

On the propagation of semiclassical Wigner functions

P P de M Rios^{1,3} and A M Ozorio de Almeida^{2,4}

¹ Laboratório Nacional de Computação Científica, Av. Getúlio Vargas 333, Petrópolis, RJ, 25651-070, Brazil

² Centro Brasileiro de Pesquisas Físicas, Rua Dr Xavier Sigaud 150, Urca, Rio de Janeiro, RJ, 22290-180, Brazil

E-mail: prios@math.berkeley.edu, ozorio@cbpf.br and ozorio@mpipks-dresden.mpg.de

Received 2 November 2001, in final form 25 January 2002

Published 8 March 2002

Online at stacks.iop.org/JPhysA/35/2609

Abstract

We establish the difference between the propagation of semiclassical Wigner functions and classical Liouville propagation. First we rediscuss the semiclassical limit for the propagator of Wigner functions, which on its own leads to their classical propagation. Then, via stationary phase evaluation of the full integral evolution equation, using the semiclassical expressions of Wigner functions, we provide the correct geometrical prescription for their semiclassical propagation. This is determined by the classical trajectories of the tips of the chords defined by the initial semiclassical Wigner function and centred on their arguments, in contrast to the Liouville propagation which is determined by the classical trajectories of the arguments themselves.

PACS numbers: 03.65.Sq, 03.65.–w

1. Introduction

The Wigner [13] function W is a representation of the quantum density operator $\hat{\rho}$ as a real function on phase space, for a nonrelativistic dynamical system with classical–quantum correspondence and n degrees of freedom. Here, all the underlying geometry is Euclidean and we can include the Wigner function within the general framework of Weyl representation of operators: $\hat{A} \rightarrow A \in C_{\mathbb{R}}^k(\mathbb{R}^{2n})$, so $W(x) = (2\pi\hbar)^{-n} \rho(x)$, where $x \equiv (p_1, \dots, p_n, q_1, \dots, q_n)$ denotes a point in phase space. In this context, the quantum evolution of the density operator $\hat{\rho}_t$ effected by an autonomous external Hamiltonian \hat{H} can be written in terms of its Weyl [12] representation as

$$\partial W(x, t)/\partial t = \{H, W_t\}(x) + O(\hbar^2) \quad (1)$$

where $W_t(x) \equiv W(x, t)$, H is the Weyl symbol of \hat{H} and $\{, \}$ is the classical Poisson bracket.

³ Present address: Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA.

⁴ Present address: Max Planck Institute for Physics of Complex Systems, Noethnitzer Str. 38, 0118, Dresden, Germany.

Even though the Wigner function W is generally nonpositive and thus cannot be taken as a classical probability density (in contrast to its Lagrangian averages $\int W(p, q) dp$, etc), equation (1) is the source of a somewhat widespread belief that the propagation of $W_t(x)$ coincides in the semiclassical limit ($\hbar \rightarrow 0^+$) with the classical ($\hbar = 0$) propagation of a Liouville probability density. In fact, for a quadratic Hamiltonian $\hat{H}^{(2)}$ the semiclassical (exact) propagation of $W_t(x)$ is indeed classical. That is, if $x_0 \rightarrow x_t$ is the classical Hamiltonian flow generated by $H^{(2)}$, then

$$W_0(x_0) \rightarrow W_t(x_t) = W_0(x_0). \quad (2)$$

For a general Hamiltonian \hat{H} (whose Weyl symbol H does not necessarily coincide with the classical Hamiltonian h) the classical propagation (2) induced by (1) remains a good approximation (semiclassically correct) if W is a fairly smooth function (as with the Weyl representation of an observable), in which case the corrections to (1) can be semiclassically ignored. Such a smoothness condition can be realized for highly mixed statistical states, in which case W will look very much like a classical probability density in phase space and not surprisingly its propagation will be nearly classical.

On the other hand, pure states are not in general represented by smooth Wigner functions. Indeed, the semiclassical limit \mathcal{W} for the Wigner function of a pure state Ψ in one degree of freedom [1] is

$$\mathcal{W}(x) \approx \mathcal{A}_\psi(x) \cos(S_\psi(x)/\hbar - \pi/4) \quad (3)$$

where $S_\psi(x)$ is the symplectic area between an arc of a (Bohr–Sommerfeld quantized) curve ψ and its corresponding chord centred at x , and \mathcal{A} is a smooth amplitude function. Such an expression for \mathcal{W} is highly oscillatory, especially at this semiclassical limit, and its successive derivatives knock off all the favourable powers of \hbar in the ‘corrections’ to (1). The important exception is when x lies very close to ψ in which case $S_\psi(x)$ is very close to zero and constant, and $\mathcal{W}(x)$ is locally smooth. However, the regions inside a closed leaf ψ , where \mathcal{W} is highly oscillatory, are of utmost importance for they neatly exhibit the nonpositive and thus nonclassical aspect of W which accounts for quantum interference and coherence phenomena. Thus, (2) provides an inadequate description of the propagation of Wigner functions at the semiclassical level.

Most of these problems with the propagation of semiclassical Wigner functions were already clearly discussed by Heller [4] in 1976, even before the semiclassical approximations for W were properly developed [1, 5]. The asymptotic expansion (1) in \hbar was then partially resummed to obtain an improved propagation, though not in a very general context. From another approach, starting with the integral expression of the evolution equation, Marinov [7] derived in 1991 a path integral representation for the propagator of Wigner functions, obtaining its semiclassical limit, which leads to (2). In this paper we develop the geometrical explanation of why the semiclassical limit of the propagation of Wigner functions cannot be derived from the semiclassical limit of their propagator.

Thus, in section 2 we present the integral equation for the Wigner evolution, defining its kernel, the Wigner propagator, whose stationary phase evaluation necessarily leads to classical propagation. Then, in section 3 we recollect the constructions of semiclassical Wigner functions which will permit, in section 4, a correct stationary phase evaluation of the full integral equation for their evolution. This leads, in section 5, to a simple geometrical prescription for the semiclassical limit of the propagation of a Wigner function in terms of the classical flow of the tips of the chords centred on the arguments of the initial semiclassical Wigner function. We conclude in section 6 with a discussion on the geometrical meaning of the difference between classical Liouville and semiclassical Wigner propagation.

2. Integral evolution and semiclassical propagator

The starting point of our analysis is the expression for the product of operators in the centre (or Weyl) representation. Thus, if $A(x)$ and $B(x)$ are the centre representations of \hat{A} and \hat{B} , then the centre representation for $\hat{A}\hat{B}$ is given [8, 10] by the integral Moyal product:

$$[AB](x) = \int dx' dx'' A(x') B(x'') \exp(i\Delta(x, x', x'')/\hbar) \tag{4}$$

where $\Delta(x, x', x'') \equiv 2(x \wedge x' + x' \wedge x'' + x'' \wedge x)$ is the symplectic area of the triangle with these midpoints. Iterating (4) we obtain the integral equation for the evolution of Wigner functions as the Weyl transform of the quantum evolution equation $\hat{\rho}_t = \hat{U}_{-t} \hat{\rho}_0 \hat{U}_t$:

$$W_t(x) = \int dx' dx'' dx''' U_{-t}(x') W_0(x'') U_t(x''') \exp(2i(x \wedge x' + x'' \wedge x''')/\hbar) \times \delta(x - x' + x'' - x''') \tag{5}$$

where $U_t(x)$ is the Weyl propagator, i.e. the Weyl transform of the unitary evolution operator \hat{U}_t . The δ -function prescribes the four points in \mathbb{R}^{2n} as vertices of a parallelogram. The phase is twice the area of this parallelogram (times \hbar^{-1}). But this is also the area of any element of a continuous family of quadrilaterals circumscribed with the given parallelogram. In other words, we can identify the symplectic area of the parallelogram with vertices at (x, x', x'', x''') as half of the symplectic area of any quadrilateral whose sides are centred on these points. This brings the product rule for three operators in line with that for two operators and, indeed, the product of any number of operators will depend on the corresponding circumscribed polygon [10]. Thus, denoting the area of a quadrilateral as a function of its midpoints, we have $\Delta_4(x, x', x'', x''') \equiv 2(x \wedge x' + x'' \wedge x''')$ and this function is well defined only on the subset $D^3 \subset (\mathbb{R}^{2n})^4$, isomorphic to $(\mathbb{R}^{2n})^3$, determined by the δ -function. Fixing one (say, the first, x) of these points, we obtain a subset $D_x^2 \subset (\mathbb{R}^{2n})^3$, isomorphic to $(\mathbb{R}^{2n})^2$. Denoting its induced measure by $d_x^2(x', x'', x''')$, we can rewrite (5) as

$$W_t(x) = \int_{D_x^2} d_x^2(x', x'', x''') U_{-t}(x') W_0(x'') U_t(x''') \exp(i\Delta_4(x, x', x'', x''')/\hbar). \tag{6}$$

Similarly, we can reparametrize the parallelogram by identifying: $x'' \equiv x_0, x' \equiv (x+x_0)/2 - \mu, x''' \equiv (x+x_0)/2 + \mu$, in which case we can use (5) to get an expression for the Wigner propagator or kernel $L_t(x_0, x)$, via its defining formula $W_t(x) = \int dx_0 L_t(x_0, x) W_0(x_0)$, in the following form:

$$L_t(x_0, x) = \int d\mu U_{-t}((x+x_0)/2 - \mu) U_t((x+x_0)/2 + \mu) \exp(2i(\mu \wedge (x-x_0))/\hbar). \tag{7}$$

Marinov [7] has derived an explicit path integral representation for the Wigner propagator $L_t(x_0, x)$. This can also be achieved by introducing in (7) the path integral representation [10] for the Weyl propagator $U_t(x)$. For this latter, there exists a well-established semiclassical limit [6, 10]:

$$U_t(x) \approx \mathcal{B}_{\gamma_t}(x) \exp\{i\hbar^{-1} [S_{\gamma_t} - E_{\gamma_t} t](x)\} \tag{8}$$

where $S_{\gamma_t}(x)$ is the symplectic area between the classical trajectory γ_t (determined by the Weyl Hamiltonian H) and the chord centred on x , with $E_{\gamma_t} = H(\gamma_t)$ being the energy of this trajectory, and where, again, \mathcal{B}_{γ_t} is a slow-varying real amplitude function. For sufficiently small times we can guarantee the existence of a single trajectory. Eventually there may be bifurcations, in which case (8) must be replaced by a sum of similar terms with appropriate Morse indices [10].

The naive semiclassical limit for the propagation of Wigner functions can be seen as a consequence of trying to get a semiclassical limit for the Wigner propagator (7) itself by stationary phase, either via its path integral representation [7] or directly by using (8). In any case we are led to a semiclassical Wigner propagator which is indeed classical, i.e. of the singular form

$$\mathcal{L}_t(x_0, x) \approx \delta(x_0 - (x)_{-t}) \quad (9)$$

where $x \rightarrow (x)_{-t}$ is the inverse of the classical Hamiltonian flow of H . Obviously, (9) implies (2).

Therefore, in order to obtain the correct semiclassical limit for the propagation of Wigner functions, one must apply stationary phase arguments to the full integral equation (5)–(6) for the Wigner evolution, using the correct semiclassical expressions for the Wigner functions themselves.

3. Semiclassical Wigner functions

For one degree of freedom, the simple form of the semiclassical Wigner function (3) depends on the existence only of a single chord centred on each point x , besides fulfilment of the semiclassical condition itself, i.e. the area $S_\psi(x)$ being large in comparison to Planck's constant \hbar . These conditions hold for points not too close to a convex leaf (curve) ψ , outside the cusped triangular curve well in the interior of ψ known as the Wigner caustic [1]. Inside the caustic there are three chords centred on each point and the semiclassical Wigner function becomes a superposition of contributions of the same form as (3), one for each chord and its associated area. Along the Wigner caustic itself, two chords coalesce and the amplitude in (3), defined as

$$\mathcal{A}_\psi(x) = (2\omega/\pi) (2\pi\hbar|\dot{x}_+ \wedge \dot{x}_-|)^{-1/2} \quad (10)$$

blows up. Here, ω is the classical frequency of motion along the curve ψ , whereas \dot{x}_\pm are the phase space velocities at the tips x_\pm of the corresponding chord centred on x . The Wigner caustic is hence the locus of the centres of the chords between points on ψ with parallel or anti-parallel tangents. The caustic condition thus leads to the vanishing of the skew-product $\dot{x}_+ \wedge \dot{x}_-$ and since this also happens when x converges onto ψ , this latter can be considered as a separate branch of the Wigner caustic, i.e. the breakdown of (3) also takes place on ψ itself. Berry [1] derived a uniform approximation based on the Airy function that is also oscillatory and asymptotically equivalent to (3) inside ψ , and rises to a smooth maximum near to this curve and decays exponentially outside.

The same picture holds for integrable systems with L degrees of freedom [11]. The novelty is that the caustic still has codimension 1, so the invariant L -torus arises as a higher singularity or catastrophe of the Wigner caustic. Another important feature is that multiple chords may be centred on points that lie arbitrarily close to the invariant torus. Still, sufficiently far within the energy-shell we retrieve the asymptotic form [9, 11]

$$\mathcal{W}(x) \approx \sum_j A_j(x) \cos\{S_j(x)/\hbar - n_j\pi/4\} \quad (11)$$

where again the actions $S_j(x)$ are bounded by the j th chord centred at x and any arc on the quantized torus (again a Lagrangian leaf) between the chord tips. The amplitudes are now given by

$$A_j(x) = (2/\pi) [(2\pi\hbar)^L |\det\{I_j^+, I_j^-\}|]^{-1/2} \quad (12)$$

and n_j is the signature of the matrix in (12). Here, the $2L$ action functions I_j^\pm are defined in terms of the L action functions I as $I_j^\pm(x) \equiv I(x \pm \xi_j/2)$. Note that if $L = 1$ (12) is identical with (10).

No immediate generalization of the semiclassical expression for pure-state Wigner function is available for chaotic or general nonintegrable systems. However, the superposition of pure Wigner functions in a classically narrow energy range ϵ determines the spectral Wigner function $W(x, E, \epsilon)$ which has the now familiar semiclassical limit [2, 10],

$$\mathcal{W}(x, E, \epsilon) \approx \sum_j \mathcal{A}_j(x) e^{-\epsilon t_j/\hbar} \cos[S_j(x)/\hbar - \gamma_j] \tag{13}$$

where again $S_j(x)$ is the symplectic area between a chord centred on x and a classical path between the chord tips. In this case, this is an arc of a classical trajectory along the energy shell, traversed in the time t_j . For x close to the energy shell, the chords are all small and there is one arc traversed in a small time and a succession of longer arcs winding around the energy shell. Moving the centre x leads to the crossing of many Wigner caustics where pairs of chords coalesce and disappear or vice versa. Along these caustics, one can again implement uniform approximations which coincide, off-caustics, with the sum (13) in which the amplitude for each chord is

$$\mathcal{A}_j(x) = 2^{L+1} \{(2\pi\hbar)(dE/dt_j)|\det[1 + M_j]|\}^{-1/2} \tag{14}$$

where M_j is the matrix for the linearized classical map near the j th trajectory arc on a special $(2L - 2)$ -dimensional section that is centro-symmetric with respect to x .

Thus we have a persistent overall picture: superposition of rapidly oscillating functions with wave vectors $\xi_j = -J[\partial S_j/\partial x]$, where J is the standard symplectic matrix in \mathbb{R}^{2L} , and amplitudes depending only on the relation between the velocity vectors at the tips of each respective chord.

4. Wigner evolution: full stationary phase

Having recollected the general form of the semiclassical expressions for the various kinds of Wigner functions, we are now in a position to correctly approximate (5)–(6) by stationary phase. The crucial point here is how to *geometrically* interpret formulae (5)–(6) in the semiclassical approximation. As mentioned earlier, the phase of the exponential in the integrand, which is twice the area of the parallelogram (times \hbar^{-1}), corresponds to the area of *any* quadrilateral circumscribed to this parallelogram. Furthermore, recall from the previous section that the argument x of the semiclassical Wigner function (3), (11), (13) corresponds to the centre of the chord whose tips lie on the energy shell or the Lagrangian leaf ψ corresponding to the quantum state Ψ , so that the phase of its oscillatory factor corresponds to the symplectic area between ψ and the chord centred on x (minus $\pi/4$). On the other hand, the semiclassical Weyl propagator (8) also has a phase corresponding to the symplectic area between the classical trajectory γ_t and the chord centred on x (minus $E_\gamma t$). Therefore, all the semiclassical terms of the integrand in (5)–(6) have oscillatory factors whose phases depend on the chords centred on their arguments. This suggests interpreting the phase Δ_4 of the integrator in (5)–(6) as the area of the circumscribed quadrilateral that *fits* all the pertinent chords discussed above. To be totally consistent, such fitting must take into account all the correct orientations, so that one should first dismember the Wigner function as $\mathcal{W} = (\mathcal{W}^+ + \mathcal{W}^-)/2$, where $\mathcal{W}^\pm(x) \approx \mathcal{A}(x) \exp\{\pm i(S(x)/\hbar - \pi/4)\}$, and propagate each member separately, then add them up for the total evolution of \mathcal{W} . The situation for a single degree of freedom is illustrated in figure 1.

The really important issue is that these perfect matchings correspond precisely to the stationary phase condition, as we now show. Let us concentrate on the configuration in figure 1(a), corresponding to the semiclassical evolution of \mathcal{W}_0^+ , determined by the leaf ψ'' ,

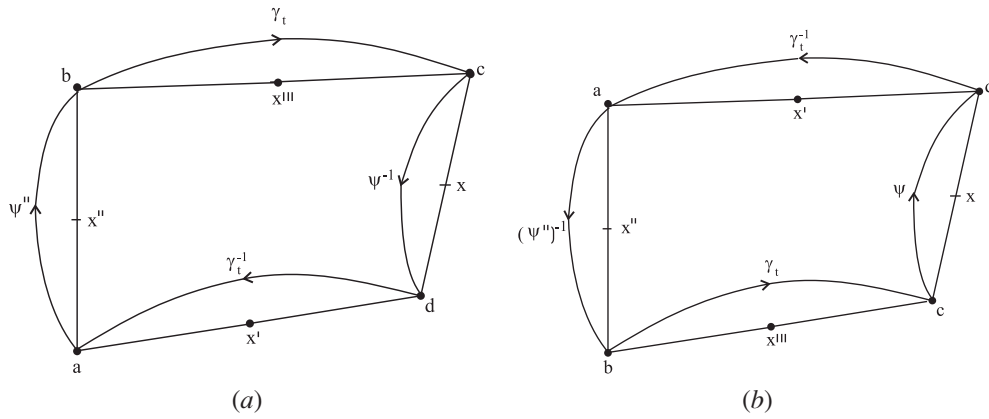


Figure 1. Stationary phase condition for the propagation of Wigner functions. (a) Single degree of freedom. The various chords of ψ'' , ψ^{-1} , γ_t and γ_{-t} centred on x'' , x , x''' and x' , respectively, match perfectly with the quadrilateral whose sides are centred on these points to yield, via equation (16), the area between ψ and the chord centred on x , the phase of the propagated Wigner function at x . (b) The same situation but with all the orientations reversed.

via formulae (5)–(6). The same analysis holds for the other case (figure 1(b)) as well as more dimensions. Inserting expressions (3) and (8) for \mathcal{W}_0 and \mathcal{U}_t in (6), we obtain

$$W_t^+(x) = \int_{D_x^2} d^2(x', x'', x''') \mathcal{B}'(x') \mathcal{A}''(x'') \mathcal{B}'''(x''') \exp \left\{ \frac{i}{\hbar} \Phi^+(x, x', x'', x''') \right\}$$

$$\Phi^+(x, x', x'', x''') = [S_{\gamma_{-t}} + E_{\gamma_{-t}} t](x') + S_{\psi''}(x'') - \pi \hbar / 4$$

$$+ [S_{\gamma_t} - E_{\gamma_t} t](x''') + \Delta_4(x, x', x'', x'''). \tag{15}$$

The stationary phase condition: $\partial \Phi^+ / \partial x' = \partial \Phi^+ / \partial x'' = \partial \Phi^+ / \partial x''' = 0$ implies that

$$\frac{\partial [S_{\gamma_{-t}} + E_{\gamma_{-t}} t]}{\partial x'} = -\frac{\partial}{\partial x'} \quad \frac{\partial S_{\psi''}}{\partial x''} = -\frac{\partial}{\partial x''} \quad \frac{\partial [S_{\gamma_t} - E_{\gamma_t} t]}{\partial x'''} = -\frac{\partial}{\partial x'''}$$

These equations mean [10] that the side of the quadrilateral Δ_4 whose midpoint is x'' coincides with the chord of $S_{\psi''}$ centred on x'' , with the same orientation as ψ'' . This is the chord from a to b in figure 1(a). Similarly for the other cases, the side of Δ_4 centred on x' is the chord connecting d to a and the side of Δ_4 centred on x''' is the chord connecting b to c . Therefore, the stationary phase condition coincides with the perfectly matching scenario. Furthermore, note that $(E_{\gamma_t} - E_{\gamma_{-t}})t$ is the area of the ‘curvilinear’ quadrilateral which is formed by the paths ψ'' , γ_t , ψ^{-1} , γ_{-t} , where ψ is the image of ψ'' under the Hamiltonian flow for a time t . It follows by direct geometrical inspection on the perfectly matching configuration that

$$\{S_{\gamma_{-t}}(x') + S_{\psi''}(x'') + S_{\gamma_t}(x''') + \Delta_4(x, x', x'', x''') - [E_{\gamma_t} - E_{\gamma_{-t}}]t = S_{\psi}(x)\}_{\text{match}} \tag{16}$$

and thus $\Phi_{\text{stat}}^+(x) = S_{\psi}(x) - \pi \hbar / 4$. And of course, $\Phi_{\text{stat}}^- = -\Phi_{\text{stat}}^+$.

Therefore, we retrieve a new Wigner function W_t whose phase is of the same general form given in (3), (11), (13) for \mathcal{W}_t , in terms of the new action $S_{\psi}(x)$. Of course, this is just as we should expect within the general rules of semiclassical self-consistency in which all integrations are carried out within the stationary phase approximation. It should now be feasible to proceed with the full stationary phase evaluation of (15) for each of the different types of Wigner functions (3), (11), (13) previously described, using the semiclassical expression [6, 10] for the amplitude \mathcal{B} of the Weyl propagator. Such a thorough verification of semiclassical self-consistency has already been carried out [9] for the pure state condition $\hat{\rho}^2 = \hat{\rho}$. It relies

on such complicated geometrical constructions that we consider more appropriate, at this point, to just assume semiclassical self-consistency for the propagated amplitudes, as well. Accordingly, each semiclassically propagated Wigner function \mathcal{W}_t retains its respective form (3), (11), (13), with its corresponding amplitude function (10), (12), (14).

5. Semiclassical propagation: geometrical prescription

The beautiful consequence of this natural result is that it is sufficient to propagate the two tips of the chord of the original Wigner function \mathcal{W}_0 , centred on $x'' \equiv x_0 \equiv (x_0^- + x_0^+)/2$, in order to obtain the value of the new Wigner function \mathcal{W}_t at $x \equiv \tilde{x}_t \equiv (x_t^- + x_t^+)/2$. Then, using (16) we obtain the new phase $S_\psi(x) \equiv S_{\psi_t}(\tilde{x}_t)$ from $S_{\psi''}(x'') \equiv S_{\psi_0}(x_0)$. Similarly for the amplitudes, the actions $\mathcal{S}_{\gamma_t}(x''') \equiv \{\mathcal{S}_{\gamma_t} - E_{\gamma_t}t\}(x''')$ and $\mathcal{S}_{\gamma_{-t}}(x') \equiv \{\mathcal{S}_{\gamma_{-t}} + E_{\gamma_{-t}}t\}(x')$ determine the matrices $M_+(x''')$ and $M_-(x')$ for the linearized classical maps from $x_0^+ \equiv x_+'' = x'''$ to $x_+''' = x_+ \equiv \tilde{x}_t^+$ and from $\tilde{x}_t^- \equiv x_- = x'_$ to $x'_+ = x_+'' \equiv x_0^-$, respectively, by [10]

$$M_\pm(y_\pm) = [1 - J(\partial^2 \mathcal{S}_{\gamma_{\pm t}}(y_\pm)/\partial y_\pm^2)] [1 + J(\partial^2 \mathcal{S}_{\gamma_{\pm t}}(y_\pm)/\partial y_\pm^2)]^{-1} \tag{17}$$

where $y_+ \equiv x'''$, $y_- \equiv x'_$. Hence, the corresponding phase-space velocity vectors for the tips of the propagated chord are obtained from the original ones by

$$\dot{x}_+ = M_+ \dot{x}_+'' \quad \dot{x}_- = M_-^{-1} \dot{x}_-'' \tag{18}$$

which gives the new amplitudes $\mathcal{A}_\psi(x) \equiv \mathcal{A}_{\psi_t}(\tilde{x}_t)$ from $\mathcal{A}_{\psi''}(x'') \equiv \mathcal{A}_{\psi_0}(x_0)$ via their defining equations (10), (12), (14). When the central action $\mathcal{S}_{\gamma_t}(y_+)$ or $\mathcal{S}_{\gamma_{-t}}(y_-)$ goes through a caustic, one may rely on the chord actions $\tilde{\mathcal{S}}_{\gamma_t}(\xi_+)$ or $\tilde{\mathcal{S}}_{\gamma_{-t}}(\xi_-)$ instead, for $\xi_\pm = \pm(x_t^\pm - x_0^\pm)$, where the chord actions are the Legendre transform of the central actions: $\tilde{\mathcal{S}}(\xi) \equiv \mathcal{F}(y(\xi), \xi)$, $\mathcal{F}(y, \xi) = \xi \wedge y - \mathcal{S}(y)$, to get [10]

$$M_\pm(\xi_\pm) = - [1 + J(\partial^2 \tilde{\mathcal{S}}_{\gamma_{\pm t}}(\xi_\pm)/\partial \xi_\pm^2)] [1 - J(\partial^2 \tilde{\mathcal{S}}_{\gamma_{\pm t}}(\xi_\pm)/\partial \xi_\pm^2)]^{-1} . \tag{19}$$

Note, however, that in this case one must be very careful in correctly counting the Maslov indices.

Therefore, our analysis clearly leads to interpreting the correct point-to-point propagation of a semiclassical Wigner function as given by the simple geometrical picture:

$$\mathcal{W}_0(x_0) \rightarrow \mathcal{W}_t(\tilde{x}_t) \tag{20}$$

where \tilde{x}_t is the midpoint of (x_t^-, x_t^+) , while (x_0^-, x_0^+) stands for the tips of the original chord centred on x_0 . This prescription based on the tips-of-the-chord flow provides a precise semiclassical evaluation of $\mathcal{W}_t(\tilde{x}_t)$ from $\mathcal{W}_0(x_0)$. Specifically, (16) is used to propagate the phase of \mathcal{W} along the path $x_0 \rightarrow \tilde{x}_t$. For short times, the smooth amplitude may be taken as approximately constant along this path, as illustrated below. For longer times the amplitude must also be properly propagated along this path via (17)–(19). In this way, (20) is straightforwardly used to obtain a quantitatively precise semiclassical propagation of \mathcal{W} ‘almost everywhere’. But, wherever \mathcal{W}_t goes through an inner Wigner caustic, its analysis must be carefully completed by uniform approximations. Even then (20) provides most valuable information because (17)–(19) can be used to determine these inner Wigner caustic points and (16) determine the contributing new phases which, from the knowledge of the corresponding caustics and uniform approximations for \mathcal{W}_0 and after a careful qualitative analysis of the flow $\psi_0 \rightarrow \psi_t$, can lead to the correct (Airy, Pearcey ... [1]) functions representing \mathcal{W}_t in these regions. The on-shell Wigner caustics are much simpler to deal with because there $x_t^- \equiv x_t^+$ and (20) coincides with (2). Thus, we can say that (20) is quantitatively precise ‘almost everywhere’ and qualitatively precise everywhere, at the semiclassical level.

As a simple partial illustration of the method, consider the Hamiltonian $h = (q^2 + p^2)^2/4$. Operator ordering is not relevant, modulo a constant and $H = h = r^4/4$. The trajectories are circles around the origin, but $\dot{\theta} = r^2$ so the classical flow of $x_0 = (r_0, \theta_0 \equiv 0)$ is $x_t = (r_t = r_0, \theta_t = r_0^2 t)$. If x_0 is the centre of the chord whose tips, lying on ψ_0 , are $x_0^- = (r_-, \alpha)$, $x_0^+ = (r_+, -\beta)$, then the new centre $\tilde{x}_t = (\tilde{r}_t, \tilde{\theta}_t)$ of (x_t^-, x_t^+) is given by $\tilde{r}_t^2 = (r_-^2 + r_+^2)/4 + r_- r_+ \cos((r_+^2 - r_-^2)t - (\alpha + \beta))/2$, $\tilde{\theta}_t = r_-^2 t + \alpha + \alpha'$, with $2\tilde{r}_t \cos(\alpha') = r_- + r_+ \cos((r_+^2 - r_-^2)t - (\alpha + \beta))$. Thus, not only $\tilde{\theta}_t \neq \theta_t$, but also $\tilde{r}_t^2 - r_t^2 \equiv \tilde{r}_t^2 - r_0^2 = r_+ r_- \{\cos((r_+^2 - r_-^2)t - (\alpha + \beta)) - \cos(\alpha + \beta)\}/2$.

Despite the oddity of this flow, $\mathcal{W}_t(\tilde{x}_t)$ is almost everywhere obtained from $\mathcal{W}_0(x_0)$ in a straightforward manner from the knowledge of x_t^\pm . Hence, modulo nontrivial Maslov changes, the phase difference $\delta S(x_0, \tilde{x}_t) = S_{\psi_t}(\tilde{x}_t) - S_{\psi_0}(x_0)$ does not depend on ψ_0 or \mathcal{W}_0 and is (via (16)) given by

$$\delta S(x_0, \tilde{x}_t) = t(r_+^4 - r_-^4)/4 + r_+ r_- \{\sin(t(r_+^2 - r_-^2)/2) \cos(t(r_+^2 - r_-^2)/2 - (\alpha + \beta))\}. \quad (21)$$

On the other hand, the new amplitude depends on the velocities of the tips of the chord determined by ψ_0 and thus depends on ψ_0 and \mathcal{W}_0 itself in an intrinsic way. However, for short enough times such that \tilde{x}_t is not too different from x_t , we can approximate the slow-varying amplitude by $\mathcal{A}_t(\tilde{x}_t) \approx \mathcal{A}_t(x_t) \approx \mathcal{A}_0(x_0)$, ‘using’ (2) in a more justified way. Together with (21) this gives a first approximation for the semiclassical propagation of *any* Wigner function, almost everywhere and for short times, under the r^4 Hamiltonian. For longer times, we obtain $\mathcal{A}_t(\tilde{x}_t)$ from $\mathcal{A}_0(x_0)$ via (10) and (18), with M_\pm determined by (17) from $S_{\pm t} = \pm r_\pm^2 [r_\pm^2 t - 2 \sin(r_\pm^2 t)]/4$ with $|y_\pm|^2 = r_\pm^2 \cos^2(r_\pm^2 t/2)$, and by (19) from the corresponding $\tilde{S}_{\pm t}$. As for formula (2), again the precise error will depend on ψ_0 and x_0 , but if x_0 is far from the leaf ψ_0 , we can approximate $S_{\psi_t}(x_t) - S_{\psi_0}(x_0) \propto t(r_+ - r_-)^3$. Thus, for any pair (ψ_0, x_0) such that r_+ and r_- differ significantly, (2) can be considerably wrong even for a very short time propagation.

6. Conclusion

We have shown that the propagation of a semiclassical Wigner function is correctly determined by the classical flow of the *tips* of the chord whose centre is the argument of the initial Wigner function, not by the classical flow of the argument itself. This reveals the irrelevance of the semiclassical limit for the Wigner propagator. Accordingly, in evaluating the integral equation for the evolution of a Wigner function, we found that satisfying the stationary phase condition was tantamount to matching four areas around an appropriate quadrilateral. However, by reducing the propagation to a single trajectory, traversed in positive and negative times, the side of the quadrilateral facing the chord of the Wigner function shrinks to a point and so it could only be matched by a zero length chord. Thus we found that the relevant trajectories for a precise semiclassical propagation are explicitly determined by the specific semiclassical Wigner function being propagated.

Indeed, one way to correctly evolve a semiclassical Wigner function \mathcal{W}_0 determined by a leaf ψ_0 in phase space is to classically evolve ψ_0 via the Hamiltonian flow: $\psi_0 \rightarrow \psi_t$ and then evaluate the new semiclassical Wigner function \mathcal{W}_t at each point from the knowledge of ψ_t via (3), (10)–(14), as shown in [3]. Thus, to obtain the value of the new semiclassical Wigner function \mathcal{W}_t at x one needs to know, at the very least, the corresponding chord of ψ_t centred on x . In fact, our analysis shows that the knowledge of the flow of each pair of points in ψ_0 is sufficient to determine the evolved semiclassical Wigner function \mathcal{W}_t at all points. Though these two prescriptions for the evolution of \mathcal{W} are equivalent, in practice the

semiclassical point-to-point propagation of \mathcal{W} is often simpler to determine, both qualitatively and quantitatively, by (16)–(19) via tips-of-the-chord flow (20).

On the other hand, mere knowledge of the Hamiltonian flow of an argument of \mathcal{W}_0 is not enough to reconstruct the new chord and hence the new value of \mathcal{W}_t . The exceptions are either linear flow or when the two tips of the corresponding chord coalesce into its centre, in other words, when the argument of \mathcal{W}_0 lies on the leaf ψ_0 . For these points the total chord flow coincides with the centre flow, thus (2) is verified in the regions of high amplitude near the leaf ψ_0 , but not otherwise for nonlinear flows. It is easy to see why the phase of $\mathcal{W}_0(x_0)$ corresponds to the symplectic area between ψ_0 and the chord centred on x_0 . Under a nonlinear Hamiltonian flow, the symplectic area of this closed circuit is preserved, however the image of the chord is no longer a straight segment and thus the symplectic area between ψ_t and a new chord centred on x_t will generally differ from the corresponding area evaluated at $t = 0$. This difference can be large, even for short times, if the tips of the initial chord are sufficiently far apart, i.e. when the argument of the initial Wigner function is sufficiently far from the initial classical leaf. Therefore, generally (2) is simply wrong, semiclassically. In sharp contrast, the prescription based on the tips-of-the-chord flow (20) provides a precise semiclassical evaluation of $\mathcal{W}_t(\tilde{x}_t)$ from $\mathcal{W}_0(x_0)$. Only when $\tilde{x}_t \equiv x_t$ do (2) and (20) coincide.

References

- [1] Berry M V 1977 *Phil. Trans. R. Soc.* **287** 237
- [2] Berry M V 1991 *Chaos and Quantum Physics (Session LII, Les Houches)* ed M J Giannoni, A Voros and J Zinn-Justin (Amsterdam: North-Holland)
- [3] Berry M V and Balazs N L 1979 *J. Phys. A: Math. Gen.* **12** 625
- [4] Heller E J 1976 *J. Chem. Phys.* **65** 1289
- [5] Heller E J 1977 *J. Chem. Phys.* **67** 3339
- [6] Marinov M S 1980 *Phys. Rep.* **60**
- [7] Marinov M S 1991 *Phys. Lett. A* **153** 5
- [8] Moyal J E 1949 *Proc. Camb. Phil. Soc. Math. Phys. Sci.* **45** 99
- [9] Ozorio de Almeida A M 1982 *Physica A* **110** 501
- [10] Ozorio de Almeida A M 1998 *Phys. Rep.* **295**
- [11] Ozorio de Almeida A M and Hannay J H 1982 *Ann. Phys., NY* **138** 115
- [12] Weyl H 1931 *Group Theory and Quantum Mechanics* (New York: Dover)
- [13] Wigner E P 1932 *Phys. Rev.* **40** 749