\boldsymbol{\xi})^2 + (\boldsymbol{l}''\cdot\boldsymbol{\xi})^2\right]\chi(\boldsymbol{\xi})\ , \quad (4.10)$$

in agreement with [10]. Compared with (4.9), we find the same second term on the right hand side. If the Lindblad operator is self-adjoint, i.e. $\boldsymbol{l}'' = \boldsymbol{0}$, this will be the only term. In this case, it is easier to develop a semiclassical theory for evolution of the density operator, which becomes exact in the case that the Hamiltonian is quadratic. This will be pursued in the following section. The first term was shown to describe dissipation in the exact linear theory [10]. Though dissipation cannot be included in a standard semiclassical theory, we will show that it is naturally accommodated within the double phase space formalism that is developed in later sections.

5. Decoherence without dissipation

In this section, all Lindblad operators $\hat{L}_k$ are restricted to be self-adjoint, so that $\mathcal{I}m\{L(\mathbf{x}_j^+(\boldsymbol{\xi}))L^*(\mathbf{x}_j^-(\boldsymbol{\xi}))\} \equiv 0$, simplifying equation (4.7). As pointed out in the introduction, this means that the system may be considered to be conservative, albeit open to a random environment.

If we further ignore the internal hamiltonian motion, or, more reasonably, restrict analysis of the decoherence process to its first stages, then we can consider the action $\sigma_j(\boldsymbol{\xi})$ to be constant in time, while the semiclassical amplitude evolves as

$$\alpha_j(\boldsymbol{\xi}, t) = \alpha_j(\boldsymbol{\xi}, 0)\exp\Big\{-\frac{t}{2\hbar}\sum_k |L_k(\mathbf{x}_j^+(\boldsymbol{\xi})) - L_k(\mathbf{x}_j^-(\boldsymbol{\xi}))|^2\Big\}\ . \quad (5.1)$$

Generally, the above equation implies a fast shrinking of the chord function to a progressively narrower neighborhood of the origin. According to the discussion in [10], this accounts for a fast loss of quantum correlations. However, for those tips, $\mathbf{x}_j^{\pm}$, that happen to be on a level curve (or level surface) of one of the real functions, $L_k(\mathbf{x})$, this will not contribute to the loss of amplitude. The condition for a chord not to decay at all is that its tips should lie on the intersection of level surfaces for all the functions, $L_k(\mathbf{x})$.

The effect of the internal Hamiltonian, $\hat{H}$, can be included by considering the limit in a process where we switch it on and off, while alternatively connecting and disconnecting the Lindblad interaction (opening and closing the system) [6]. It is thus pictured as a periodic markovian system in the limit of small periods, as in the periodization of hamiltonian systems in [31]. Both the tips of the chord, $\mathbf{x}_j^{\pm}(\boldsymbol{\xi})$, will evolve classically as $\mathbf{x}_j^{\pm}(\boldsymbol{\xi}, t)$ according to the Hamilton-Jacobi equation (3.10) for a time $\tau/2$, implying in the temporal evolution of a given chord, $\boldsymbol{\xi}$, as $\overline{\boldsymbol{\xi}}_j(\boldsymbol{\xi}, t)$ and for the centre, $\mathbf{x}_j(\boldsymbol{\xi})$, the motion $\overline{\overline{\mathbf{x}_j}}(\boldsymbol{\xi}, t)$, according to equations (3.11) and (3.12). Then, at each opening of the system for a further period of $\tau/2$, the amplitude evolves according to (5.1). Naturally, one must multiply both the open and the closed terms of the Lindblad equation by a factor of two, to make up for the reduced time in which either of them acts. In the limit as $\tau \to 0$ of an infinite number of *closing* and *opening* operations, we obtain the full semiclassical evolution of the chord function in a region free of caustics, as $\chi(\boldsymbol{\xi}, t) = \sum_j \chi_j(\boldsymbol{\xi}, t)$, with

$$\chi_j(\boldsymbol{\xi}, t) = {\chi_j}^0(\boldsymbol{\xi}, t)\ \ \exp\Big[\frac{-1}{2\hbar}D\{\mathbf{x}_j^+(-t), \mathbf{x}_j^-(-t)\}\Big], \quad (5.2)$$

where $\mathbf{x}_j^{\pm}(-t)$ is short for $\mathbf{x}_j^{\pm}(\boldsymbol{\xi}, -t)$ and ${\chi^0}_j(\boldsymbol{\xi}, t)$ denotes the semiclassical propagation for a time t of the j-branch of the chord function for the corresponding closed system (with all

$\widehat{L}_k = 0$). The decay in amplitude for each branch of the chord function is determined by the *decoherence functional* over trajectory pairs,

$$D\{\mathbf{x}^+(t), \mathbf{x}^-(t)\} := \sum_k \int_0^t dt' |L_k(\mathbf{x}^+(t')) - L_k(\mathbf{x}^-(t'))|^2 \ , \tag{5.3}$$

where $\mathbf{x}^{\pm}(0) = \mathbf{x}(\boldsymbol{\xi}) \pm \boldsymbol{\xi}/2$. Hence, it is the pair of backward trajectories ending at a given pair of chord tips on $\mathcal{L}(t)$ that determine the decrease in amplitude. The square root of the decoherence functional is a kind of time dependent measure of *distance* between any pair of points $(\mathbf{x}^+, \mathbf{x}^-)$, as pointed out by Strunz [16].

Concerning the derivation of the above semiclassical expression (5.2), note that the chord function entering into the master equation is here the semiclassical chord function of a pure state (2.16), which is valid away from chord caustics. But far from the origin (a chord caustic), the damping factor in (5.2) is a non-oscillatory function. Hence, in a first approximation, it may be considered as a new factor of the semiclassical amplitude $\alpha_j(\boldsymbol{\xi})$ in (5.2), even though the exponent is divided by $\hbar$. Thus, when this modified expression for the chord function is inserted into the master equation, it is still the chord action function from (2.16) that defines the semiclassical evolution of the decaying chord function (5.2), as long as all pertinent integrals are computed via the *real* stationary phase method. Accordingly, an improvement to (5.2) could in principle be obtained by computing all pertinent integrals via the complex steepest descent method. This improvement is at present being investigated.

The simplest case is where the Lindblad operators are all linear functions of position and momenta, $L_k(\mathbf{x}) = \boldsymbol{l}_k \cdot \mathbf{x}$. Then expression (5.2) simplifies, because

$$\Big| L_k(\mathbf{x}_j^+(\boldsymbol{\xi}, -t')) - L_k(\mathbf{x}_j^-(\boldsymbol{\xi}, -t')) \Big|^2 = |\boldsymbol{l}_k \cdot \overline{\boldsymbol{\xi}}_j(\boldsymbol{\xi}, -t')|^2 \ , \tag{5.4}$$

where the explicit dependence on $\boldsymbol{\xi}$ is emphasized in the r.h.s. Generally the evolution of each $\boldsymbol{\xi}_j$ results from the hamiltonian flow of the tips $\mathbf{x}_j^{\pm}(\boldsymbol{\xi})$, so that the evolution is j-dependent. However, if the internal Hamiltonian is a homogeneous quadratic, then the evolution of the chord is just given by [10]

$$\dot{\boldsymbol{\xi}} = \mathbf{J}\, \frac{\partial H}{\partial \boldsymbol{\xi}} \tag{5.5}$$

and is therefore j-independent. Furthermore, the internal dynamics of the chord function is then Liouvillian: ${\chi^0}_j(\boldsymbol{\xi}, t) = \chi_j(\boldsymbol{\xi}(-t))$. In this way, all j-branches of the semiclassical chord function can be combined into a single evolution, so that

$$\chi(\boldsymbol{\xi}, t) = \chi^0(\boldsymbol{\xi}(-t)) \exp\Big\{ -\frac{1}{2\hbar} \sum_k \int_0^t dt' |\boldsymbol{l}_k \cdot \boldsymbol{\xi}(-t')|^2 \Big\}. \tag{5.6}$$

It is remarkable that this simple expression for the semiclassical evolution of an open system is actually exact and valid for any initial chord function (pure or mixed), under the above hypothesis for $\hat{L}$ and $\hat{H}$ [10]. Thus, no matter how fraught with quantum correlations the initial state might be, the infinite product of gaussian exponentials in (5.6), or the more general exponential of the decoherence functional in (5.2) progressively squeezes them out. This process by which the large chords are quenched proceeds irreversibly, since $D\{\mathbf{x}_j^+(-t), \mathbf{x}_j^-(-t)\}$ is a nondecreasing function of time.

The semiclassical expression (5.2) generalizes the simple exact solution (5.6), when the Hamiltonian is not quadratic, for chords that never lie close to caustics throughout the evolution. If all Lindblad operators are linear (refer to (5.4)), the squeezing of each term of the chord function still results from a product of Gaussians, but these depend on the particular branch of the chord action function, according to each specific evolution $\overline{\boldsymbol{\xi}}_j(\boldsymbol{\xi}, t)$,

as explained above. For nonlinear Lindblad operators, we retain the qualitative picture in which the evolving chord function is squeezed onto the origin by the decoherence functional, although $D\{\mathbf{x}_j^+(t), \mathbf{x}_j^-(t)\}$ is no longer a quadratic function of $\boldsymbol{\xi}$.

The only possibility for it not to increase arises if the pairs of classical trajectories generated by $H(\mathbf{x})$ lie along a level submanifold of the linear Lindblad-Weyl function, $L(\mathbf{x})$, that is, when the Poisson bracket $\{L, H\} = 0$, which holds when the operators $\widehat{L}$ and $\widehat{H}$ commute. For more than one Lindblad operator, there is no damping when the classical trajectories lie on the intersection of all L_k level submanifolds, that is, when $\{L_k, H\} = 0$, for all k. The specific evolution for each kind of classical dynamics (elliptic, parabolic or hyperbolic) is studied in [10].

In view of the full evolution expression (5.2), we can justify the rough qualitative description of decoherence that neglects the internal dynamics (5.1), because the chord function decays exponentially with time, in the domain where (5.2) is valid (which excludes a neighborhood of the origin). Since we may assume that generically $\{L_k, H\} \neq 0$, it follows that after a relatively short time (with respect to the internal dynamics) the chord function becomes negligible, except in a region of small chords, where (5.2) is not appropriate. A generalization of the quadratic case (5.6) into the region of small chords is still required and is the subject of ongoing work, but this is achieved indirectly in section 8.

The semiclassical evolution of the Wigner function for open conservative markovian systems is obtained by the Fourier transform of expressions (5.2) or (5.6) for the semiclassical evolution of the chord function. Each term of the sum, $W(\mathbf{x}, t) = \sum_j W_j(\mathbf{x}, t)$, is given by a convolution integral of the unitarily evolving branch of the semiclassical Wigner function (2.18) with the Fourier transform of the decaying amplitude term. This diffusive window, which coarse-grains the Wigner function, will broaden with time, as its inverse Fourier transform narrows down the range of the chord function. The window will be gaussian and this description of the evolution of the Wigner function becomes exact in the case of a quadratic Hamiltonian [10].

The semiclassical expression (5.2) can only be inserted into the Fourier transform (2.10) for chords that are far from caustics, which precludes small chords. For large chords, the convolution integral can be evaluated by (real) stationary phase, because the decaying amplitude term is a smooth function of $\boldsymbol{\xi}$ far from the origin. We then obtain a superposition of terms of the same form as (2.18), each of them corresponding to a different branch of the centre action function $S_j(\mathbf{x}, t)$. However, as with the chord function, the amplitude $a_j(\mathbf{x})$ now acquires a new time-dependent factor, so that we have the complete analogue of equation (5.2) for the semiclassical evolution of the Wigner function as

$$W_j(\mathbf{x}, t) = {W^0}_j(\mathbf{x}, t) \;\; \exp\Big[\frac{-1}{2\hbar} D\{\mathbf{x}_j^+(-t), \mathbf{x}_j^-(-t)\}\Big], \tag{5.7}$$

where ${W^0}_j(\mathbf{x}, t)$ denotes semiclassical propagation for a time t of the j-branch of the Wigner function as a closed system. The amplitude also decays according to the decoherence functional (5.3) that quenches the contribution of long chords. However, this is now determined by the choice of centre, $\mathbf{x}$, rather than the chord, $\boldsymbol{\xi}$, i.e. here the chord tips are $\mathbf{x}^{\pm}(\mathbf{x}, t)$. One should note that the inclusion of the new gaussian factors in (5.2) and (5.7) within the respective semiclassical amplitudes preserves their equality (2.21).

The semiclassical expression (5.7) was derived earlier in [6], solely within the Weyl representation. However, it does not lead to the exact evolution of the Wigner function in the simple case of linear Lindblad and quadratic Hamiltonian operators as does its chord similar (5.2). It follows that the theory for markovian open systems does not extend the general semiclassical covariance of various representations of unitary evolutions with respect

to Fourier transforms performed by stationary phase. Here, the chord representation has a decided advantage, because an essential qualitative feature is destroyed if the Fourier transform to the Wigner function is approximated by stationary phase.

Thus, although (5.7) well describes the initial stages of decoherence, it fails to address the diffusive process that sets in at longer time scales. In fact, there is no suggestion of the *decoherence time* threshold at which the initial pure-state Wigner function becomes positive definite, as always happens in the case of linear Lindblad and quadratic Hamiltonian operators [32, 10]. This is the time which takes an initial state, represented by a Dirac-delta function in phase space, to evolve into a Gaussian with the width of a pure coherent state. At this time, any initial pure Wigner function evolves into a positive-definite phase space distribution, which is indistinguishable from a Husimi function. In the general case of nonquadratic Hamiltonians, we can still define a local decoherence time as that which it takes the quenching factor in (5.2) to shrink to the extent that it has the same area (or volume) as a coherent state. Beyond this time, its Fourier transform will coarse-grain away the fine oscillations of the Wigner function.

6. Semiclassical evolution in double phase space

WKB theory and its generalization to higher dimensions [7, 14] relates the solution, $\langle q|\psi(t)\rangle$, of a Schrödinger equation to the corresponding classical Hamiltonian trajectories in the phase space $\mathbf{x} = (p, q) \in \mathbf{R^{2N}}$. This Schrödinger solution is the $|q\rangle$ representation of the unitarily evolving state, associated to a lagrangian submanifold $\mathcal{L}_\psi$, which is described within the Weyl formalism in section 3. This submanifold is more commonly described (locally) as a graph of the classical function, $p(q) = \frac{\partial S}{\partial q}(q)$, which maps $q \in \mathbf{R^N}$ onto $p \in \mathbf{R^N}$ (which are also lagrangian coordinate subspaces, satisfying (2.15)). The classical action $S(q)$ is both the generating function for $\mathcal{L}_\psi$ and the oscillating phase of the quantum wave $\langle q|\psi\rangle$.

Analogously, the linear operators, $\widehat{A}$, that act on the quantum Hilbert space, form a vector space $|A\rangle\!\rangle$, for which the dyadic operators $|\boldsymbol{Q}\rangle\!\rangle = |q^-\rangle\langle q^+|$ constitute a complete basis. Thus, defining the *Hilbert-Schmidt product*:

$$\mathrm{tr}\, \widehat{A}^\dagger \widehat{B} = \langle\!\langle A|B\rangle\!\rangle, \tag{6.1}$$

we can interpret the ordinary position representation of the operator $\widehat{A}$ as

$$\langle q^+|\widehat{A}|q^-\rangle = \mathrm{tr}\, |q^-\rangle\langle q^+|\widehat{A} = \langle\!\langle \boldsymbol{Q}|A\rangle\!\rangle, \tag{6.2}$$

in close analogy to a wave function. It is then natural to relate a *double Hilbert space* of $|ket\rangle\langle bra|$ states to a double phase space: $\{\mathbf{X}\} = \{\mathbf{x}^-\} \times \{\mathbf{x}^+\}$, where $\mathbf{x}^\pm = (p^\pm, q^\pm)$ (see e.g.[11, 12], or [33] for non-vectorial cases). The operator $|\boldsymbol{Q}\rangle\!\rangle$ should then correspond to the lagrangian subspace, $\boldsymbol{Q} = constant$, in the double phase space. This does hold, within a minor adaptation, due to the presence of the adjoint operator in the definition of the Hilbert-Schmidt product, or, more directly, the fact that, in ordinary Hilbert space, $\langle bras|$ are adjoint to $|kets\rangle$. Accordingly, if we define $\boldsymbol{Q} = (q^-, q^+)$, we should define $\boldsymbol{P} = (-p^-, p^+)$ as conjugate coordinates on the double phase space $\{\mathbf{X} = (\boldsymbol{P}, \boldsymbol{Q})\}$. This is equivalent to changing the sign of the symplectic structure on $\mathbf{R^{2N}} = \{\mathbf{x}^-\}$.

In this way, we include among the set of lagrangian submanifolds in double phase space, all the graphs of canonical transformations on single phase space, $\mathbf{x}^- \mapsto \mathbf{x}^+ = \mathbf{C}(\mathbf{x}^-)$. That is, we may rewrite the definition of a canonical transformation as

$$\oint_\Gamma \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{Q} = 0, \tag{6.3}$$

where Γ is any curve defined on the $(2N)$-dimensional submanifold, $\Lambda_{\mathbf{C}}$, which is the graph of the canonical transformation $\mathbf{C}$ on the $(2N)$-dimensional space $\{\mathbf{x}^- = (q^-, p^-)\}$, within the $(4L)$-dimensional double phase space, $\mathbf{R^{4N}} = \{\mathbf{X} = (\boldsymbol{P}, \boldsymbol{Q})\}$. If θ is a parameter along Γ, then $\Gamma(\theta) = (\gamma^-(\theta), \gamma^+(\theta))$, where $\gamma^-(\theta) \mapsto \gamma^+(\theta) = \mathbf{C}(\gamma^-(\theta))$, and we may consider the curves $\gamma^\pm$ as projections of the curve Γ. Going back to the operational meaning of this construction, if $\mathcal{L}^-$ is the lagrangian manifold corresponding to a quantum state $|\psi^-\rangle$, and $\mathbf{C}$ a canonical transformation, then $\mathcal{L}^+ = \mathbf{C}(\mathcal{L}^-)$ can be interpreted as the lagrangian manifold of some $|\psi^+\rangle$ state, and the whole operation corresponds to a unitary quantum operator, $\widehat{U}_{\mathbf{C}} : |\psi^-\rangle \mapsto |\psi^+\rangle$.

Besides portraying the graph of a canonical transformation as a Lagrangian submanifold, the product of a Lagrangian submanifold, $\mathcal{L}^-$ in $\{\mathbf{x}^-\}$ with any another submanifold $\mathcal{L}^+$ in $\{\mathbf{x}^+\}$, $\Lambda = \mathcal{L}^- \times \mathcal{L}^+$, is also Lagrangian in double phase space, but projects singularly onto either of the factor spaces $\{\mathbf{x}^\pm\}$. In the case that both submanifolds are tori, we obtain a double phase space torus, as if we had doubled the number of degrees of freedom. If $N = 1$, it will be a 2-dimensional product torus [12] (taking care with the sign of p^-, in the present construction).

If both Lagrangian submanifolds in single phase space correspond to the same state, i.e. $|\psi^-\rangle = |\psi^+\rangle$, then we represent the corresponding pure state density operator, $\widehat{\rho}_\psi = |\psi\rangle\langle\psi| = |\Psi\rangle\!\rangle$, in the $|\boldsymbol{Q}\rangle\!\rangle$ representation as

$$\langle\!\langle \boldsymbol{Q}|\Psi\rangle\!\rangle = \langle q^+|\psi\rangle\langle\psi|q^-\rangle. \tag{6.4}$$

Therefore, its simplestsemiclassical approximation can be expressed as a superposition of terms of the form

$$\langle\!\langle \boldsymbol{Q}|\Psi\rangle\!\rangle = A_j(\boldsymbol{Q}) \exp[i\mathcal{S}_j(\boldsymbol{Q})/\hbar], \tag{6.5}$$

with

$$\mathcal{S}_j(\boldsymbol{Q}) = \int_{\mathbf{0}}^{\boldsymbol{Q}} \boldsymbol{P}_j(\boldsymbol{Q}') \cdot \mathrm{d}\boldsymbol{Q}' = \int_0^{q_+} p_j^+ \cdot \mathrm{d}q^+ - \int_0^{q_-} p_j^- \cdot \mathrm{d}q^-. \tag{6.6}$$

Again, this is in strict analogy with the construction of semiclassical product states of higher degrees of freedom [12].

The next step is a change of lagrangian coordinates in double phase space:

$$(\boldsymbol{P}, \boldsymbol{Q}) \mapsto (\mathbf{x}, \mathbf{y}) \ , \ \mathbf{x} = \frac{\mathbf{x}^+ + \mathbf{x}^-}{2} \ , \ \mathbf{y} = \mathbf{J}(\mathbf{x}^+ - \mathbf{x}^-) = \mathbf{J}\boldsymbol{\xi}. \tag{6.7}$$

Here, $\mathbf{J}$ is the constant symplectic matrix in single phase space and is used to *canonize* the initial $\pi/4$ rotation on $(\mathbf{x}^-, \mathbf{x}^+)$ that introduces the pair of lagrangian coordinates $(\mathbf{x}, \boldsymbol{\xi})$ on double phase space. Thus, the pair of conjugate variables $(\mathbf{x}, \mathbf{y})$ also accounts for the sign change in the p^- coordinate. We should bear the discomfort that the canonical coordinate in double phase space is $\mathbf{y}$, while the geometrically meaningful variable in single phase space is $\boldsymbol{\xi}$, the trajectory *chord*, which has $\mathbf{x}$ as its *centre*. It is also possible to choose the variable, $\boldsymbol{\xi}$, as the conjugate to $\mathbf{x}$, instead of $\mathbf{y}$, but at the cost of writing the symplectic form on double phase space in a noncanonical way, leading to less familiar expressions for Hamilton's equations and some other elements of the semiclassical theory (see [33] for some of these expressions).

If we consider the *horizontal* Lagrangian subspaces $\mathbf{y} = \textit{constant}$, each is identified with an element of the group of phase space translations, which includes the identity, since the identity subspace is defined as $\boldsymbol{\xi} = 0$. On the other hand, the *vertical* subspace, $\mathbf{x} = \mathbf{0}$, defines the canonical reflection through the origin, $\mathbf{x}^- \mapsto \mathbf{x}^+ = -\mathbf{x}^-$ (or inversion), since all the chords for this transformation are centred on the origin (see [17] or [12] for further discussion.)

We can now, in analogy to (6.6), interpret the centre action $S(\mathbf{x})$ in the semiclassical Wigner function (2.18) as

$$S(\mathbf{x}) = \int^{\mathbf{x}} \mathbf{y}(\mathbf{x}') \cdot \mathrm{d}\mathbf{x}' = \int^{\mathbf{x}} \boldsymbol{\xi}(\mathbf{x}') \wedge \mathrm{d}\mathbf{x}'. \tag{6.8}$$

The integral is evaluated along a path on the Lagrangian submanifold $\Lambda_\psi$in double phase space, corresponding to $\widehat{\rho}_\psi$ in double phase space, from some point on its intersection with the $\mathbf{x}$-plane, which reproduces the single torus $\mathcal{L}_\psi$. This integral is independent of the path on Λ_ψ, because Λ_ψ is Lagrangian. We thus obtain the chord (2.19) by taking the derivative of (6.8).

The chord function is the Fourier transform of $W(\mathbf{x})$. If this transform of the semiclassical Wigner function is performed within the stationary phase approximation, the semiclassical expression for the chord function has a phase, $\sigma(\boldsymbol{\xi})/\hbar$, such that the chord action, $\sigma(\boldsymbol{\xi})$, is the Legendre transform of the centre action, $S(\mathbf{x})$. It can be defined directly in terms of a similar integral to (6.8), with the roles of $\mathbf{x}$ and $\boldsymbol{\xi}$ reversed:

$$\sigma(\boldsymbol{\xi}) = \int_0^{\boldsymbol{\xi}} \mathbf{x}(\boldsymbol{\xi}') \wedge \mathrm{d}\boldsymbol{\xi}' = -\int_0^{\mathbf{J}\boldsymbol{\xi}} \mathbf{x}(\mathbf{y}') \cdot \mathrm{d}\mathbf{y}' = \sigma'(\mathbf{y}). \tag{6.9}$$

The action $\sigma(\boldsymbol{\xi})$ is, of course, the same as appeared in the semiclassical theory for the chord function (2.16). When this theory is transported into double phase space, it is often simpler to deal with $\sigma'(\mathbf{y})$. Then, within this formalism, the semiclassical expression, for (each branch of) the pure state Wigner function or chord function, assumes a generalized WKB form, derived by Van Vleck [34].

So as to treat the unitary evolution of the density operator, which preserves the purity of the state, $|\psi\rangle\langle\psi|$, we need to consider the corresponding classical evolution of both the tips of each chord, $\mathbf{x}^-$ and $\mathbf{x}^+$, lying on a $2N$-dimensional lagrangian torus. Taking account of the sign change of p^-, in the definition of double phase space, we find that the double phase space Hamiltonian must be [15]

$$I\!H_U(X) = H(\mathbf{x}^+) - H(\mathbf{x}^-) = H(\mathbf{x} - \mathbf{J}\mathbf{y}/2) - H(\mathbf{x} + \mathbf{J}\mathbf{y}/2). \tag{6.10}$$

This Hamiltonian dynamics evolves lagrangian submanifolds in double phase space, which correspond to pure density operators that obey the Liouville-Von Neumann equation. The explicit formulae for the semiclassical evolution of the Wigner function are given in [8, 15], whereas the evolving action (3.10) of the chord function is presented in [17]. Reinterpreted as an evolving action in double phase space, (3.10) assumes the form of an ordinary Hamilton-Jacobi equation for $\sigma'(\mathbf{y})$:

$$\frac{\partial \sigma}{\partial t}(\boldsymbol{\xi}, t) = \frac{\partial \sigma'}{\partial t}(\mathbf{y}, t) = I\!H(-\frac{\partial \sigma'}{\partial \mathbf{y}}, \mathbf{y}) = I\!H(\mathbf{J}\frac{\partial \sigma}{\partial \boldsymbol{\xi}}, \mathbf{J}\boldsymbol{\xi}). \tag{6.11}$$

The difficulty lies in the caustics of the initial state, which require more sophisticated semiclassical treatment.

According to the discussion in section 3, the evolution of the amplitudes, in the decomposition of either the Wigner function or the chord function, relies on the previous specification of translated action variables, for the lagrangian manifold, $\mathcal{L}$, corresponding to a semiclassical state. The corresponding lagrangian submanifold is now a product, $\Lambda = \mathcal{L}^- \times \mathcal{L}^+$. Thus, the quantized double torus is defined as the intersection of all the level submanifolds of the $2N$ variables $\mathcal{I}_n^\pm$, defined by (2.20), or

$$\mathcal{I}_n^\pm(\mathbf{X}) = \mathcal{I}_n(\mathbf{x} \mp \mathbf{J}\mathbf{y}). \tag{6.12}$$

The fact that we are dealing with projection operators restricts the Bohr-level for each pair of variables $(\mathcal{I}_n^+(\mathbf{X}), \mathcal{I}_n^-(\mathbf{X}))$ to be the same. ¶

Let us then consider the family of actions, $S(\mathbf{x}, \mathcal{I}^\pm)$ or $\sigma(\boldsymbol{\xi}, \mathcal{I}^\pm)$, evolving classically for all possible constant values of the action variables, $\mathcal{I}^\pm$. This is known as the complete solution of the Hamilton-Jacobi equation [28]. Then, a simple extension to double phase space of the usual canonical formalism implies that

$$|\det \frac{\partial^2 \sigma'(\mathbf{y}, \mathcal{I}^\pm, t)}{\partial \mathbf{y} \partial \mathcal{I}^\pm}| = |\det \frac{\partial \mathcal{I}^\pm}{\partial \mathbf{x}}|^{-1} \quad \text{and} \quad |\det \frac{\partial^2 S(\mathbf{x}, \mathcal{I}^\pm, t)}{\partial \mathbf{x} \partial \mathcal{I}^\pm}| = |\det \frac{\partial \mathcal{I}^\pm}{\partial \mathbf{y}}|^{-1}. \tag{6.13}$$

Combining (6.13) with the expressions (2.21) and (2.22) for the semiclassical amplitudes, leads to

$$a(\mathbf{x}, t) = |\det \frac{\partial^2 S(\mathbf{x}, \mathcal{I}^\pm, t)}{\partial \mathbf{x} \partial \mathcal{I}^\pm}|^{1/2} = |\det \frac{\partial^2 \sigma'(\mathbf{y}, \mathcal{I}^\pm, t)}{\partial \mathbf{y} \partial \mathcal{I}^\pm}|^{1/2} = \alpha'(\mathbf{y}, t) = \alpha(\boldsymbol{\xi}, t), \tag{6.14}$$

where this equality (6.14) between the centre and the chord amplitudes holds only at a specific double phase space point $\mathbf{X} = (\mathbf{x}, \mathbf{y})$ on the double torus Λ, as pointed out in the discussion at the end of section 2. Therefore, the amplitudes of the evolving Wigner and chord functions are entirely determined by the complete solution of the respective Hamilton-Jacobi equations, in full analogy to the ordinary evolution of WKB semiclassical states in the Schrödinger formalism, derived by Van Vleck [34].

It should be pointed out that our use of double phase concerns only the semiclassical approximation to the evolution generated by the Liouville-Von Neumann equation and susequently the full Lindblad equation. At each instant, either the Wigner function, or the chord function are defined in the standard way, as the traces (2.6), or (2.9), in terms of the single phase space of centres, $\mathbf{x}$, or chords $\boldsymbol{\xi}$, respectively. Though it may be tempting to define an enlarged quantum evolution for superoperators in direct correspondence with double phase space, no such generalization is treated here.

Now, we finally turn to the semiclassical theory for markovian evolution, as discussed in the previous section. We immediately recognize in (5.3) the same structure as that of the double phase space Hamiltonian (6.10), that is:

$$\mathbb{L}(\mathbf{X}) = L(\mathbf{x}^+) - L(\mathbf{x}^-) = L(\mathbf{x} - \mathbf{J}\mathbf{y}/2) - L(\mathbf{x} + \mathbf{J}\mathbf{y}/2). \tag{6.15}$$

This double phase space Lindblad function is the basic ingredient in the decoherence functional, which is now defined along a single trajectory in double phase space, generated by $\mathbb{H}_U(\mathbf{X})$:

$$D\{\mathbf{X}(t)\} := \sum_k \int_0^t dt' \, |\mathbb{L}_k(\mathbf{X}(t'))|^2 \, . \tag{6.16}$$

The square root of this functional can now be interpreted as a time dependent *length* of the double phase space vector, $\mathbf{X}(t)$, with $D\{(\mathbf{x}, \mathbf{y} = 0)\} = 0$ for all time, instead of a *distance* between a pair of of single phase space points.

In conclusion, we can interpret the conservative semiclassical evolution of the chord function (5.2) entirely within the double phase space picture. Indeed, this has assumed the same form as general Van Vleck evolution, with the centre variables $\mathbf{x}$ playing the role of positions, while $\mathbf{y} = \mathbf{J}\boldsymbol{\xi}$ stand for the momenta. Following this analogy, the Wigner function substitutes the Schrödinger wave function and the chord function is its Fourier transform.

The only new element that has been added is the action of the decoherence functional: The amplitude of the chord function away from the origin progressively decays in time. This

¶ If this condition is relaxed, the present semiclassical theory is immediately extended to include the propagation of dyadic operators corresponding to different eigenstates of N commuting operators.

quenching of the long chords can only be partially incorporated in the semiclassical Wigner function, obtained from the alternative projection of the same Lagrangian submanifold, $\Lambda(t)$, in the limit of very short times. Indeed, it is only through this submanifold that we can ascribe specific chords to each centre $\mathbf{x}$. For finite times, it is better to calculate the Wigner function as a full convolution, according to the discussion in the previous section.

So far, that is, in the semiclassical theory for quantum unitary evolution, the concept of double phase space may be considered to be somewhat redundant, because everything can be described in terms of pairs of hamiltonian trajectories in single phase space. However, and this is the fundamental point, for dissipative markovian systems, we can identify the dissipative term in the full semiclassical master equation (4.7), that is,

$$I\!H_L(\mathbf{X}) := \sum_k \mathcal{I}m\ L_k(\mathbf{x}_j^+)L_k^*(\mathbf{x}_j^-) = \sum_k \mathcal{I}m\ L_k(\mathbf{x}-\mathbf{J}\mathbf{y}/2)L_k^*(\mathbf{x}+\mathbf{J}\mathbf{y}/2)\ , \tag{6.17}$$

as a new term of the double phase space Hamiltonian. Indeed, the introduction of (3.9), together with the generalized version of (4.7) for several Lindblad operators, into the chord representation of the master equation (1.1) results in the semiclassical evolution equation,

$$\begin{aligned}\frac{\partial\chi_j}{\partial t}(\boldsymbol{\xi},t) = \Big\{ -\frac{i}{\hbar}\Big[H(\mathbf{x}_j^+) - H(\mathbf{x}_j^-) + \sum_k \mathcal{I}m\Big(L_k(\mathbf{x}_j^+)L_k^*(\mathbf{x}_j^-)\Big)\Big] \\ - \frac{1}{2\hbar}\sum_k \Big|L_k(\mathbf{x}_j^+) - L_k(\mathbf{x}_j^-)\Big|^2\Big\}\ \chi_j(\boldsymbol{\xi},t),\end{aligned} \tag{6.18}$$

for each branch of the chord function, recalling that $\mathbf{x}^\pm \equiv \mathbf{x}^\pm(\boldsymbol{\xi}) = \mathbf{x} \pm \boldsymbol{\xi} = \mathbf{x} \mp \mathbf{J}\mathbf{y}$.

In this way, for open dissipative systems, we can consider the total Hamiltonian function on double phase space $\mathbf{R^{2N}} \times \mathbf{R^{2N}}$ as given by

$$I\!H(X) = I\!H_U(X) + I\!H_L(X)\ , \tag{6.19}$$

where $I\!H_U(\mathbf{X})$ and $I\!H_L(\mathbf{X})$ are given respectively by equations (6.10) and (6.17), for functions $H(\mathbf{x})$ and $L_k(\mathbf{x})$ on simple phase space $\mathbf{R^{2N}}$. The particular combination of Lindblad functions, which we have recognized in (6.17) as a new term in the double Hamiltonian, also appears as the integrand of the *phase functional* in Strunz's path inegral [16], but there, in the absense of double phase space, it lacks an interpretation. Indeed, it is the very fact that $I\!H_L(\mathbf{X})$ cannot be related to a Hamiltonian in simple phase space in the same way as $I\!H_U(\mathbf{X})$, which now establishes the double phase space formalism as wholly indispensable. Therefore, for open dissipative systems, a trajectory of the full Hamiltonian $I\!H(\mathbf{X})$ in double phase space is not equivalent to a pair of trajectories of a Hamiltonian in simple phase space, as in the semiclassical theory for closed systems.

7. Dissipative semiclassical evolution

To simplify our study, we here restrict the Lindblad operators to be linear functions of positions and momenta, that is, given by (4.8). The dissipative term (6.17) in the double phase space Hamiltonian is then rewritten as

$$I\!H_L(X) = \gamma\ \mathbf{x}\cdot\mathbf{y} = \gamma\ \boldsymbol{\xi}\wedge\mathbf{x}, \tag{7.1}$$

defining the dissipation coefficient,

$$\gamma = \sum_k \boldsymbol{l}'_k \wedge \boldsymbol{l}''_k. \tag{7.2}$$

The contribution of this term to Hamilton's equations in double phase space is

$$\dot{\mathbf{x}} = -\gamma\ \mathbf{x} \quad , \quad \dot{\mathbf{y}} = \gamma\ \mathbf{y} \quad , \quad (\dot{\boldsymbol{\xi}} = \gamma\ \boldsymbol{\xi}), \tag{7.3}$$

so that the dissipative evolution in double phase space is always hyperbolic: contraction on the lagrangian subspace $\mathbf{R^{2n}} \equiv \{\mathbf{x}\}$ (the identity plane, i.e. the centre phase space) and expansion on its conjugate $\mathbf{R^{2n}} \equiv \{\mathbf{y}\}$ (and hence of the chord space), or vice versa. The fact that these are precisely the subspaces which support the Weyl and the chord representation singles them out as privileged choices for the description of quantum markovian processes.

For the case where $\hat{L}$ is the annihilation operator $\hat{L} = \hat{a}$, then $\boldsymbol{l}' = (0, 2^{-1/2})$ and $\boldsymbol{l}'' = (2^{-1/2}, 0)$, so $\gamma > 0$ and the centre motion is contractive, while the spacing between neighboring chords expands with time. In the case where $\hat{L}$ is the creation operator $\hat{a}^\dagger$, the opposite happens. The optical master equation (1.2) combines both the creation and annihilation operators, but in such a way that $\gamma > 0$, so the centre motion is contractive (strictly dissipative). In the example of a two level atom coupled to a bath of photons in a single field mode, this indicates that, although stimulated emission compensates absorption, spontaneous emission leads to an irreversible loss.

The double phase space formulation maintains the correspondence of the evolving density operator, $\widehat{\rho}(t)$, to a time-dependent lagrangian submanifold, $\Lambda(t)$, even in the presence of dissipation. It is true that, unlike $I\!H_U(\mathbf{X})$, the new dissipative term, $I\!H_L(\mathbf{X})$, of the double phase space hamiltonian destroys the factorization of the initial double lagrangian submanifold into single phase spaces. However, the generalized Van Vleck form of the evolution, which was shown to hold for phases and amplitudes in the previous section, is in no way restricted to product tori. Thus, there is no obstacle to the immediate generalization of the present theory.

The evolving lagrangian submanifold, $\Lambda(t)$, can in principle be described by an action function that measures its symplectic area with respect to any (double phase space) lagrangian coordinate subspace, such as the position subspace, $\boldsymbol{Q} = (q^-, q^+)$, even though we can no longer describe $\Lambda(t)$ locally as $p^\pm = \pm\, \partial s(q,t)/\partial q^\pm$, with the same action function, $s(q)$, for q^- and q^+. Nonetheless, the Weyl and the chord representations, in terms of the double phase space variables $\mathbf{x}$ and $\boldsymbol{\xi} = -\mathbf{J}\mathbf{y}$ respectively, have privileged roles: The former continues to be interpreted as a quasiprobability, whereas the evolution of the latter exhibits decoherence and diffusion in a specially simple form, by merely quenching the amplitude of the various branches of the chord action function (6.9).

The evolution of each branch is then obtained from the same Hamilton-Jacobi equation (6.11) as before, except that now we replace the closed double Hamiltonian (6.10) by the full markovian double Hamiltonian (6.19). Spelled out in terms of the single space functions, for $I\!H_L(X)$ given by (7.1), this becomes

$$\frac{\partial\sigma}{\partial t}(\boldsymbol{\xi},t) = H(\mathbf{J}\frac{\partial\sigma}{\partial\boldsymbol{\xi}} + \frac{\boldsymbol{\xi}}{2}) - H(\mathbf{J}\frac{\partial\sigma}{\partial\boldsymbol{\xi}} - \frac{\boldsymbol{\xi}}{2}) - \gamma\boldsymbol{\xi}\cdot\frac{\partial\sigma}{\partial\boldsymbol{\xi}} \quad . \tag{7.4}$$

Recalling the simple form of the Hamilton-Jacobi equation (6.11) in double phase space, the evolution of the lagrangian submanifold, $\Lambda(t)$, is just given by $\mathbf{x}(\mathbf{y},t) = -\partial\sigma'(\mathbf{y},t)/\partial\mathbf{y}$.

The semiclassical approximation for the evolving chord function is still given by (5.2) for each of the branches of the chord function. The action function evolves according to (7.4), whereas the amplitude of the decoherenceless factor, ${\chi^0}_j(\boldsymbol{\xi},t)$ is specified by (6.14) . In the general case, where the Hamiltonian is not quadratic, the closed evolution of the decoherenceless factor of the chord function in (5.2) is not obtained from the single phase space trajectories, $\boldsymbol{\xi}(-t)$, generated by the classical Hamiltonian, $H(\mathbf{x})$, i.e. ${\chi_j}^0(\boldsymbol{\xi},t) \neq {\chi_j}^0(\boldsymbol{\xi}(-t))$.

Having reinterpreted the decoherence functional (5.3) in double phase space as (6.16), we now obtain the full semiclassical markovian evolution for each branch of the chord function, including dissipation, in the same form (5.2) as before, even though the evolving ingredients can no longer be interpretated in single phase space. Because of the linearity assumed for

all the Lindblad operators (4.8), the decoherence functional (6.16) or (5.3) takes the explicit form,

$$D\{\mathbf{x}_j^+(t),\mathbf{x}_j^-(t)\}=\sum_k\int_0^t dt'\Big[|\boldsymbol{l}'_k\cdot\overline{\boldsymbol{\xi}}_j(\boldsymbol{\xi},t')|^2+|\boldsymbol{l}''_k\cdot\overline{\boldsymbol{\xi}}_j(\boldsymbol{\xi},t')|^2\Big], \tag{7.5}$$

for each branch of the chord function. It is the classical evolution of the full double phase space vector, $\mathbf{X}_j=(\mathbf{x}_j^-(\boldsymbol{\xi}),\ \mathbf{x}_j^+(\boldsymbol{\xi}))$, or, in alternative coordinates, $\mathbf{X}_j=(\mathbf{x}_j(\boldsymbol{\xi}),\ \mathbf{y}=\mathbf{J}\boldsymbol{\xi})$, that determines the decoherence functional, i.e. $\overline{\boldsymbol{\xi}}_j(\boldsymbol{\xi},-t')$ is obtained by multiplying the $\mathbf{y}$-component of $\mathbf{X}_j(t)$ by $-\mathbf{J}$.

It is only the further restriction to a quadratic Hamiltonian that forces all the chord projections in double phase space to evolve in the same way, independently of each centre, $\mathbf{x}_j(\boldsymbol{\xi})$. Then the evolution of all the semiclassical branches can again be united into (5.6), with the only difference that now $\boldsymbol{\xi}(t)$ is obtained from the expansive linear equation [10]:

$$\dot{\boldsymbol{\xi}}=\mathbf{J}\frac{\partial H}{\partial\boldsymbol{\xi}}+\gamma\,\boldsymbol{\xi}. \tag{7.6}$$

This is just one of Hamilton's equations for the full double phase space Hamiltonian (6.19). Since (5.6) coincides with the exact solution of the Lindblad equation for quadratic $H(\mathbf{x})$ and linear $L(\mathbf{x})$, it follows that the present semiclassical theory is exact in this limit, even in the presence of dissipation.

There is a subtle distinction to be noted in the derivation of the semiclassical evolution of a dissipative Markovian system, with respect to the theory in section 5. The approximation there could be interpreted as the short time limit of a conceivable periodic system, wherein the internal Hamiltonian and the Lindblad operators were alternatively turned on and off. The full dissipative approximation is now derived in the same way, if we include the new dissipative part in the double Hamiltonian. However, this same mathematical procedure is now devoid of its interpretation as a conceivable periodic system, because we cannot physically switch off the decohering part of the Lindblad equation without eliminating the dissipative part of the Hamitonian: It is possible to have decoherence without dissipation, but not the other way around. In any case, the dissipative approximation is also exact in the quadratic limit.

The evolution of the semiclassical Wigner function now follows through the derivation in section 5: The Fourier transform of (5.2) will be a convolution of ${W_j}^0(\mathbf{x},t)$, the Fourier transform of ${\chi_j}^0(\boldsymbol{\xi},t)$, with a widening window which coarse-grains over the interferences of the Wigner function. Beyond the decoherence time, the Fourier transform of the chord function, i.e. the Wigner function becomes smooth and classical-like. From then on, the classical though diffusive motion on this centre space is given by Hamilton's (single phase space) equations, with the addition of the purely dissipative term (7.3):

$$\dot{\mathbf{x}}=\mathbf{J}\frac{\partial H}{\partial\mathbf{x}}-\gamma\mathbf{x}. \tag{7.7}$$

The reason is that this is an invariant subspace for unitary double phase evolution [17], a property which is not altered by dissipation. Since the decoherence functional has, at this stage, effectively cancelled all large chords, the physical interest is concentrated on this plane. In this regime the evolution of the decoherenceless factor of the Wigner function in (5.7), can be pictured as purely classical, ${W^0}_j(\mathbf{x},t)={W^0}_j(\mathbf{x}(-t))$, with $\mathbf{x}(-t)$ obtained from (7.7).

It might seem strange that the semiclassical solution of the dissipative master equation for the chord function becomes exact in the case of linear Lindblad phase space functions and a quadratic Hamiltonian . The latter condition is familiar on its own, but the exact Lindblad term (4.10) is not in the same form as the semiclassical approximation (4.9), so that it is not

evident that the same limiting behaviour is obtained. However, let us, in this case, reinterpret the discrepant term as part of a quantized Hamiltonian superoperator for a Schrödinger-like equation corresponding to double phase space. Then $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ will be operators corresponding respectively to positions and momenta in double phase space, such that $\widehat{\mathbf{x}} = i\hbar\frac{\partial}{\partial \mathbf{y}}$. It follows that $\chi(-\mathbf{J}\mathbf{y})$ can then be interpreted as a *double wave function*, so that

$$(\boldsymbol{l}' \wedge \boldsymbol{l}'')\ \boldsymbol{\xi} \cdot \frac{\partial \chi}{\partial \boldsymbol{\xi}} = (\boldsymbol{l}' \wedge \boldsymbol{l}'')\ \mathbf{y} \cdot \frac{\partial}{\partial \mathbf{y}}\ \ \chi(-\mathbf{J}\mathbf{y}) = \frac{i}{\hbar}(\boldsymbol{l}' \wedge \boldsymbol{l}'')\ \widehat{\mathbf{y}} \cdot \widehat{\mathbf{x}}\ \ \chi(-\mathbf{J}\mathbf{y}), \tag{7.8}$$

and the action of this Hamiltonian superoperator on a semiclassical branch of the chord function, to first order in $\hbar$,

$$\frac{i}{\hbar}(\boldsymbol{l}' \wedge \boldsymbol{l}'')\ \widehat{\mathbf{y}} \cdot \widehat{\mathbf{x}}\ \ \chi_j(-\mathbf{J}\mathbf{y}) = (\boldsymbol{l}' \wedge \boldsymbol{l}'')\ \frac{\partial \sigma_j}{\partial \boldsymbol{\xi}} \wedge \boldsymbol{\xi}\ \ \chi_j(\boldsymbol{\xi}), \tag{7.9}$$

is the same as in (4.9).

8. Semiclassical markovian propagators

The present semiclassical picture for markovian evolution of the density operator has dealt directly with the chord function and its Fourier transform, the Wigner function. The problem that must now be addressed is that, whereas the decoherence functional quickly quenches the contribution of large chords, the remaining *classical* small chord region lies in the neighborhood of a caustic for both the chord function and the Wigner function. In other words, the region of the lagrangian submanifold, $\Lambda(t)$, lying close to the centre subspace $\mathbf{y} = 0$ in double phase space, projects singularly onto both the $\mathbf{x}$-subspace and the $\mathbf{y}$-subspace. Hence, the direct semiclassical theory above is only applicable to the initial stages of markovian evolution, before the long chords are mostly quenched. [+] Though the further processes of dissipation and diffusion proceed continuously through the initial stage and beyond, their direct description require an advanced semiclassical treatment of markovian evolution in the caustic region, beyond the scope of the present theory.

One way out of this problem is to consider alternative lagrangian submanifolds, which do not have caustics, at least initially. This is the approach adopted in [17]. Instead of the submanifold $\Lambda(t)$, which corresponds to $\widehat{\rho}(t)$, but has undesirable caustics, we evolve the submanifolds $\mathbf{x} = \textit{constant}$, or $\mathbf{y} = \textit{constant}$, corresponding respectively to unitary reflection, or translation operators, $\widehat{R}_{\mathbf{x}}$, or $\widehat{T}_{\boldsymbol{\xi}}$. The $\mathbf{x} = \textit{constant}$ submanifold has no caustic in its $\mathbf{y}$-projection, while the $\mathbf{y} = \textit{constant}$ submanifold is free from caustics in its $\mathbf{x}$-projection. Furthermore, a finite time must pass before the evolution generated by the double Hamiltonian can bend either of these submanifolds sufficiently to produce caustics, until which time both the chord representation of $\widehat{R}_{\mathbf{x}}(t)$ and the centre representation of $\widehat{T}_{\boldsymbol{\xi}}(t)$ will be represented in the simple semiclassical forms (2.18) and (2.16). In other words, $\tilde{R}_{\mathbf{x}}(\boldsymbol{\xi}, t)$, the chord representation of the reflection operator, and $T_{\boldsymbol{\xi}}(\mathbf{x}, t)$, the centre representation of the translation operator, respectively, will both have a single semiclassical branch.

In this way we obtain a semiclassical approximation to the evolution, whether unitary or markovian, by inserting the approximate evolved operators in the exact relations [17],

$$W(\mathbf{x}, t) = \int \frac{\mathrm{d}\boldsymbol{\xi}}{(2\pi\hbar)^L}\ \chi(\boldsymbol{\xi})\ T_{\boldsymbol{\xi}}(\mathbf{x}, t), \tag{8.1}$$

[+] On the other hand, it is the essentially nontransversal intersection of the double torus, $\Lambda = \mathcal{L}(\mathbf{x}^-) \times \mathcal{L}(\mathbf{x}^+)$, with the central subspace, along $\mathcal{L}(\mathbf{x})$, that allows for a nontrivial Wigner function even after decoherence, such as is observed in the quadratic limit. (The single torus, $\mathcal{L}(\mathbf{x})$, has the same form as the factor tori in the respective spaces, $\mathbf{x}^{\pm}$.)

or

$$\chi(\boldsymbol{\xi},t) = \int \frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^L}\, W(\mathbf{x})\, 2^N\, \tilde{R}_{\mathbf{x}}(\boldsymbol{\xi},t). \tag{8.2}$$

Unlike these *mixed* propagators, which involve both centres and chords, the semiclassical expressions for *direct* propagators of Wigner functions [35] necessarily involve uniform approximations through caustics, even in the simple case of unitary evolution.

In the present context, it is (8.2) that should be chosen, because the markovian evolution of a reflection operator, initially represented by

$$2^N\, \tilde{R}_{\mathbf{x}}(\boldsymbol{\xi}) = \exp\Big(\frac{i}{\hbar}\mathbf{x}\wedge\boldsymbol{\xi}\Big), \tag{8.3}$$

is approximated semiclassically within the chord representation in the same way as a single branch of the chord function, $\chi(\boldsymbol{\xi})$ in (2.16). This holds for arbitrary Hamiltonians, together with linear Lindblad operators, self-adjoint or not. Thus, the approximate evolution of (8.3) has precisely the same form as (5.2), i.e.

$$\tilde{R}_{\mathbf{x}}(\boldsymbol{\xi},t) = \tilde{R}^0_{\mathbf{x}}(\boldsymbol{\xi},t)\ \ \exp\Big\{-\frac{1}{2\hbar}\sum_k \int_0^t dt' |\boldsymbol{l}_k\cdot\overline{\boldsymbol{\xi}}(-t')|^2\Big\}, \tag{8.4}$$

where $\overline{\boldsymbol{\xi}}(t)$ is the trajectory of the initial chord, $\overline{\boldsymbol{\xi}}(0)=\boldsymbol{\xi}$, obtained from the $\mathbf{y}$ component of the evolving double phase space vector, $\mathbf{X}=(\mathbf{x},\ \mathbf{y}=\mathbf{J}\boldsymbol{\xi})$, and the Lindblad coefficients (4.8) are in general complex. Here, the decoherenceless factor is

$$\tilde{R}^0_{\mathbf{x}}(\boldsymbol{\xi},t) = \tilde{R}^0_{\mathbf{x}}(-\mathbf{J}\mathbf{y},t) = \frac{1}{2^N}\Big|\frac{\partial^2\sigma'_{\mathbf{x}}(\mathbf{y},t)}{\partial\mathbf{y}\partial\mathbf{x}}\Big|^{1/2}\exp\Big(\frac{i}{\hbar}\sigma'_{\mathbf{x}}(\mathbf{y},t)\Big). \tag{8.5}$$

The evolution of $\tilde{R}^0_{\mathbf{x}}(\boldsymbol{\xi},t)$ does not coincide with the unitary evolution presented in [17], because of the dissipative term in the Hamiltonian. In short, we merely substitute the transported action variables $\mathcal{I}^{\pm}$, that previously defined the evolving lagrangian surface in double phase space, by $\mathbf{x}$ in (6.14) and the chord action, $\sigma_{\mathbf{x}}(\boldsymbol{\xi},t)=\sigma'_{\mathbf{x}}(\mathbf{y},t)$ is governed by the Hamilton-Jacobi equation (7.4). Evidently, the initial action is $\sigma'_{\mathbf{x}}(\mathbf{y},0)=\mathbf{x}\cdot\mathbf{y}$, so that the initial semiclassical expression coincides with (8.3). It should be noted that, at the chord origin, $\tilde{R}^0_{\mathbf{x}}(\boldsymbol{\xi},t)=2^{-N}$ for all times, and multiplication by the exponential of the decoherence functional does not alter this value. Therefore, normalization is preserved, according to (2.14).

The conditions for the derivation of (8.4) must now be analyzed. Recall that the exponential of the decoherence functional was assumed to be a smooth (non-oscillatory) function in our previous derivation of the semiclassical chord function (5.2). For small chords, which are now in focus, this smoothness assumption is falsified as the decoherence time is reached, i.e. the time for the volume of the decoherence factor to shrink to that of a coherent state. This establishes the duration beyond which this chord propagator is valid. Nonetheless, the evolved propagator can be reexpressed in terms of (static) reflection operators, i.e. in the Weyl representation and this whole procedure can then be iterated indefinitely.

It is precisely in this region, where the direct semiclassical approximation for the chord function (5.2) is singular, that the much simpler form,

$$2^N\, \tilde{R}^0_{\mathbf{x}}(\boldsymbol{\xi},t) = \exp\Big(\frac{i}{\hbar}\mathbf{x}(t)\wedge\boldsymbol{\xi}\Big), \tag{8.6}$$

can be employed [17]. Here, $\mathbf{x}(t)$ is the trajectory, issuing from $\mathbf{x}(0)=\mathbf{x}$, that integrates (7.7) in single phase space, because of the restriction to the invariant subspace $\mathbf{y}=0$. This approximation implies that the evolved reflection operator is still represented by a vertical subspace in double phase space. This is indeed true for motion generated by a quadratic

Hamiltonian and even a general closed double Hamiltonian, $I\!H_U(X)$, of the form (6.10), leads to a submanifold whose tangent space is vertical at the identity subspace, throughout the motion, as discussed in [17]. Fortunately, the new open dissipative term, $I\!H_L(X)$, of the double Hamiltonian (7.1) preserves this feature.

To evaluate the corresponding approximation for the decoherence functional, we expand the double Hamiltonian (6.19) as

$$I\!H(\mathbf{x},\mathbf{y}) = \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}) \wedge \mathbf{y} + \gamma\, \mathbf{x}\cdot\mathbf{y}, \tag{8.7}$$

which holds to second order in $\mathbf{y}$. Then, Hamilton's equation for the $\mathbf{y}$-motion in the neighborhood of the invariant centre subspace is just

$$\dot{\mathbf{y}}(t) \;=\; [2\mathbf{J}\mathbf{H}_2(\mathbf{x}(t)) + \gamma]\,\mathbf{y}, \tag{8.8}$$

where the independent centre motion $\mathbf{x}(t)$ is determined by the non-hamiltonian classical equation (7.7) and $\mathbf{H}_2(\mathbf{x})$ is the Hessian matrix for $H(\mathbf{x}(t))$. The quadratic form $\mathbf{y}\cdot\mathbf{H}_2(\mathbf{x}(t))\mathbf{y}$ can be interpreted as a local Hamiltonian for the motion transverse to the centre subspace, though it receives an extra boost from the dissipation coefficient, γ. Given an initial point $\mathbf{X} = (\mathbf{x},\ \mathbf{y}=\mathbf{J}\boldsymbol{\xi})$, whose orbit is assumed to remain close to the identity subspace within the time t, we thus obtain its chord evolution as

$$\boldsymbol{\xi}_t(\mathbf{x},\boldsymbol{\xi}) = \mathbf{G}_t(\mathbf{x})\,\boldsymbol{\xi}, \tag{8.9}$$

where the classical propagation matrix is

$$\mathbf{G}_t(\mathbf{x}) := \lim_{N\to\infty} \prod_n^N \exp\Big[\frac{t}{N}[2\mathbf{J}\mathbf{H}_2(\mathbf{x}(\frac{nt}{N})) + \gamma]\Big], \tag{8.10}$$

which, for small times is approximately

$$\mathbf{G}_t(\mathbf{x}) \approx\; \exp\Big[\int_0^t \mathrm{d}t'\ (2\mathbf{J}\mathbf{H}_2(\mathbf{x}(t')) + \gamma)\Big]. \tag{8.11}$$

The only difference between this approximation and the exact evolution for the quadratic case in [10] is the $\mathbf{x}$-dependence for $\mathbf{H}_2$ and, hence, the need for an integral in the definition of $\mathbf{G}_t$. Furthermore, the present definition incorporates the dissipation coefficient, γ, from the full double Hamiltonian (6.19).

We are dealing here with trajectories that remain very close to the invariant centre subspace, where $D\{\mathbf{x}(t),\ \mathbf{y}=0\} = 0$. In other words, within the short chord approximation, the pair of trajectories, which define the decoherence functional, become indistinguishable from a single classical trajectory, so that

$$D\{\mathbf{x}(t), \mathbf{y}\to 0\} = \frac{\boldsymbol{\xi}\cdot\mathbf{M}_t(\mathbf{x})\boldsymbol{\xi}}{2}, \tag{8.12}$$

a simple quadratic form in the variables transverse to this centre subspace, specified by the evolving matrix,

$$\mathbf{M}_t(\mathbf{x}) = \sum_k \int_0^t \mathrm{d}t'\ \mathbf{G}_{t'}^T(\mathbf{x})\mathbf{l}_k\mathbf{l}_k^T\mathbf{G}_{t'}(\mathbf{x}). \tag{8.13}$$

Here $(.)^T$ denotes the transpose of a matrix, or a vector. This quenching exponent has exactly the same form as the exact one for quadratic Hamiltonians [10] along the centre subspace. The only difference lies in the dependence of the classical chord propagator $\mathbf{G}_t(\mathbf{x})$ on the local quadratic approximations of the Hamiltonian along the trajectory $\mathbf{x}(t)$, which can be taken

either in the forward direction, starting at $\mathbf{x}$, or backwards from $\mathbf{x}(t)$. In conclusion, the small chord approximation for the mixed propagator is

$$2^N\ \tilde{R}_{\mathbf{x}}(\boldsymbol{\xi},t) = \exp\Big(\frac{i}{\hbar}\mathbf{x}(t)\wedge\boldsymbol{\xi}\Big)\ \exp\Big(-\frac{\boldsymbol{\xi}\cdot\mathbf{M}_t(\mathbf{x})\boldsymbol{\xi}}{2\hbar}\Big). \tag{8.14}$$

This can now be inserted into (8.2), to obtain

$$\chi(\boldsymbol{\xi},t) = \int\frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^L}\,W(\mathbf{x})\ \exp\Big(\frac{i}{\hbar}\mathbf{x}(t)\wedge\boldsymbol{\xi}\Big)\exp\Big(-\frac{\boldsymbol{\xi}\cdot\mathbf{M}_t(\mathbf{x})\boldsymbol{\xi}}{2\hbar}\Big). \tag{8.15}$$

Here, we cannot use the semiclassical approximation (2.18) for the initial Wigner function, $W(\mathbf{x})$, in the above integral, because it is singular in the small chord region, but improved uniform approximations [24, 36] are valid in this range. A correct initial normalization of the Wigner funtion guarantees the normalization of the chord function for any subsequent time, according to (2.14).

It is now worthwhile to review the full construction required to follow the markovian evolution of a given initial pure state. By inserting its Wigner function into (8.2) along with the semiclassically evolved propagator, specified by (8.4) and (8.5), we obtain the evolution of the chord function for a time interval of the order of the decoherence time. The amplitudes of all long chords will be strongly quenched in this evolved chord function, so that its Fourier transform, the evolved Wigner function, becomes smooth and positive. This can be further evolved by inserting it into (8.2) again. But now there will be no large chord contribution, even right from the start, because at $t = 0$, equation (8.2) reduces to the inverse of the Fourier transform that we have just made. Therefore, the much simpler evolution given by the small chord approximation (8.15) is now adequate. The markovian evolution never *dequenches* the large chord amplitudes, so that all further iterations of this procedure may safely rely on the small chord approximation.

Let us now describe the evolutions that can be calculated within the small chord approximation entirely within the Weyl representation. In the limit of small propagation time, $\mathbf{M}_t \to 0$, the Fourier transform of the evolving chord function is exactly $W(\mathbf{x}',t) = W(\mathbf{x}(-t))$, the classical propagation of the Wigner function. This is the limit where the Wigner-Wigner propagator becomes merely $\delta(\mathbf{x}' - \mathbf{x}(t))$. Unless the Hamiltonian is quadratic, it is only valid because the long chords that have already been quenched out of the propagator. For longer times, it is also possible to perform the Fourier transformation of (8.15) exactly, so as to obtain the Wigner function as an evolving convolution with a gaussian that broadens from an initial Dirac δ-function. Following [10], this is given by

$$W(\mathbf{x}',t) = \int\frac{\mathrm{d}\mathbf{x}}{(2\pi\hbar)^L}\,\frac{W(\mathbf{x})}{\sqrt{\det\mathbf{M}_t(\mathbf{x})}}\ \exp\Big(-\frac{(\mathbf{x}-\mathbf{x}'(-t))\cdot\mathbf{M}'_t(\mathbf{x})(\mathbf{x}-\mathbf{x}'(-t))}{2\hbar}\Big), \tag{8.16}$$

in which $\mathbf{x}'(-t)$ is the backward trajectory of the evaluation point, $\mathbf{x}'$. It is important to emphasize here, once again, that this is a trajectory not of the single Hamiltonian $H(\mathbf{x})$, but of equation (7.7), which adds the dissipative term, $-\gamma\mathbf{x}$, to Hamilton's equation for the single Hamiltonian $H(\mathbf{x})$. The matrix

$$M'_t(\mathbf{x}) := -\mathbf{J}\mathbf{M}_t(\mathbf{x})^{-1}\mathbf{J} \tag{8.17}$$

characterizes the Fourier transform of the exponential of the decoherence functional in (8.15). This is now a broadening gaussian window, which coarse-grains the classical evolution of the initial Wigner function. The same coarse-graining, which accounted for the initial loss of quantum coherence of an initial pure state, can now be interpreted as resulting from a classical Langevin equation for Brownian motion acting on a purely classical probability distribution in phase space [10].

It is tempting to extrapolate the approximate small chord evolution (8.15) beyond the decoherence time. After all, the effect of this process is precisely to eliminate the contribution of long chords, so that the passage to (8.16) becomes more valid. Indeed, there would be no obvious contradiction if the validity of (8.15) were to extend beyond the range allowed by our derivation, as is true in the quadratic case. If we do insert (8.14) into (4.4), the integral to be evaluated becomes

$$(\hat{L}\widehat{R}_{\mathbf{x}}\hat{L}^{\dagger} - \frac{1}{2}\hat{L}^{\dagger}\hat{L}\widehat{R}_{\mathbf{x}} - \frac{1}{2}\widehat{R}_{\mathbf{x}}\hat{L}^{\dagger}\hat{L})(\boldsymbol{\xi}') =$$
$$\int \frac{d\boldsymbol{\xi}''d\mathbf{x}''}{(2\pi\hbar)^N} \exp\left(\frac{i}{\hbar}\mathbf{x}(t)\wedge\boldsymbol{\xi}''\right) \exp\Big(-\frac{\boldsymbol{\xi}''\cdot\mathbf{M}_t(\mathbf{x})\boldsymbol{\xi}''}{2\hbar}\Big)\exp\Big(\frac{i}{\hbar}[\mathbf{x}''\wedge(\boldsymbol{\xi}'-\boldsymbol{\xi}'')]\Big)$$
$$\Big\{L(\mathbf{x}''+\frac{\boldsymbol{\xi}'}{2})L^*(\mathbf{x}''-\frac{\boldsymbol{\xi}''}{2})-\frac{1}{2}[L(\mathbf{x}''+\frac{\boldsymbol{\xi}''}{2})L^*(\mathbf{x}''+\frac{\boldsymbol{\xi}'}{2})+L(\mathbf{x}''-\frac{\boldsymbol{\xi}'}{2})L^*(\mathbf{x}''-\frac{\boldsymbol{\xi}''}{2})]\Big\}. \tag{8.18}$$

Recalling that the Lindblad functions $L(\mathbf{x})$ are assumed linear, we can now substitute $\mathbf{x}''$ by $i\hbar\mathbf{J}\partial/\partial\boldsymbol{\xi}'$ within the brackets $\{\}$ in (8.18) and then integrate over this variable to obtain $\delta(\boldsymbol{\xi}'-\boldsymbol{\xi}'')$. Thus, in this case, the integral of (8.18) becomes

$$(\hat{L}\widehat{R}_{\mathbf{x}}\hat{L}^{\dagger} - \frac{1}{2}\hat{L}^{\dagger}\hat{L}\widehat{R}_{\mathbf{x}} - \frac{1}{2}\widehat{R}_{\mathbf{x}}\hat{L}^{\dagger}\hat{L})(\boldsymbol{\xi}') =$$
$$\Big(-i(\boldsymbol{l}'\wedge\boldsymbol{l}'')[\mathbf{x}(t)\wedge\boldsymbol{\xi}'-\boldsymbol{\xi}'\cdot\mathbf{M}_t(\mathbf{x})\boldsymbol{\xi}'] - \frac{1}{2}[(\boldsymbol{l}'\cdot\boldsymbol{\xi}')^2+(\boldsymbol{l}''\cdot\boldsymbol{\xi}')^2]\Big)\, R_{\mathbf{x}}(\boldsymbol{\xi}',t), \tag{8.19}$$

if we neglect semiclassically small terms arising from the derivatives of the Lindblad functions.

This is almost in the same form as (4.9), except for the term $i\gamma\ \boldsymbol{\xi}'\cdot\mathbf{M}_t(\mathbf{x})\boldsymbol{\xi}'$, which is not semiclassically small. Nonetheless, this term does disappear in the nondissipative case, so allowing us to carry through an analogous derivation to that of the semiclassical chord function in section 5. The result is that in this case the small chord approximation (8.15) is valid and the closed formula for long term evolution of the Wigner function (8.16) can be legitimately extrapolated far beyond the initial period for decoherence. In the general dissipative case with nonquadratic Hamiltonians, the qualitative picture of the Markovian evolution, resulting from repeated iterations of (8.16) is the same, but a single closed formula is not yet available.

9. Discussions

The point of view of this paper is that semiclassical Wigner functions or chord functions stand to double phase space as do semiclassical position and momentum wave functions to single phase space. In analogy to the more familiar theory, each of these conjugate representations is defined in its own subspace and contains complete information concerning the quantum state, be it pure or mixed. However, mixed systems demand a density operator description, rather than as states in Hilbert space, which can be provided by the Wigner function or the chord function. For strictly unitary evolution, it is still possible to restrict consideration to the single phase spaces on which these functions are originally defined, but not for general quantum markovian processes.

No attempt has here been made to expand the framework of quantum mechanics itself. Both the Wigner function and the chord function are particular choices of representation for the evolving density operator, in simple phase space. It is only the identification of the various terms in these representations of the Lindblad equation with those of standard semiclassical wave function evolution, that leads to a generalized WKB-like solution for quantum markovian motion, in double phase space.

The full double phase space Hamiltonian would have an unfamiliar form if it were to be considered as the generator of motion for a mechanical system in an ordinary multidimensional

phase space, but, once these peculiarities are understood, there results a qualitative picture for generalized semiclassical evolution that is pleasingly intuitive: The decoherence functional quickly quenches the contribution of all large chords, just as in the exact quadratic case [10]. Hence, after a short *decoherence time*, we may restrict the analysis to the neighbourhood of the centre subspace $\mathbf{y} = 0$ in double phase space. The fact that this is the subspace, where the Wigner function is supported, indicates that the conjugate pair of the Weyl and the chord representations constitute a privileged frame for the study of markovian evolution for quantum systems.

The various semiclassical representations of the density operator are derived from a single Lagrangian submanifold in double phase space, with its hamiltonian evolution. It so happens that the product nature of this manifold implies that its intersection with the identity subspace coincides with a caustic for both the Wigner function and the chord function. We have shown in section 8 that a simple adaptation of the present theory, obtained by choosing a set of more favourable lagrangian submanifolds, furnishes an optimum semiclassical propagator. This allows for appropriate small chord approximations, leading back to a closed formula for the evolution of the Wigner function (8.16). Thus, we again establish contact with the exact markovian theory for quadratic Hamiltonians.

Notwithstanding that the evolution of our propagator is derived in the semiclassical approximation, it will transport any kind of pure or mixed Wigner function. These may be initial semiclassical states, but also (squeezed) coherent states, *Schrödinger cat states*, or whatever. Though the analysis is harder in the intermediate stage, between the initial loss of quantum coherence and asymptotic classical motion, it can be conjectured that the Wigner function becomes positive everywhere, even if it is doubtful that the time for this will be independent of the initial state, as in the quadratic case [32, 10].

The fact that the present semiclassical theory is exact in the case of a quadratic internal Hamiltonian, even in the presence of linear Lindblad operators, can be considered as an indication that it provides a useful generalization of this simple case. It will certainly be necessary to make detailed comparisons of the approximate semiclassical evolution to the direct integration of the exact equation in the case of non quadratic Hamiltonians. However, the integration of the multidimensional partial differential master equation is a considerable enterprise, specially for the highly oscillatory Wigner functions described by semiclassical theory. An alternative is to resort to quantum monte carlo methods, such as in [3], but then the comparison is merely between alternative approximations.

Our analysis has dealt only with Lindlad operators that are linear functions of position and momentum operators. This may be justified by picturing these operators as the quantum variables that are responsible for the coupling to the environment, as in the derivations of our standard example (1.2). Weak coupling, as assumed for a markovian theory, often implies that a linear approximation is valid, but this need not be so. Perhaps it has been the very difficulty of obtaining a fully reliable general picture of the evolution of the density operator of an open system that has so far hampered the study of systems described by nonlinear Lindblad operators. Further generalization of the present theory to include this possibility is the subject of ongoing work.

It is early to predict whether the insight provided within a full semiclassical theory of non-unitary evolution will reveal features which are beyond our present intuition. This can only be decided by analyzing examples of increasing complexity within the present framework, or by answering harder mathematical questions of the formalism itself (see [37], for instance), or perhaps both. For the present, we have gained in understanding how dissipation and diffusion are emmerging properties of the single continuous markovian evolution in double phase space,

which, in its initial stages, is best described as decoherence. It is only when the contribution of long chords is quenched onto the neighbourhood of the centre subspace that the motion can meaningfully be described as dissipative.

Acknowledgments

Partial financial support from Millenium Institute of Quantum Information, FAPERJ, PROSUL, CNPq and UNESCO/IBSP Project 3-BR-06 is gratefully acknowledged.

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