ON (NON)COMMUTATIVE PRODUCTS OF FUNCTIONS ON THE SPHERE

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Abstract. We investigate the commutativity of global products of functions on $\mathbf{S}^2$ from the point of view of a construction started in [RT] and named the skewed product. We complete the construction of the skewed product of functions on $\mathbf{S}^2$ and show that it is $\mathbf{Z}_2$-graded commutative and nontrivial only as a product of functions with correct parity under the antipodal mapping. These properties are valid for a more general class of integral products of functions on the sphere, with integral kernel of a special WKB-type that is natural from semiclassical considerations. We argue that our construction provides a simple geometrical explanation for an old theorem by Rieffel [Rf] on equivariant strict deformation quantization of the two-sphere.

Introduction

An integral product of functions on a simply connected symplectic symmetric space M has been defined in [RT]. This product, which we name *skewed product* of functions on M, is a mathematical invention. The motivation for its construction is to provide a simple geometric framework that allows for generalizing to some other symplectic manifolds the product of functions on $\mathbf{R}^{2n}$ that has been defined by Weyl, von Neumann, Groenewold and Moyal [Wl][vN][Gr][Ml], with emphasis on its integral formulation [vN][Gr], which is better suited to more careful treatments of oscillatory functions [Rs].

The inspiration for the construction of the skewed product comes, on the one hand, from Berry's "center" description in semiclassical mechanics [By] and, on the other hand, from the prequantization approach developed by Kostant and Souriau [Ko][So]. Via symplectic groupoids [WX], these two geometric guidelines are brought together.

This name "skewed product" has been coined to stress the distinction of its construction from the better known product defined via formal deformation quantization, which is named "star product" [B-S], and the product defined within the context of quantum theory via symbol mapping homomorphism, which is named "twisted product" in [VG] (though these two products are often confused and named star product in the general literature).

As opposed to twisted products, no Hilbert space structure (and its quantum theory) is required for the construction of the skewed product. Also, as opposed to formal deformation quantization, associativity is not imposed beforehand. Its construction is purely geometrical, fairly simple, and the product agrees with an ansatz by Weinstein [Wn] for an integral kernel of a special WKB-type which is related, on the one hand, to the product of von Neumann and Groenewold and, on the other hand, to the composition of central generating functions of canonical relations on symmetric symplectic spaces [Mv][RO].

Accordingly, it should at least be expected that the skewed product provides valuable information on "quantum" products at the semiclassical limit ($\hbar \to 0^+$), or as a framework for studying "quantizations" ($0^+ \to \hbar$) of symmetric symplectic spaces.

The construction of the skewed product, as presented in [RT], is well defined and unique when M is a simply connected symplectic symmetric space of noncompact type, but it is also possible to be carried out, at least partially, in the compact case.

In this paper, after reviewing the partial construction in [RT], we complete the construction of the skewed product of functions on the 2-sphere, showing that the global skewed product $\star^n$ of functions on $\mathbf{S}^2$ is $\mathbf{Z}_2$-graded commutative, $f \star^n g = (-1)^n g \star^n f$, and is nontrivial only as a product on the subspace of functions satisfying $f(-\alpha) = (-1)^n f(\alpha)$, where n is the prequantization integer and $-\alpha$ is antipodal to α (Theorems 1 and 2).

In fact, the main reason why a well defined skewed product in an infinite dimensional space of functions on the sphere must satisfy these properties, also implies that they are satisfied quite independently of the particular amplitude function for an integral kernel of WKB-type, just as long as the kernel of WKB-type is symmetric, $SU(2)$ invariant, and its phase is a half-integral multiple of a midpoint triangular area (Corollary 3).

In light of an old result by Rieffel [Rf], we discuss the sharp contrast between our result and the noncommutative property of any star product of formal power series in $\hbar$ with coefficients in the space of smooth functions on the sphere [MO][Md]. We argue that our construction provides a simple geometrical explanation of Rieffel's theorem.

The skewed product of functions on the 2-sphere

(Partial) construction of the skewed product of functions.

The skewed product of functions on a simply connected symmetric symplectic space (M, ω, ∇) has been defined in [RT]. The defining elements of the construction of the skewed product are: (i) prequantization of the pair groupoid; (ii) central polarization; (iii) central polarized sections and central decomposition; (iv) product of sections and holonomy of the central decomposition; (v) averaging procedure. We now summarize the construction:

(i) Denoting by ρ_+ and ρ_- the two canonical projections $M \times M \to M$, the *pair* (symplectic) *groupoid* over M is the manifold $M \times M$ with symplectic form $\rho_+^*\omega - \rho_-^*\omega =: (\omega, -\omega)$. If $\pi : Y \to M$ is a prequantization of M with connection θ, the manifold $Y \times Y$ is a principal torus bundle over $M \times M$. Taking the quotient of the diagonal action of $\mathbf{S}^1$, we obtain a principal $\mathbf{S}^1$-bundle $[Y] = (Y \times Y)/\mathbf{S}^1 \to M \times M$. We denote points in $[Y]$ as $[y_1, y_2]$ with $y_i \in Y$. The induced $\mathbf{S}^1$-action is $e^{i\phi} \cdot [y_1, y_2] = [e^{i\phi} \cdot y_1, y_2] = [y_1, e^{-i\phi} \cdot y_2]$. The 1-form $(\theta, -\theta)$ induces a connection form $[\theta] \equiv [\theta, -\theta]$ on $[Y]$, whose curvature is $(\omega/\hbar, -\omega/\hbar)$. $[Y]$ is a prequantization of $M \times M$ which is an $\mathbf{S}^1$ extension of the pair groupoid, for the space of unities as the diagonal section $\varepsilon_0 : M \to [Y]$, $\varepsilon_0(m) = [y, y]$ with $y \in Y$ s.t. $\pi(y) = m$, which is horizontal for the connection $[\theta]$, and the groupoid product reading $[x, y] \odot [y, z] = [x, z]$. We let $[L] \to M \times M$ be the associated complex line bundle with connection and compatible hermitian structure, so that we can identify $[Y] \subset [L]$.

(ii) For the complete affine connection ∇ on TM, geodesic inversion is a symplectomorphism, so we define the map: $F : TM \to M \times M$; $F(m, v) = (\exp_m(-v), \exp_m(v))$, where $\exp_m : T_mM \to M$ denotes the geodesic flow at time $t = 1$, starting at $m \in M$, in the direction of $v \in T_mM$. F is a diffeomorphism in a neighborhood of the zero section of TM, so we define $U \subset TM$ as the maximal connected and open neighborhood of the zero section on which F *is* a diffeomorphism, and its image $V = F(U) \subset M \times M$ (if ∇ has no closed geodesics, then $U = TM$ and $V = M \times M$). On TM there is a natural foliation $\mathcal{F}_v$ whose leaves are the fibres T_mM. Restricting $\mathcal{F}_v$ to U defines a polarization of U, with pullback symplectic form $\Omega = F^*(\omega, -\omega)$, which we call the *vertical* or *central polarization.*

(iii) The section ε_0 of $[Y]$ gets transformed to a section ϵ_0 of the pullback bundle $(B, \Theta) = F^*([Y], [\theta])$ above M seen as the zero section of TM. Since the fibers T_mM are simply connected and since the pullback connection Θ has null curvature on these fibers, ϵ_0 extends to a global section $\sigma : TM \to B$ which is horizontal on leaves of $\mathcal{F}_v$. Denote by σ_0 the restriction of σ to U. Pushing it by F we obtain a section s_0 over V, which, $\forall (m_1, m_2) \in V$ gives $s_0(m_1, m_2) = [x, y]$ where x and y are the end points of a horizontal curve in Y above the geodesic on M between m_1 and m_2 whose pre-image $F^{-1}(m_1, m_2)$ is in U. We call *central polarized sections*, the sections of $[L]$ above V that are covariantly constant along $\mathcal{P} = F_*(\mathcal{F}_v|_U)$. Viewing $[Y] \subset [L]$, the section s_0 constructed above is $\mathcal{P}$-constant and, since s_0 is a smooth nowhere vanishing section, central polarized sections are in bijection to functions f that are constant on the leaves of $\mathcal{P}$, i.e., with functions on $M = V/\mathcal{P}$. The identification is $s = f \cdot s_o$, or, more precisely, $s(m_1, m_2) = f(m_{12}) \cdot s_0(m_1, m_2)$, where m_{12} is the midpoint of the geodesic on M, between m_1 and m_2, s.t. $F^{-1}(m_1, m_2) = (m_{12}, v) \in U$. More generally, any section $s : V \to [L]$ can be decomposed as $s = \phi \cdot s_o$, where ϕ is a function on V. We call this the *central decomposition*, that separates the phase dependence along the fibers T_mM which is obtained by parallel transporting the identity section $\varepsilon_0 \cong \epsilon_0$.

(iv) Any $p \in [L]$ can be written in a unique way as $p = \lambda[x, y]$ with $\lambda \in [0, \infty)$ and $[x, y] \in [Y] \subset [L]$. Now, for $p_i = \lambda_i[x_i, y_i]$ such that $\pi(y_1) = \pi(x_2)$ we define $p_1 \odot p_2 = \lambda_1\lambda_2[x_1, y_1] \odot [x_2, y_2]$. With this extended quasi-groupoid structure (quasi because now not every element has an inverse), we construct a *product of two sections* s^1 and s^2 of $[L]$ by $(s^1 \circledcirc s^2)(m_1, m_3) = \int_M s^1(m_1, m_2) \odot s^2(m_2, m_3)\, dm_2$, where dm_2 is the Liouville measure on (M, ω). For two $\mathcal{P}$-constant sections $s^i = f_i \cdot s_0$, each f_i a function on M, we get $(s^1 \circledcirc s^2)(m_1, m_3) = \int_M f_1(m_{12}) f_2(m_{23}) s_0(m_1, m_2) \odot s_0(m_2, m_3)\, dm_2$, in which m_{jk} denotes the midpoint of the geodesic between m_j and m_k, with $F^{-1}(m_j, m_k) = (m_{jk}, v_{jk}) \in U$. However, $s_0(m_1, m_2) \odot s_0(m_2, m_3) = \lambda s_0(m_1, m_3)$, where $\lambda \in \mathbf{S}^1$ is the *holonomy over the geodesic triangle* with vertices (m_1, m_2, m_3) and all geodesics with pre-images in U, so that $(f_1 \cdot s_0 \circledcirc f_2 \cdot s_0)(m_1, m_3) = \phi(m_1, m_3) \cdot s_0(m_1, m_3)$, where ϕ is a function on V. In this way, we have associated to two functions f_1, f_2 on M a new function ϕ on $V \subset (M \times M)$.

(v) In order to get a new central polarized section from the product of two central polarized sections, i.e., in order to associate to two functions f_1 and f_2 on M a new function $f_1 \star f_2$ on M, we integrate (average) ϕ over the leaves of $\mathcal{P}$. In terms of the fibres of TM we get $(f_1 \star f_2)(m) = \int_{U_m} dv\ \phi(F(m, v))$, where $U_m = T_mM \cap U$. We choose the measure $d_{U_m}(v) \equiv dv$ on U_m s.t. $d_M(m) \wedge d_{U_m}(v) = d_U(m, v)$, where $d_M(m)$ is the Liouville measure on (M, ω) and $d_U(m, v)$ is the Liouville measure on (U, Ω). This choice for dv defines our *averaging procedure.* If ∇ has no closed geodesics, $U = TM$ and, in this case, $f_1 \star f_2$ defined above is *the skewed product* of f_1 and f_2. Otherwise, we call $f_1 \star f_2$ a *partial* skewed product, because we have used only a (big) neighborhood $U \subset TM$ of the zero section.

A partial skewed product of functions on the sphere.

Skewed products can be written entirely in terms of midpoint geometrical data (using that M is prequantized). When $M = \mathbf{R}^{2n}$, it coincides with the product of von Neumann and Groenewold [RT]. Coming to the sphere, let α, β, γ be three points on $\mathbf{S}^2$, also thought as three unit vectors in $\mathbf{R}^3$, and let $n = Area(\mathbf{S}^2)/2\pi\hbar = 2/\hbar \in \mathbf{Z}^+$ be the prequantization integer. A *partial* skewed product of two functions f_1 and f_2 on $\mathbf{S}^2$ is given by [RT]:

$$f_1 \star f_2\ (\gamma)\ = \iint_{W_\gamma} f_1(\alpha) f_2(\beta)\ \mathrm{e}^{inS(\alpha,\beta,\gamma)/2}\, A(\alpha,\beta,\gamma)\, d\alpha\, d\beta\ . \tag{1}$$

We now identify the various elements in formula (1) above. First,

$$S(\alpha,\beta,\gamma) = 2\,\mathrm{Arg}\Big(\eta\sqrt{1-\det(\alpha\beta\gamma)^2} + i\det(\alpha\beta\gamma)\Big)\ , \tag{2}$$

is the symplectic area of a geodesic triangle with (α,β,γ) as midpoints and sides whose lengths are all smaller than or equal to π. Not all triple of midpoints satisfy this restriction on the lengths of the sides of the corresponding midpoint triangle. Given any $\gamma \in \mathbf{S}^2$, only those (α,β) in $W_\gamma \subset \mathbf{S}^2 \times \mathbf{S}^2$ satisfy this restriction, where

$$W_\gamma = \{(\alpha,\beta) \in \mathbf{S}^2 \times \mathbf{S}^2 \mid \mathrm{sign}\langle\alpha|\beta\rangle_{_E} = \mathrm{sign}\langle\beta|\gamma\rangle_{_E} = \mathrm{sign}\langle\gamma|\alpha\rangle_{_E}\ \}\ , \tag{3}$$

with $\langle\alpha|\beta\rangle_{_E}$ denoting the usual scalar product of vectors α,β in $\mathbf{R}^3$. Accordingly, in formula (2) η is this sign of the three scalar products (for triangles with at most one side bigger than π, then η is the same as the majority of signs among the three scalar products).

Now, if one (and therefore all) of these three scalar products is not zero, there is a bijection G^{-1} that takes the three midpoints (α,β,γ) in the restricted set above to the three vertices (a,b,c) of the geodesic triangle with all sides smaller than π. Then, A is the jacobian of this transformation, that is, if da, $d\alpha$, etc, is the canonical measure on $\mathbf{S}^2$,

$$(G^{-1})^*(dadbdc) = A(\alpha,\beta,\gamma)d\alpha d\beta d\gamma\ ,$$

$$A(\alpha,\beta,\gamma) = 16\left|\langle\alpha|\beta\rangle_{_E}\cdot\langle\beta|\gamma\rangle_{_E}\cdot\langle\gamma|\alpha\rangle_{_E}\right|\cdot\Big(1-\det(\alpha\beta\gamma)^2\Big)^{-5/2}\ . \tag{4}$$

We also note that the integral kernel $K = Ae^{inS/2}$ is *symmetric*, in the sense that

$$K(\alpha,\beta,\gamma) = K(\gamma,\alpha,\beta) = \overline{K}(\gamma,\beta,\alpha)\ . \tag{5}$$

Accordingly, we say that the integral product given by formula (1) is *symmetric*, in this sense, which must not be confused with commutative. This symmetric property for the skewed product is general and not particular for the case of the sphere. The same can be said of the geometrical interpretation for the phase, the amplitude and the domain of integration of the skewed product, which are valid in general [RT].

Finally, we say that the integral product given by formula (1) defines a *restricted* skewed product, because the integration is carried over a proper subset of $\mathbf{S}^2 \times \mathbf{S}^2$. This is also true of *the* skewed product of functions on the noncompact hyperbolic plane $\mathbf{H}^2$ [RT], even though that skewed product is not partial because, in that case, $U = T\mathbf{H}^2$.

Extending the construction: antipodal midpoints.

We now notice that the restriction (3) is unnatural. This is so because the amplitude function (4) does not depend on this restriction, in sharp contrast with the relation between the amplitude and the restriction for the case of the hyperbolic plane [RT]. Thus, we question if the product (1) is unique, if it can be written equivalently in other domains, or if it can be nontrivially extended to a larger domain in $\mathbf{S}^2\times\mathbf{S}^2$ in a well defined and unique way. We shall investigate these questions from the point of view of a natural generalization of the averaging procedure. To do so, we must deal with the lack of uniqueness in determining spherical triangles, as determined by midpoints. This, in turn, is rooted in the non uniqueness of geodesics connecting any two points on the sphere, not just those that are antipodal.

If two points on $\mathbf{S}^2$ stand in antipodal relation, there is an infinity of geodesics, with an infinity of directions, connecting these points. Then, the whole equator of these points is the set of their possible midpoints. On the other hand, if two points a and b in $\mathbf{S}^2$ do not stand in antipodal relation, there still exists an infinity of geodesics connecting these points, but they all have the same direction. Thus, there are only two possible midpoints for this pair of points: the midpoint α of the shortest geodesic connecting a to b and its antipodal $-\alpha$, which is the midpoint of the geodesic that composes with the shortest one (with reverse orientation) to form a big circle on the sphere. Borrowing from the terminology in [RO], we say that these two geodesics are weakly equivalent to each other.

All other geodesics connecting a and b are strongly equivalent to one of these, in the sense that their midpoints coincide. In what follows, it will become clear that strongly equivalent geodesics can be treated as the same, therefore we shall refer to these two strongly inequivalent geodesics as *the short* and *the long* geodesic connecting a to b.

Looking at $V \subset (\mathbf{S}^2 \times \mathbf{S}^2)$, the set of all pairs of points in $\mathbf{S}^2$ that are not antipodal, we can identify the pre-image $U \equiv U_0 \subset T\mathbf{S}^2$ as the set of all tangent vectors $\tau = (m, v)$ whose lengths $|v|$ are smaller than $\pi/2$. However, we can also identify another pre-image in $T\mathbf{S}^2$, $U_1 \neq U \equiv U_0$, which is the set of all tangent vectors $\tau = (m, v)$ whose lengths $|v|$ satisfy $\pi/2 < |v| \leq \pi$. Although $F(U_1) = V$, the inverse is ill defined because all vectors of length π based at α have as image the pair $(-\alpha, -\alpha)$. In order to define a bijection, we must identify all such tangent vectors τ_1 and τ_2 under the equivalence relation $\tau_1 \sim \tau_2$ if $m_1 = m_2$ and $|v_1| = |v_2| = \pi$. Then, $\widetilde{U_1} = (U_1/\sim)$ stands in bijection to V and thus to $U \equiv U_0$. In what follows, it will become clear that $\widetilde{U_1}$ is the only other space that needs to be considered for generalizing our construction. Clearly, the vertical polarization is natural to $\widetilde{U_1}$, but remember that, if α is the base of the element $\tau \in U_0$, s.t. $F(\tau) = (a, b) \in V$, then $-\alpha$ is the projection of $\tau' \in \widetilde{U_1}$, s.t. $F(\tau') = (a, b) \in V$.

Unique composition of central polarized sections.

The next step of the construction in [RT] to be generalized, pulling back the prequantized bundle to $\widetilde{U_1}$, depends on the choice of a trivializing section ϵ_1 over the new pre-image of the diagonal in $\mathbf{S}^2 \times \mathbf{S}^2$ and its extension to a horizontal section σ_1 over $\widetilde{U_1}$ that pushes forward under F to two new sections $\varepsilon_1 : M \to [Y]$ and $s_1 : V \to [Y]$, respectively.

The natural choice for ϵ_1 and its extension to σ_1 is explained in proposition 6.1' of [RO] and is such that, as σ_0 is the restriction to $U \equiv U_0 \subset T\mathbf{S}^2$ of a single section $\sigma : T\mathbf{S}^2 \to B$, σ_1 is the restriction of this same section σ to $\widetilde{U_1}$, in such a way that the pushed forward section s_1 satisfies $s_1(a, b) = [x', y']$, where x' and y' are the endpoints of a horizontal curve above the long geodesic between a and b.

The fact that σ_1 is well defined on $\widetilde{U_1}$, not just on U_1, is a consequence of the prequantization condition so that, given the positive integer $n = Area(\mathbf{S}^2)/2\pi\hbar = 2/\hbar \in \mathbf{Z}^+$, it is clear that $\varepsilon_1(a) \cong s_1(a,a) = (-1)^n \cdot s_0(a,a) \cong (-1)^n \cdot \varepsilon_0(a)$, since $(-1)^n$ is the holonomy along any great circle. It follows that $\epsilon_1(-a,[\pi]) = (-1)^n \cdot \epsilon_0(a,0)$ and this relation extends naturally so that, if $\tau \in U_0$ and $\tau' \in \widetilde{U_1}$ are s.t. $F(\tau) = F(\tau') = (a,b) \in V$, then

$$s_1(a,b) = (-1)^n \cdot s_0(a,b) \ .$$

Now, $\mathcal{P} = F_*(\mathcal{F}_v|_{U_0}) = F_*(\mathcal{F}_v|_{\widetilde{U}_1})$. If $s : V \to [L]$ is $\mathcal{P}$-constant, s can be decomposed as $s(a,b) = f(\alpha) \cdot s_0(a,b)$, where f is a function on $V/\mathcal{P} = \mathbf{S}^2$ and α is the midpoint of the short geodesic connecting a to b. But then, s can equally well be decomposed as $s(a,b) = \widetilde{f}(-\alpha) \cdot s_1(a,b)$, where $\widetilde{f}$ is also a function on $V/\mathcal{P} = \mathbf{S}^2$, since $-\alpha$ is the midpoint of the long geodesic connecting a to b. It follows from the above that we must have:

$$\widetilde{f}(-\alpha) = (-1)^n \cdot f(\alpha) \ , \ \forall \alpha \in \mathbf{S}^2 \ . \tag{6}$$

Therefore, to any $\mathcal{P}$-constant section $s : V \to [L]$, we associate two functions f and $\widetilde{f}$ on $V/\mathcal{P} = \mathbf{S}^2$ satisfying (6). Clearly, (6) is a very strong relation, for, if g is a function of definite parity with respect to the antipodal mapping, i.e., if $g(-\alpha) = (-1)^k \cdot g(\alpha)$, k integer, then, from (6) it follows that $\widetilde{g}(-\alpha) = (-1)^n \cdot g(\alpha) = (-1)^n \cdot (-1)^k \cdot g(-\alpha) = (-1)^k \cdot \widetilde{g}(\alpha)$, so that $\widetilde{g}$ has the same parity as g and furthermore $\widetilde{g} = \pm g$ is such that

$$\widetilde{g} = g \ \Leftrightarrow \ g(\alpha) = (-1)^n \cdot g(-\alpha) \ , \tag{7a}$$

$$\widetilde{g} = -g \ \Leftrightarrow \ g(\alpha) = -(-1)^n \cdot g(-\alpha) \ . \tag{7b}$$

Accordingly, the space of functions on $\mathbf{S}^2$ satisfying (7a) or (7b), respectively, will be called the *n-even* or *n-odd* subspace of $\mathcal{F}un(\mathbf{S}^2)$ and denoted by $\mathcal{F}un^n_+(\mathbf{S}^2)$ or $\mathcal{F}un^n_-(\mathbf{S}^2)$, respectively. For every $n \in \mathbf{Z}^+$, we have $\mathcal{F}un(\mathbf{S}^2) = \mathcal{F}un^n_+(\mathbf{S}^2) \oplus \mathcal{F}un^n_-(\mathbf{S}^2)$.

Thus, if we concentrate on functions of definite parity with respect to the antipodal map, then it is clear from (7) that in practice we associate a single function g to any $\mathcal{P}$-constant section $s : V \to [L]$ of definite parity. Since functions without definite parity can always be decomposed into such, if g_+ or g_- satisfies (7a) or (7b), respectively, then

$$f = g_+ + g_- \ \Leftrightarrow \ \widetilde{f} = g_+ - g_- \ . \tag{8}$$

In this way, we uniquely associate a single pair of n-even and n-odd functions (g_+, g_-) on $\mathbf{S}^2$ to any $\mathcal{P}$-constant section $s : V \to [L]$, with the two choices of associating this pair of functions to a polarized section, as presented above.

Continuing the construction, we now generalize the *composition of central polarized sections*. Remember the sequence of steps: we start with two polarized sections s^1 and s^2, multiply them to get a new unpolarized section $s^1 \circledcirc s^2$, then average over the fibers to get a final central polarized, or $\mathcal{P}$-constant section $< s^1 \circledcirc s^2 >$.

When the association of polarized sections with functions is unique, as in spaces without closed geodesics, this procedure yields a unique skewed product of functions. However, now to any polarized section s^1 we have two choices of association: $s^1(a,b) = f_1(\alpha) \cdot s_0(a,b)$ and $s^1(a,b) = \widetilde{f}_1(-\alpha) \cdot s_1(a,b)$, with f and $\widetilde{f}$ related by (6) and (8).

Thus, the skewed product of functions associated to $< s^1 \circledcirc s^2 >$ can now be written in $2^3 = 8$ ways. But it is not difficult to see that (6) and (8) guarantee that these are all equivalent, in accordance with the uniqueness of $< s^1 \circledcirc s^2 >$.

Eight partial skewed products of functions on the sphere.

We have just seen that if one starts with two central polarized sections $s^1, s^2 : V \to [L]$, their composition is uniquely defined. However, if one starts with two functions f_1, f_2 on the sphere, there would seem to be a greater freedom in obtaining their skewed product.

More specifically, given a generic function f_1 on $\mathbf{S}^2$, we can generically associate two distinct polarized sections $[s^1]_0$ and $[s^1]_1$ from V to $[L]$ by $[s^1]_0(a,b) = f_1(\alpha) \cdot s_0(a,b)$ and $[s^1]_1(a',b') = f_1(\alpha) \cdot s_1(a',b')$. In this context, the two "types" of sections, 0 or 1, actually refer to two distinct central polarized sections. No equivalence condition, at this point.

Therefore, given two generic functions f_1 and f_2 on $\mathbf{S}^2$, there are now 4 distinct central polarized sections to be multiplied. Each of these product sections can be of type 0 or 1 and, since we are now taking the functions themselves to be the basic entities, each type of product section, for each of the 4 products, could define a distinct function on $\mathbf{S}^2$.

It follows that there are, in principle, 8 different ways of obtaining a partial skewed product $f_1 \star f_2$ and that these 8 products of functions are not necessarily the same. Instead, these 8 products divide into 4 groups of "conjugate" pairs of products, which are:

$$\begin{aligned}
&(f_1 \star f_2)_{000} \ \& \ (f_1 \star f_2)_{111} \ , \\
&(f_1 \star f_2)_{001} \ \& \ (f_1 \star f_2)_{110} \ , \\
&(f_1 \star f_2)_{010} \ \& \ (f_1 \star f_2)_{101} \ , \\
&(f_1 \star f_2)_{100} \ \& \ (f_1 \star f_2)_{011} \ ,
\end{aligned} \tag{9}$$

according to the 8 ways of composing polarized sections associated to f_1 and f_2:

$$\begin{aligned}
&< [[s^1]_0 \circledcirc [s^2]_0]_0 > \ \& \ < [[s^1]_1 \circledcirc [s^2]_1]_1 > \ , \\
&< [[s^1]_0 \circledcirc [s^2]_0]_1 > \ \& \ < [[s^1]_1 \circledcirc [s^2]_1]_0 > \ , \\
&< [[s^1]_0 \circledcirc [s^2]_1]_0 > \ \& \ < [[s^1]_1 \circledcirc [s^2]_0]_1 > \ , \\
&< [[s^1]_1 \circledcirc [s^2]_0]_0 > \ \& \ < [[s^1]_0 \circledcirc [s^2]_1]_1 > \ .
\end{aligned} \tag{10}$$

Here, $< [[s^1]_1 \circledcirc [s^2]_0]_1 >$ stands for the following operation: f_1 is associated to a polarized section $[s^1]_1$ while f_2 is associated to $[s^2]_0$; these polarized sections multiply (see [RO]) into an unpolarized section of "type" 1 that is $[[s^1]_1 \circledcirc [s^2]_0]_1$, which after averaging over the fibers becomes a polarized section $< [[s^1]_1 \circledcirc [s^2]_0]_1 >$, to which is associated the function $(f_1 \star f_2)_{101}$. Similarly for all other cases. The "standard" case $< [[s^1]_0 \circledcirc [s^2]_0]_0 >$ with $(f_1 \star f_2)_{000}$ is the case that was considered previously, resulting in equation (1).

Thus, the construction of each "nonstandard" partial product $(f_1 \star f_2)_{\eta\nu\rho}$, where $\eta, \nu, \rho, \in \{0,1\}$, is a natural generalization of the standard one so that it can be written as:

$$(f_1 \star f_2)_{\eta\nu\rho}(m) = \iint_{W_m^{\eta\nu\rho}} f_1(m') f_2(m'') K_{\eta\nu\rho}(m', m'', m) dm' dm'' \ ,$$

where each integral kernel $K_{\eta\nu\rho}(m', m'', m) = A_{\eta\nu\rho}(m', m'', m)\, \mathrm{e}^{inS_{\eta\nu\rho}(m',m'',m)/2}$.

From the definition of the polarized sections s_0 and s_1 it follows that $S_{\eta\nu\rho}(m', m'', m)$ is the symplectic area of the geodesic triangle with midpoints m', m'', m, whose side through m' is short ($< \pi$) if $\eta = 0$ or long ($> \pi$) if $\eta = 1$, and so on for the others.

Similarly, $W_m^{\eta\nu\rho} \subset \mathbf{S}^2 \times \mathbf{S}^2$ is the third restriction (to m) of $W^{\eta\nu\rho} \subset \mathbf{S}^2 \times \mathbf{S}^2 \times \mathbf{S}^2$ and this can be identified as the space of geodesic triangles as determined by the midpoints, whose side through the first midpoint is short ($< \pi$) if $\eta = 0$ or long ($> \pi$) if $\eta = 1$, and so on for the others. Accordingly, $A_{\eta\nu\rho}(m', m'', m)dm'dm''dm$ is the natural volume form on this space, obtained similarly to the standard case.

However, aside from a set of measure zero in $W^{\eta\nu\rho}$, each nonstandard geodesic triangle $\overline{\Delta}^{\eta\nu\rho}(m', m'', m) \in W^{\eta\nu\rho}$ has well defined vertices $a, b, c \in \mathbf{S}^2$ and corresponds uniquely to a standard geodesic triangle $\overline{\Delta}^{000}(\widetilde{m'}, \widetilde{m''}, \widetilde{m})$, where $\widetilde{m'} = m'$ if $\eta = 0$, or $\widetilde{m'} = -m'$ if $\eta = 1$, and so on for the others. It follows that each $W^{\eta\nu\rho}$ is isomorphic to $W^{000} \equiv W$ and that their volume forms are the same, that is, $A_{\eta\nu\rho} \equiv A$ is given by formula (4), because each change $m' \to \widetilde{m'}$, etc, could only multiply A by ± 1, but $A_{\eta\nu\rho}$ is a nonnegative function.

The global skewed product of functions on the sphere.

So, what is to be done of these 8 partial skewed products of functions? At this point, it is important to remember how each partial skewed product is defined:

Starting with two functions on the sphere, each function is associated to a central polarized section and these sections are multiplied into a new section that is *not* polarized. This unpolarized section is associated, not to a function on the sphere, but to a continuous family of functions on the sphere, parametrized by the points on each leaf of $\mathcal{P}$, or in other words, each vector in (a subset of) the tangent space over a point on the sphere.

It is only after *averaging* over this continuous family of functions on the sphere, or in other words, over each leaf of $\mathcal{P}$, or similarly over each (subset of the) tangent space over a point, that a new function on the sphere is defined. In this way, the *averaging procedure* is fundamental to the construction of each partial skewed product of functions.

Now, we have just found out that, starting with two generic function on the sphere, we have further obtained a discrete family of functions on the sphere, according to the eight possible ways of producing a partial skewed product. Therefore, in order to obtain a single function on the sphere, the natural thing to do is to *average* over this family of functions, naturally generalizing the *averaging procedure* to this discrete family.

In order to do this, we start by focusing attention momentarily on the first column of (9) and (10). There, we note that all geodesic triangles involved have at most one side that is long. For such triangles, formula (2) applies directly [RO], so that

$$S_{\eta\nu\rho} \equiv S \ (formula\ (2)) \ , \ \forall(\eta\nu\rho) \in \{(000), (001), (010), (100)\} \ . \tag{11}$$

Furthermore, if $(\eta\nu\rho) \in \{(000), (001), (010), (100)\}$, by direct inspection we also note that the four $W_m^{\eta\nu\rho}$ are mutually disjoint, except for sets of measure zero, and $W_m^{000} \cup W_m^{001} \cup W_m^{010} \cup W_m^{100} = \mathbf{S}^2 \times \mathbf{S}^2$. This is seen from the spherical trigonometric relation $\cos(y_1)/\cos(z_1) = \cos(y_2)/\cos(z_2) = \cos(y_3)/\cos(z_3)$, where y_1 is half the length of side 1 and z_1 is the distance between the midpoints of the other sides, and so on [RO]. Thus, W^{001} is determined by: $\operatorname{sign}\langle m'|m\rangle_E = \operatorname{sign}\langle m''|m\rangle_E = -\operatorname{sign}\langle m'|m''\rangle_E$, and so on.

It follows from all of the above that we can average the four partial products obtained in the left column of (9), associated to the left column of (10), into a single global product defined on $\mathbf{S}^2 \times \mathbf{S}^2$ which we shall denote as $(f_1 \star f_2)_{[\mathbf{0}]}$ and is given by:

$$(f_1 \star f_2)_{[\mathbf{0}]}(m) = \frac{1}{4} \iint_{\mathbf{S}^2 \times \mathbf{S}^2} f_1(m') f_2(m'') A(m', m'', m) \, \mathrm{e}^{inS(m',m'',m)/2} \, dm'dm'' \ ,$$

where S and A are given by formulas (2) and (4), respectively.

This is still only half of the story, of course, because we also have to take into account the right column of (9) and (10). To do so, we must understand precisely the relation between *conjugate* midpoint triangles, that is, geodesic triangles with the same midpoints (which do not belong to a degenerate set).

More explicitly, if $(m', m'', m) \in W^{000}$ determines a unique standard geodesic triangle with vertices (a, b, c), then (m', m'', m) also determine a geodesic triangle with three long sides and vertices $(-a, -b, -c)$. To see this, just prolong the short geodesic $\overline{ab}$ to a long geodesic $\overline{-a-b}$, and so on. The short triangle with vertices (a, b, c) and the long triangle with vertices $(-a, -b, -c)$ have the same midpoints (m', m'', m) so they are conjugate to each other. It follows immediately that $W^{000} \equiv W^{111}$. Similarly, $W^{001} \equiv W^{110}$, $W^{010} \equiv W^{101}$ and $W^{100} \equiv W^{011}$. Thus each pair of conjugate products in each line of (9) is defined in a same domain $W_m^{\eta\nu\rho} \subset \mathbf{S}^2 \times \mathbf{S}^2$, with the same amplitude function A.

As for the holonomy of each composition, since it does not depend on f_1 or f_2 and since $s_1(-a, -b) = (-1)^n \cdot s_0(-a, -b)$, then, starting with $(m', m'', m) \in W^{000}$, clearly the holonomy over the long triangle with vertices $(-a, -b, -c)$ is equal to $(-1)^n$ times the holonomy over the short triangle with these same vertices, whose midpoints are $(-m', -m'', -m)$, which is written as $e^{inS(-m',-m'',-m)/2}$, where S is given by equation (2). But from (2), $\eta(-m', -m'', -m) = \eta(m', m'', m)$ and $det(-m', -m'', -m) = -det(m', m'', m)$, so

$$e^{inS(-m',-m'',-m)/2} = e^{-inS(m',m'',m)/2} \ . \tag{12}$$

Therefore, the holonomy over the long triangle whose midpoints are (m', m'', m) is obtained from the holonomy over the short triangle with the same midpoints by

$$e^{inS_{111}(m',m'',m)/2} = (-1)^n \cdot e^{-inS_{000}(m',m'',m)/2} \ .$$

And the same analysis applies when comparing any pair of conjugate products, i.e., when comparing the right to the left column in each line of (9) or (10), that is:

$$\begin{gathered} e^{inS_{\bar\eta\bar\nu\bar\rho}(m',m'',m)/2} = (-1)^n \cdot e^{-inS_{\eta\nu\rho}(m',m'',m)/2} \ , \\ \eta + \bar\eta = \nu + \bar\nu = \rho + \bar\rho = 1 \ (modulo\ 2) \ . \end{gathered} \tag{13}$$

It follows from the above that we can average the 4 products obtained in the right column of (9), associated to the right column of (10), into another single global product defined on $\mathbf{S}^2 \times \mathbf{S}^2$ which we shall denote as $(f_1 \star f_2)_{[\mathbf{1}]}$ and is given by:

$$(f_1 \star f_2)_{[\mathbf{1}]}(m) = \frac{(-1)^n}{4} \iint_{\mathbf{S}^2 \times \mathbf{S}^2} f_1(m') f_2(m'') A(m', m'', m)\, \mathrm{e}^{-inS(m',m'',m)/2}\, dm' dm'' \ ,$$

where again A and S are given by formulas (4) and (2), respectively.

Note that $(f_1 \star f_2)_{[\mathbf{0}]}$ and $(f_1 \star f_2)_{[\mathbf{1}]}$ are not the same, instead they are related by

$$(f_1 \star f_2)_{[\mathbf{1}]} = (-1)^n \cdot (f_2 \star f_1)_{[\mathbf{0}]} \ . \tag{14}$$

The final step of the construction is, of course, to average the two global products which were obtained by averaging each of the columns in (9). In this way, we arrive at *the global skewed product* of functions on the sphere, denoted by $\star^n$, which is the average over all partial skewed products on the sphere, given by ($\eta, \nu, \rho \in \{0,1\}$):

$$f_1 \star^n f_2 = \frac{1}{8}\sum_{\eta\nu\rho}(f_1 \star f_2)_{\eta\nu\rho} = \frac{1}{2}\left\{ \ (f_1 \star f_2)_{[\mathbf{0}]} + (f_1 \star f_2)_{[\mathbf{1}]} \ \right\} \ . \tag{15}$$

We say that the global skewed product is $\mathbf{Z}_2$-*graded commutative* because

$$f_1 \star^n f_2 = (-1)^n \ f_2 \star^n f_1 \ , \tag{16}$$

as follows immediately from (14) and (15). Our results are summarized below:

Theorem 1: *For the 2-sphere, the partial skewed product of functions given by formula* (1)*, where $n = 2/\hbar \in \mathbf{Z}^+$, with A and S given by formulas* (4) *and* (2)*, respectively, extends naturally to the global skewed product $\star^n$ which is uniquely obtained by averaging over the eight possible partial skewed products of functions on the sphere. The global skewed product is $\mathbf{Z}_2$-graded commutative* (16) *and is given explicitly by ($k \in \mathbf{Z}^+$)*:

$$f_1 \star^{2k} f_2 \ (m) = \iint_{\mathbf{S}^2\times\mathbf{S}^2} f_1(m')f_2(m'')\frac{1}{4}A(m',m'',m)\cos\{kS(m',m'',m)\}dm'dm'' \ , \tag{17a}$$

$$f_1 \star^{2k-1} f_2 \ (m) = \iint_{\mathbf{S}^2\times\mathbf{S}^2} f_1(m')f_2(m'')\frac{i}{4}A(m',m'',m)\sin\{(k-\frac{1}{2})S(m',m'',m)\}dm'dm'' \tag{17b}$$

Skewed products of functions with definite parity.

We now focus attention on functions with definite parity, i.e. n-even functions in $\mathcal{F}un^n_+(\mathbf{S}^2)$ and n-odd functions in $\mathcal{F}un^n_-(\mathbf{S}^2)$, satisfying (7a) and (7b), respectively.

Start with a n-even function $g_1 \in \mathcal{F}un^n_+(\mathbf{S}^2)$. For such a function, $\widetilde{g_1} = g_1$ so that the two polarized sections $[s^1]_0$ and $[s^1]_1$ from V to $[L]$ given by $[s^1]_0(a,b) = g_1(\alpha)\cdot s_0(a,b)$ and $[s^1]_1(a',b') = g_1(\alpha)\cdot s_1(a',b')$ are actually the same. Therefore, if we multiply two such functions, it is particularly important to understand how all partial products relate to each other and to the global product given by formula (17).

It turns out that, in order to compare the partial products, it is necessary to express how the midpoint triangular area $S(m',m'',m)$ changes as some of its arguments are changed under the antipodal mapping. From formula (2), if only one of the arguments is changed (say $(m',m'',m) \mapsto (m',m'',-m)$, for instance), then it is clear that $\eta \mapsto -\eta$ and $det \mapsto -det$, so that $S/2 \mapsto S/2 \pm \pi$ and therefore

$$e^{inS(m',m'',-m)/2} = (-1)^n \cdot e^{inS(m',m'',m)/2} \ . \tag{18}$$

On the other hand, if two of the arguments are changed (for instance, $(m',m'',m) \mapsto (-m',-m'',m)$), then $\eta \mapsto -\eta$ but $det \mapsto det$, so that

$$e^{inS(-m',-m'',m)/2} = (-1)^n \cdot e^{-inS(m',m'',m)/2} \ . \tag{19}$$

Finally, combining these two transformations we recover the case for when the three arguments are changed, as previously discussed, which is given by formula (12).

Now, let's start by comparing $(g_1 \star g_2)_{010}$ to $(g_1 \star g_2)_{000}$, when $g_1, g_2 \in \mathcal{F}un_+^n(\mathbf{S}^2)$. From

$$(g_1 \star g_2)_{010}(m) = \iint_{W_m^{010}} g_1(m')g_2(m'')A(m',m'',m)\, e^{inS(m',m'',m)/2}\, dm'dm'' \ ,$$

we notice that, if $(m',m'') \in W_m^{010}$ then $(m',-m'') \in W_m^{000}$, so that $\iint_{W_m^{010}} dm'dm''... = \iint_{W_m^{000}} dm'd(-m'')...$ and thus, using formulas (7a) and (18), we can write

$$(g_1 \star g_2)_{010}(m) = \iint_{W_m^{000}} g_1(m')g_2(-m'')A(m',-m'',m)\, e^{inS(m',-m'',m)/2}\, dm'd(-m'') \ ,$$

so that $(g_1 \star g_2)_{010} \equiv (g_1 \star g_2)_{000}$. A similar analysis shows that $(g_1 \star g_2)_{100} \equiv (g_1 \star g_2)_{000}$.

We now compare $(g_1 \star g_2)_{001}$ to $(g_1 \star g_2)_{000}$. Starting from

$$(g_1 \star g_2)_{001}(m) = \iint_{W_m^{001}} g_1(m')g_2(m'')A(m',m'',m)\, e^{inS(m',m'',m)/2}\, dm'dm'' \ ,$$

we notice that, if $(m',m'') \in W_m^{001}$ then $(-m',-m'') \in W_m^{000}$, so that $\iint_{W_m^{001}} dm'dm''... = \iint_{W_m^{000}} d(-m')d(-m'')...$ and thus, using formulas (7a) and (19), we write $(g_1 \star g_2)_{001}(m) =$

$$= (-1)^n \cdot \iint_{W_m^{000}} g_1(-m')g_2(-m'')A(-m',-m'',m)\, e^{-inS(-m',-m'',m)/2}\, d(-m')d(-m'') \ ,$$

so that $(g_1 \star g_2)_{001} \equiv (-1)^n \cdot (g_2 \star g_1)_{000}$.

On the other hand, if $(m',m'',m) \in W^{001}$ then $(m',m'',-m) \in W^{000}$, so that $W_m^{001} \equiv W_{-m}^{000}$, hence $\iint_{W_m^{001}} dm'dm''... = \iint_{W_{-m}^{000}} dm'dm''...$ and thus, using formula (18), we write

$$(g_1 \star g_2)_{001}(m) = (-1)^n \cdot \iint_{W_{-m}^{000}} g_1(m')g_2(m'')A(m',m'',-m)\, e^{inS(m',m'',-m)/2}\, dm'dm'' \ ,$$

yielding $(g_1 \star g_2)_{001}(m) = (-1)^n \cdot (g_1 \star g_2)_{000}(-m)$. Combining the two results we have that

$$(g_2 \star g_1)_{000}(m) = (g_1 \star g_2)_{000}(-m) \ , \tag{20}$$

a formula that can also be obtained directly from formulas (1), (7a) and (12).

Now, if we follow through with the averaging procedure, we obtain from formula (13) that $(g_1 \star g_2)_{111} \equiv (-1)^n \cdot (g_2 \star g_1)_{000}$ and, combining with the previous results, we obtain from formula (13) that $(g_1 \star g_2)_{101} \equiv (-1)^n \cdot (g_2 \star g_1)_{010} \equiv (-1)^n \cdot (g_2 \star g_1)_{000}$, also $(g_1 \star g_2)_{011} \equiv (-1)^n \cdot (g_2 \star g_1)_{100} \equiv (-1)^n \cdot (g_2 \star g_1)_{000}$ and finally $(g_1 \star g_2)_{110} \equiv (-1)^n \cdot (g_2 \star g_1)_{001} \equiv (g_1 \star g_2)_{000}$. Therefore, averaging all partial products yields:

$$g_1 \star^n g_2 = \frac{1}{8} \sum_{\eta\nu\rho} (g_1 \star g_2)_{\eta\nu\rho} = \frac{1}{2} \left\{ (g_1 \star g_2)_{000} + (-1)^n \cdot (g_2 \star g_1)_{000} \right\} \tag{21}$$

and it follows immediately from (20) (and (7a)) that $g_1 \star^n g_2 \in \mathcal{F}un^n_+(\mathbf{S}^2)$. Explicitly:

$$g_1 \star^n g_2\ (m) = \frac{1}{2}\left\{(g_1 \star g_2)_{000}(m) + (-1)^n \cdot (g_1 \star g_2)_{000}(-m)\right\}\ , \tag{22}$$

or even more explicitly, with $W_m \equiv W^{000}_m$ given by formula (3) and for $k \in \mathbf{Z}^+$:

$$g_1 \star^{2k} g_2\ (m) = \iint_{W_m} g_1(m')g_2(m'')A(m',m'',m)\cos\{kS(m',m'',m)\}dm'dm''\ , \tag{23a}$$

$$g_1 \star^{2k-1} g_2\ (m) = \iint_{W_m} g_1(m')g_2(m'')iA(m',m'',m)\sin\{(k-\frac{1}{2})S(m',m'',m)\}dm'dm'' \tag{23b}$$

Of course, it also follows directly from formulas (17) and (18) that for any $f_1, f_2 \in \mathcal{F}un(\mathbf{S}^2)$ their global skewed product belongs to $\mathcal{F}un^n_+(\mathbf{S}^2)$. Therefore, we now study the products of functions of definite parity when at least one belongs to $\mathcal{F}un^n_-(\mathbf{S}^2)$.

Start with $g_1, g_2 \in \mathcal{F}un^n_-(\mathbf{S}^2)$. Similar considerations, using formulas (7b), (12), (18) and (19), show that in this case $(g_1 \star g_2)_{010} \equiv -(g_1 \star g_2)_{000}$ and $(g_1 \star g_2)_{100} \equiv -(g_1 \star g_2)_{000}$, while again $(g_1 \star g_2)_{001} \equiv (-1)^n \cdot (g_2 \star g_1)_{000}$. From formula (13) we again obtain that $(g_1 \star g_2)_{111} \equiv (-1)^n \cdot (g_2 \star g_1)_{000}$, while $(g_1 \star g_2)_{101} \equiv (-1)^n \cdot (g_2 \star g_1)_{010} \equiv -(-1)^n \cdot (g_2 \star g_1)_{000}$, also $(g_1 \star g_2)_{011} \equiv (-1)^n \cdot (g_2 \star g_1)_{100} \equiv -(-1)^n \cdot (g_2 \star g_1)_{000}$ and finally $(g_1 \star g_2)_{110} \equiv (-1)^n \cdot (g_2 \star g_1)_{001} \equiv (g_1 \star g_2)_{000}$. It follows that these eight products average to zero, that is, if $g_1, g_2 \in \mathcal{F}un^n_-(\mathbf{S}^2)$ then $g_1 \star^n g_2 \equiv 0$.

If $g_1 \in \mathcal{F}un^n_+(\mathbf{S}^2)$ and $g_2 \in \mathcal{F}un^n_-(\mathbf{S}^2)$, then we obtain that $(g_1 \star g_2)_{010} \equiv -(g_1 \star g_2)_{000}$, but $(g_1 \star g_2)_{100} \equiv (g_1 \star g_2)_{000}$ and $(g_1 \star g_2)_{001} \equiv -(-1)^n \cdot (g_2 \star g_1)_{000}$. Also, $(g_1 \star g_2)_{111} \equiv (-1)^n \cdot (g_2 \star g_1)_{000}$ and $(g_1 \star g_2)_{101} \equiv (-1)^n \cdot (g_2 \star g_1)_{010} \equiv -(-1)^n \cdot (g_2 \star g_1)_{000}$, but $(g_1 \star g_2)_{011} \equiv (-1)^n \cdot (g_2 \star g_1)_{100} \equiv (-1)^n \cdot (g_2 \star g_1)_{000}$ and finally $(g_1 \star g_2)_{110} \equiv (-1)^n \cdot (g_2 \star g_1)_{001} \equiv -(g_1 \star g_2)_{000}$. Thus, again, these eight products average to zero. Similarly if $g_1 \in \mathcal{F}un^n_-(\mathbf{S}^2)$ and $g_2 \in \mathcal{F}un^n_+(\mathbf{S}^2)$. In both cases, $g_1 \star^n g_2 \equiv 0$. In summary:

Theorem 2: *The global skewed product on $\mathcal{F}un(\mathbf{S}^2)$, which is given by formula* (17)*, is nontrivial only as a product of n-even functions: $\mathcal{F}un^n_+(\mathbf{S}^2) \times \mathcal{F}un^n_+(\mathbf{S}^2) \to \mathcal{F}un^n_+(\mathbf{S}^2)$, so that $\mathcal{F}un(\mathbf{S}^2) \times \mathcal{F}un^n_-(\mathbf{S}^2) \to 0$ and $\mathcal{F}un^n_-(\mathbf{S}^2) \times \mathcal{F}un(\mathbf{S}^2) \to 0$. Furthermore, on $\mathcal{F}un^n_+(\mathbf{S}^2) \subset \mathcal{F}un(\mathbf{S}^2)$, the subspace of functions satisfying* (7a)*, the global skewed product coincides with the restricted skewed product given by formulas* (21), (22) *and* (23)*, which is the $\mathbf{Z}_2$-graded commutative version of the partial skewed product given by formula* (1).

Generalized skewed products of functions on the sphere.

We have seen that although the composition of central polarized sections is uniquely defined, the corresponding skewed product of functions seemed to be non unique. This apparent lack of uniqueness of the skewed product would originate in the lack of uniqueness in associating, to each function, a central polarized section. Thus, in order to obtain a unique skewed product of functions we averaged over all possibilities, in line with the averaging procedure that was used to define each partial skewed product. In so doing, we have obtained a global skewed product which is $\mathbf{Z}_2$-graded commutative.

However, for functions in $\mathcal{F}un^n_+(\mathbf{S}^2)$, that is, functions of definite parity satisfying formula (7a), there is no lack of uniqueness in associating, to each function, a central polarized section. Therefore, it would be only natural that their skewed product should be uniquely associated to a central polarized section, in other words, that their skewed product should also belong to $\mathcal{F}un_+^n(\mathbf{S}^2)$. This is guaranteed by averaging, for the composition of central polarized sections, the two ways of associating a function to the composed section, according to formula (6). This is exactly what is explicitly stated by formula (22), which, furthermore, asserts the equivalence between all other pairs of skewed products and, specially, the equivalence of any well-defined skewed product on $\mathcal{F}un_+^n(\mathbf{S}^2)$ to the global skewed product of functions, which, for all practical purposes, is also defined only on $\mathcal{F}un_+^n(\mathbf{S}^2)$.

In this way, we have seen that the two ways of assuring that the skewed product of functions on the sphere is unique and well defined, turn out to be the same.

Furthermore, we note that the specific form of the amplitude function (formula (4)) is not relevant to the $\mathbf{Z}_2$-graded commutativity of the skewed product of functions on the sphere. As long as the amplitude function is a real nonnegative symmetric function (formula (5)) which is invariant under the $SU(2)$ diagonal action on $\mathbf{S}^2\times\mathbf{S}^2\times\mathbf{S}^2$, it is the relation between holonomies over conjugate midpoint triangles (formula (13)) that implies the skewed product to be nontrivial only on $\mathcal{F}un_+^n(\mathbf{S}^2)$ and to be $\mathbf{Z}_2$-graded commutative.

Now, if another integral product of functions on $\mathbf{S}^2$ is to be defined so that the amplitude function is different from the one defining the skewed product (and could depend on n, but not exponentially, say, polinomially in $1/n$), but such that the phase of the oscillatory function is defined via midpoint triangles on $\mathbf{S}^2$, then, since for any triple of points in $\mathbf{S}^2$ there is no a priori reason to choose one midpoint triangle over another and so all midpoint triangles should be considered equally, this implies the more general result stated below:

Corollary 3: *Given $n\in\mathbf{Z}^+$, the skewed product $\star^n$ on $\mathcal{F}un(\mathbf{S}^2)$ is well defined only because it is $\mathbf{Z}_2$-graded commutative, $f\star^n g=(-1)^n g\star^n f$, and is nontrivial only as a product on the subspace $\mathcal{F}un_+^n(\mathbf{S}^2)$ of functions satisfying $f(m)=(-1)^n f(-m)$, where $-m$ is antipodal to m. This is also true for any generalized skewed product $\tilde{\star}^n$ of the form $f\ \tilde{\star}^n g\ (m)=\iint_{\mathbf{S}^2\times\mathbf{S}^2}\tilde{K}_n(m,m',m'')f(m')g(m'')dm'dm''$, where the integral kernel $\tilde{K}_n$ is symmetric (formula (5)) and $SU(2)$ invariant, of the form $\tilde{K}_n=\tilde{A}_n\exp(in\mathcal{S}/2)$, where $\tilde{A}_n$ is a nonnegative real function on $\mathbf{S}^2\times\mathbf{S}^2\times\mathbf{S}^2$ (possibly expanded in powers of $1/n$) and $\mathcal{S}(m,m',m'')$ is the symplectic area of a geodesic triangle with (m,m',m'') as midpoints. Because $\exp(in\mathcal{S}/2)$ is double valued (formula (13)) and counting all possibilities equally, then $\tilde{K}_{2k}=\tilde{A}_{2k}\cos(kS)$ and $\tilde{K}_{2k-1}=i\tilde{A}_{2k-1}\sin((k-\frac{1}{2})S)$, with S given by formula (2), which implies that $\tilde{\star}^n$ is nontrivial only on $\mathcal{F}un_+^n(\mathbf{S}^2)$ and is $\mathbf{Z}_2$-graded commutative.*

Discussion

Formal star products on the sphere are not $\mathbf{Z}_2$-graded commutative, neither commutative [MO][Md]. However, an old result by Rieffel [Rf], based on a theorem by Wassermann [Wa], asserts that any associative $SU(2)$-equivariant *strict* deformation of the pointwise product of functions on the sphere must be commutative.

A $SU(2)$-equivariant formal star product on the sphere does not satisfy Rieffel's strict deformation conditions because it is a product of formal power series in $\hbar$ with coefficients in the space of smooth functions on the sphere, which, however, generally does not converge as power series and, therefore, is not defined as a product in any infinite dimensional normed $\mathcal{C}^*$-algebra which deforms the usual $\mathcal{C}^*$-algebra of functions on the sphere.

On the other hand, for any $n = 2/\hbar \in \mathbf{Z}^+$, the skewed product is, in principle, defined as a product on some infinite dimensional subspace of the space of *functions* on the sphere, $\mathcal{F}un(\mathbf{S}^2)$, which, in principle, could be identified as the same space for when $n = \infty$ ($\hbar = 0$). However, the skewed product on $\mathcal{F}un(\mathbf{S}^2)$ is $\mathbf{Z}_2$-graded commutative and is nontrivial only as a product on $\mathcal{F}un^n_+(\mathbf{S}^2)$. So, how does it relate to Rieffel's theorem?

Well, although it is $\mathbf{Z}_2$-graded commutative, it is a deformation (n close to ∞) of the pointwise product of functions on the sphere only when it is commutative ($n = 2k$, $k \in \mathbf{Z}^+$), in which case it is a deformation of the usual $\mathcal{C}^*$-subalgebra of functions on the sphere satisfying $f(m) = f(-m)$, where $-m$ is antipodal to m.

In other words, the skewed product of functions on the sphere seems to provide an explicit example of Rieffel's theorem for strict deformation of the pointwise product of functions on $\mathbf{S}^2$, for functions satisfying $f(m) = f(-m)$. Moreover, the skewed product of functions on $\mathbf{S}^2$ seems to provide a $SU(2)$-equivariant "anticommutative strict deformation" of the Poisson bracket of functions on $\mathbf{S}^2$, for functions satisfying $f(m) = -f(-m)$, given some appropriate definition of "strict deformation" of anticommutative products.

Not so fast, one should argue, because we have not studied the convergence of the skewed product and identified an appropriate infinite dimensional subspace of $\mathcal{F}un^n_+(\mathbf{S}^2)$ where the product converges for every (even or odd) n. And this must be done, for the skewed product.

However, in this respect we have much extra freedom, because, if we approach this problem in the context of generalized skewed products, as in Corollary 3, we are allowed to modify the amplitude function in great generality so as to define a rich infinite dimensional subspace of the space of n-even functions on $\mathbf{S}^2$ in which a generalized skewed product converges for every (even or odd) n. That is, with so much freedom, it is natural to assume that some generalized skewed products are, either for every even, or for every odd n, analytically well defined as products on infinite dimensional subspaces of the space of functions on the sphere satisfying either $f(m) = f(-m)$, or $f(m) = -f(-m)$, respectively.

But, one should still question, how about associativity? Again, if we look at this question in the context of generalized skewed products, then, for the commutative product, we could question whether there are some amplitude functions $\tilde{A}_n$ for which each respective convergent product is associative for every even n. But, for the anticommutative product, associativity makes no sense and it should translate, instead, into a question about the Jacobi identity for some $\tilde{A}_n$, for every odd n. But now these questions are not so simple, because they may be tied up with the determination of the function spaces. And, of course, these questions are not simpler in the context of *the* skewed product, properly.

However, we remind that associativity, Jacobi identity, or other algebraic properties play no direct role in the $\mathbf{Z}_2$-graded commutativity of the skewed product, or generalized skewed products. Therefore, the $\mathbf{Z}_2$-graded commutativity of generalized skewed products proposes that Rieffel's obstruction to $SU(2)$-equivariant strict deformation quantization is more general in scope. In other words, it states that more general $SU(2)$-equivariant products that deform the pointwise product in an infinite dimensional space of functions on the sphere are commutative. Likewise, that more general $SU(2)$-equivariant products that deform the Poisson bracket in an infinite dimensional space of functions on the sphere are anticommutative. In this respect, $\mathbf{Z}_2$-graded commutativity being independent of associativity or Jacobi identity, the investigation of these or other related algebraic properties of the skewed product, or generalized skewed products, can be carried out independently.

On the other hand, how relevant is the special form of integral kernel for the skewed product, or a generalized skewed product as in Corollary 3? Well, we argue that this form of integral product is very relevant in the limit $n \to \infty$.

First, if we picture this limit as the limit of spheres with smaller and smaller curvatures (for an identification of riemannian curvature as proportional to $1/n$), we should expect that an integral product of functions on the sphere, with symmetric kernel, should get more and more similar to the von Neumann - Groenewold product of functions on $\mathbf{R}^2$.

Second, regardless of pictures, in this limit $n \to \infty$, classical data should become predominant, so, for instance, one expects that stationary phase evaluation of an integral product of two oscillatory functions f and g with phases ϕ and γ, respectively, should yield a new oscillatory function h whose phase η is, at least to lowest order in $1/n$, related to ϕ and γ by the rule of composition of central generating functions on the sphere [RO], just as happens on $\mathbf{R}^{2n}$. And so on, for multiple products one expects the corresponding classical composition of spherical midpoint triangles into spherical midpoint quadrilaterals, etc [RO].

So, it is reasonable to assume that $SU(2)$-equivariant symmetric integral products on an infinite dimensional space of functions on the sphere should take the form of a generalized skewed product, as $n \to \infty$. And this provides a framework for possible strict deformation quantizations of $\mathbf{S}^2$. Therefore, since we can assume that a $SU(2)$-equivariant strict deformation quantization of the sphere should be expressible as a generalized skewed product in the limit $n \to \infty$ and since it is the relation between the areas of conjugate midpoint triangles, as given by formula (13), that is responsible for the $\mathbf{Z}_2$-graded commutativity of generalized skewed products, we deduce that the relation between the areas of conjugate midpoint triangles presents a simple geometrical explanation for Rieffel's theorem on the obstruction to $SU(2)$-equivariant strict deformation quantization of the sphere.

Furthermore, this theorem extends to a proposition on the commutativity of more general deformations of the pointwise product and the anticommutativity of more general deformations of the Poisson bracket, on infinite dimensional spaces of functions on the sphere.

On the other hand, one could feel tempted to deduce from all this that we should forget about products on infinite dimensional spaces of functions on the sphere and focus on formal deformation quantization of $\mathbf{S}^2$, or twisted products of spherical symbols, only.

However, in the case $M = \mathbf{R}^{2n}$, the skewed product coincides with the twisted product and implies deformation quantization, for admissible symbols (the Moyal product). Thus, which product (if any) is to be given preference when $M \neq \mathbf{R}^{2n}$ and the products differ considerably? We have seen in this paper that for $M = \mathbf{S}^2$ the skewed product differs considerably from any product in the context of formal deformation quantization, but is in line with Rieffel's obstruction to any $SU(2)$-equivariant strict deformation quantization of the sphere. On the other hand, both are products motivated by mutually compatible products for $M = \mathbf{R}^{2n}$, and both are inspired by semiclassical ($n \to \infty$) considerations.

Work is in progress on the relationship between the skewed product of functions on the sphere and twisted products on *finite* dimensional spaces of spherical symbols [Bn][VG], which are products within the context of a quantum theory, properly.

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