MOVING FRAMES FOR COTANGENT BUNDLES*

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Cartan's moving frames method is a standard tool in Riemannian geometry. We set up the machinery for applying moving frames to cotangent bundles and its sub-bundles defined by nonholonomic constraints.

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1. Introduction

This paper has a very modest scope: we present our "operational system" for Hamiltonian mechanics on cotangent bundles $M = T^*Q$, based on moving frames. In a related work [10], we present some concrete examples to convey the algorithmical nature of this formalism.

A powerful tool in Riemannian geometry is the "method of moving frames", introduced by Élie Cartan. However, it actually appeared earlier in Lagrangian

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mechanics (Poincaré, 1901)[1], later referred as the "quasi-coordinates" method. Cartan himself advocated applying moving frames in mechanics [5], in particular using his equivalence method. See [9] for a modern exposition of Cartan's paper.

When we use a moving frame and its dual coframe, the canonical symplectic form Ω on T^*Q deviates from the Darboux format. This is not bad: we use this feature to encode information about the system.

Moving frames are natural when dealing with Lie groups and with constrained systems, either *vakonomic* or *nonholonomic* (see [1] for background). Linear constraints define a distribution $\mathcal{E}$ of s-dimensional planes $E_q \subset T_qQ$, where Q is an n-dimensional configuration space, $s < n$.

2. Basic formalism

2.1. Coframe coordinates for T^*Q

Let $\{\epsilon_I = a_{IK}\, dq_K, I = 1, \dots, n\}$ a local coframe on Q. We denote by $\{e_J = b_{LJ}\partial/\partial q_L\}$ the dual frame, defined by $\epsilon_I(e_J) = \delta_{IJ}$. The matrices A and B are inverses.

DEFINITION 1. We call *quasi-velocities* (respectively, *quasi-momenta* [2]) the coordinates (u, q) on TQ (respectively (m, q) on T^*Q) defined by

$$u_q = u_I\, e_I\,, \qquad p_q = m_I \epsilon_I\,. \tag{1}$$

Rules of transformation are readily obtained:

$$p_J = m_I\, \epsilon_I\Big(\frac{\partial}{\partial q_J}\Big) = m_I a_{IJ}\,, \qquad m_J = p_I\, dq_I(e_J) = p_I b_{IJ}\,. \tag{2}$$

It is easy to write ω in terms of the trivialization (m, q) of T^*Q,

$$\omega = pdq = m_I \epsilon_I\,. \tag{3}$$

This is the "canonical misunderstanding": the expression $m_I\epsilon_I$ is now a 1-form on T^*Q in coordinates (m, q). The same expression (see (1)) denotes an element $p_q = m\epsilon(q) = m_I\epsilon_I(q) \in T^*_qQ$. (We use heavier notation when we feel necessary. We could add a superscript # when thinking of $\epsilon_J^{\#}$ either as a 1-form on Q, or its pullback to T^*Q. For the latter a double superscript could be used. However,

[1] We thank Larry Bates (personal communication): moving frames were introduced by Euler [7]. Certainly moving frames were understood by the caveman who invented the wheel.

[2] "Quasi-momenta" can be abbreviated without guilt by "momenta": the angular momenta m_I correspond to e_I = infinitesimal rotations in $\mathbb{R}^3$.

we will try to keep the notation as simple as possible.) The basic idea of this work is to write the canonical 2-form in a non-Darboux format. The following is obvious and will be explored in Theorem 2.

THEOREM 1. *The canonical 2-form in T^*Q writes as*

$$\Omega = d\omega = dp \wedge dq = dm_I \wedge \epsilon_I + m_I d\epsilon_I\,. \tag{4}$$

2.2. Earnest coordinate vector fields and coframes

We associate to the local trivialization (m, q), where $p_q = m_I \epsilon_I$, the lifted coframe for T^*Q given by

$$\{\epsilon_I\,,\, dm_I\}\,. \tag{5}$$

We will now describe the corresponding dual basis of vector fields on T^*Q. *It turns out that it is not* $\{e_I\,,\, \partial/\partial m_I\}$. In the correct version, the first set will acquire a fiber component, and will be denoted e_I^*.

DEFINITION 2. We call *earnest coordinate vector fields* for T^*Q the coordinate frame associated to the parameterization (m, q):

$$X^\epsilon_{q_I} = \frac{\partial}{\partial q_I}_{|(m\,\text{fixed})}\,, \qquad \frac{\partial}{\partial m_I} \equiv \epsilon_I\,. \tag{6}$$

These vector fields are dual to the forms $\{dq_I\,,\, dm_I\}$, the differentials of the coordinate functions. The identification $\partial/\partial m_I \equiv \epsilon_I(q)$, a vertical vector field in T^*Q, is the usual identification of a vector space with its tangent space (here, $T_{p_q}(T_q^*Q) \equiv T_q^*Q$).

We claim that denoting $\partial/\partial q_I$ without subscript, is misleading. The vector fields $X^\epsilon_{q_I} = \partial/\partial q_I|_{(m\,\text{fixed})}$ and $\partial/\partial q_{I|(p\,\text{fixed})}$ are different! Throughout this work we reserve unsubscripted notation $\partial/\partial q_I$ for the vector field corresponding to the *standard* coordinates (p, q) for T^*Q. Thus we write $e_J = b_{LJ}\partial/\partial q_L$ thinking of it as a vector field in T^*Q, assuming the standard (p, q) parameterization.

In fact, we must go back to a standard "Advanced Calculus" class. If (q, p) and (q, m) are two sets of coordinates on a fibered manifold, the notation $\partial/\partial q_I$ in the two coordinate systems is ambiguous: they differ by a vertical component[3]. This could be surprising at first sight since the forms dq_I in the coframes $\{dq_I, dp_I\}$ and $\{dq_I, dm_I\}$ are the same. They are simply the differentials of the functions

[3]Differential forms are more reliable than vector fields in this regard. Perhaps this is another "feminine" property of forms. Prof. S. S. Chern insists that forms are of feminine gender, vectors masculine.

$q_I \circ \pi : T^*Q \to \mathbb{R}$ ($\pi : T^*Q \to Q$ is the bundle projection and $q_I : Q \to \mathbb{R}$ is the I-th coordinate function).

We introduce matrix notation. We write the (dual) pair frame-coframe in Q as a row array and column array, respectively:

$$e = (e_1, \dots, e_n), \qquad \epsilon = \begin{pmatrix} \epsilon_1 \\ \cdots \\ \epsilon_n \end{pmatrix}, \quad (\epsilon \cdot e = I_n). \tag{7}$$

Write

$$\epsilon_I = a_{IJ} dq_J, \quad \text{that is,} \quad \epsilon = A dq, \quad A = (a_{IJ}), \tag{8}$$

and we recall

$$(e_1, \dots, e_n) = (\partial/\partial q_1, \dots, \partial/\partial q_n) B, \qquad B = A^{-1}. \tag{9}$$

Then $m_I \epsilon_I = p_J dq_J$ implies (as we already saw) $p_J = m_I a_{IJ}$.

LEMMA 1. (The importance of being earnest). *Assume ϵ and dq related by* (8). *The corresponding coframes in T^*Q are related by*

$$\begin{pmatrix} dq_I \\ dp_I \end{pmatrix} = \begin{pmatrix} I & 0 \\ \Lambda & A^\dagger \end{pmatrix} \begin{pmatrix} dq_J \\ dm_J \end{pmatrix}, \tag{10}$$

where

$$\Lambda_{IJ} = m_K \partial a_{KI} / \partial q_J. \tag{11}$$

The corresponding dual frames in T^*Q are related by

$$\left(X^\epsilon_{q_J} \, \partial/\partial m_J \right) = (\partial/\partial q_I \, \partial/\partial p_I) \cdot \begin{pmatrix} I & 0 \\ \Lambda & A^\dagger . \end{pmatrix} \tag{12}$$

Explicitly,

$$X^\epsilon_{q_J} = \partial/\partial q_J + m_K (\partial a_{KI}/\partial q_J) \frac{\partial}{\partial p_J}, \qquad \partial/\partial m_J = a_{JI} \partial/\partial p_I. \tag{13}$$

Summarizing: the vector fields $X^\epsilon_{q_I}$ and $\partial/\partial q_I$ are different, however, their difference is a vertical vector field, their projections over TQ by $\pi_* : T(T^*Q) \to TQ$ coincide. We say that $X^\epsilon_{q_I}$ acquires a *spiritual component* relative to the standard coordinates (p, q).

2.3. Extended frame $\{e_I^*, \frac{\partial}{\partial m_I}\}$ for $T(T^*Q)$ and coframe $\{\epsilon_I, dm_I\}$ for $T^*(T^*Q)$

We now change part of the coordinate basis $X^{\epsilon}_{q_I}$ to vectors e_I^*. The superscript $*$ is a reminder that $e_I^* \in T(T^*Q)$, not TQ, and also a reminder that it has a spiritual component. A simple computation gives the following result.

LEMMA 2.

$$\begin{pmatrix} dq_I \\ dp_I \end{pmatrix} = \begin{pmatrix} B & 0 \\ \Lambda B & A^\dagger \end{pmatrix} \begin{pmatrix} \epsilon_L \\ dm_L \end{pmatrix}, \tag{14}$$

Dualizing, we get

$$(e_J^* \ \partial/\partial m_J) = (\partial/\partial q_I \ \partial/\partial p_I) \cdot \begin{pmatrix} B & 0 \\ \Lambda B & A^\dagger \end{pmatrix}. \tag{15}$$

In short, the transformation rules for the moving frame in T^*Q are given (in shorthand notation), in terms of the standard coordinates (p, q) by:

$$e^* = \partial/\partial q B + \partial/\partial p \Lambda B = e + \partial/\partial p \Lambda B, \tag{16}$$

$$\partial/\partial m = \partial/\partial p A^\dagger, \qquad (\text{equivalently} \quad \epsilon = A dq). \tag{17}$$

The last equality is due to the identifications $\partial/\partial m_I = \epsilon_I$, $\partial/\partial p_I = dq_I$. The extended moving coframe in $T^*_{p_q}(T^*Q)$ is ϵ_I, dm_I, dual to $e_I^*, \partial/\partial m_I \in T_{p_q}(T^*Q)$. The importance of being earnest: the frames $\{e_I\}$ and $\{\epsilon_J\}$ are dual in $V = T_qQ, V^* = T_q^*Q$. The frames $\{\frac{\partial}{\partial m_I}\}$ and $\{dm_J\}$ are dual in $W = T_q^*Q, W^* = (T_q^*Q)^*$, but $\{e_I, \frac{\partial}{\partial m_I}\}$ and $\{\epsilon_J, dm_J\}$ are *NOT* dual in $T_{(p,q)}T^*Q, T^*_{(p,q)}T^*Q$. The basic reason is that $T_{(p,q)}T^*Q \neq T_qQ \times T_q^*Q$.

3. Symplectic form in $\{e^*, \partial/\partial m\}$ and Poisson brackets in $\{\epsilon, dm\}$

After this quite dull preparation, we are finally able to write down a more interesting formula.

THEOREM 2. *In the basis $\{e^*, \partial/\partial m\}$, the canonical symplectic form $\Omega = dp \wedge dq = dm_I \wedge \epsilon_I^{\#} + m_I d\epsilon_I^{\#}$ becomes*

$$[\Omega]_{\{e^*, \partial/\partial m\}} = \begin{pmatrix} E & -I \\ I & 0 \end{pmatrix} \tag{18}$$

with

$$E_{JK} = m_I d\epsilon_I(e_J, e_K) = -m_I \epsilon_I [e_J, e_K]. \tag{19}$$

Proof: We use Theorem 1 and Cartan's magic formula for differentiating 1-forms[4]. By duality, the first term $dm_I \wedge \epsilon_I^\#$ yields a familiar matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The "magnetic block" E (E for Euler) results from employing Cartan's formula

$$\begin{aligned} d\epsilon_I^\#(e_J^*, e_K^*)\,(\text{in}\,T^*Q) &= d\epsilon_I(e_J, e_K)\,(\text{in}\,Q) \\ &= e_J\epsilon_I(e_K) - e_K\epsilon_I(e_J) - \epsilon_I[e_J, e_K] \end{aligned} \tag{20}$$

and we observe that the first two terms vanish. □

As the Poisson structure is a skew-symmetric tensor of type $(0,2)$, it operates on two elements of $T^*_{p_q}(T^*Q)$. It is natural to use the basis $\{\epsilon_I\,,\, dm_I\}$.

THEOREM 3. *The Poisson bracket matrix relative to* ϵ_I, dm_I *is*

$$[\Omega]^{-1} = [\Lambda] = \begin{pmatrix} 0_n & I_n \\ -I_n & E \end{pmatrix}. \tag{21}$$

Equivalently,

$$\Lambda = \sum_I e_I^* \wedge \frac{\partial}{\partial m_I} + \sum_{1\le J<K\le n} E_{JK}\,\frac{\partial}{\partial m_J} \wedge \frac{\partial}{\partial m_K}. \tag{22}$$

We now observe that

$$\Lambda = \sum_I e_I^* \wedge \frac{\partial}{\partial m_I} - \frac{1}{2}\sum_{1\le I,J\le n} E_{IJ}\,\frac{\partial}{\partial m_J} \wedge \frac{\partial}{\partial m_I} = \tilde{e}_I \wedge \frac{\partial}{\partial m_I},$$

where

$$\tilde{e}_I = e_I^*(q) - \frac{1}{2}E_{IJ}\frac{\partial}{\partial m_J}.$$

The Poisson $(0,2)$ tensor can also be written as

$$\Lambda = \sum_I \frac{\partial}{\partial q_I} \wedge \frac{\partial}{\partial p_I} = \sum_I e_I \wedge \frac{\partial}{\partial m_I}. \tag{23}$$

[4] Cartan's formula is the deepest fact used in this paper.

The last equality is ridiculous. As $e_I = b_{JI}\partial/\partial q_J$, $\frac{\partial}{\partial m_I} = \frac{\partial}{\partial p_J} a_{IJ}$ (see (17)) and since $A = B^{-1}$ we have

$$\sum_I e_I \wedge \frac{\partial}{\partial m_I} = b_{JI}\frac{\partial}{\partial q_J} \wedge \frac{\partial}{\partial p_K} a_{KI} = a_{IK}\, b_{JI}\,\frac{\partial}{\partial q_J} \wedge \frac{\partial}{\partial p_K}$$
$$= \delta_{KJ}\,\frac{\partial}{\partial q_J} \wedge \frac{\partial}{\partial p_K} = \frac{\partial}{\partial q_J} \wedge \frac{\partial}{\partial p_J}.$$

Thus one could guess that $\tilde{e}_I$ equals e_I, but a brute force calculation gives

$$\tilde{e}_I = e_I + m_K \frac{\partial a_{KL}}{\partial q_R}(b_{RI}b_{LJ} + b_{RJ}b_{LI})\frac{\partial}{\partial m_J}\,. \tag{24}$$

The second term will not contribute when wedging with $\partial/\partial m_I$ and performing the summation.

4. Examples

4.1. Lie groups and KAKS bracket

Let the configuration space be a Lie group $Q = G$, e_I and ϵ_I dual left-invariant vector fields and forms. Let the structure constants be defined by $[e_J, e_K] = c^I_{JK}\, e_I$. Then

$$E_{JK} = m_I d\epsilon_I(e_J, e_K) = -m_I \epsilon_I[e_J, e_K] = -m_I c^I_{JK} \tag{25}$$

does not depend on $g \in G$. Write

$$V^a = X^a_J e^*_J + z^a_J \epsilon_J \in T_{p_g}(T^*G)\,, \qquad a = 1, 2,$$

so

$$\Omega(V^1, V^2) = (X^1, z^1)\begin{pmatrix} E & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} X^2 \\ z^2 \end{pmatrix} = X^2 z^1 - X^1 z^2 + X^1 E X^2\,. \tag{26}$$

We denote $X^a_J\, e_J(\mathrm{id}) = L_{g^{-1}}(\pi_*\, V^a_{p_g})$ simply as $X^a \in \mathcal{G}$ and therefore

$$X^1 E X^2 = -m_I \epsilon_I(g)[X_1, X_2]_g = -(L_g)^*(p_g)[X_1^{\mathrm{left}}, X_2^{\mathrm{left}}]_{\mathrm{id}}\,. \tag{27}$$

What if we replace left by right-invariant vector fields f_I and forms θ_I? The basic formula stays the same,

$$\Omega(V_1, V_2) = X^2 z^1 - X^1 z^2 + X^1 E X^2\,,$$

but now $X^a_J f_J(\mathrm{id}) = R_{g^{-1}}(\pi_* V^a_{p_g})$ and

$$X^1 E X^2 = -m_I \theta_I(g)[X_1, X_2]_g = -(R_g)_*(p_g)[X_1^{\text{right}}, X_2^{\text{right}}]_e\,.$$

where

$$[X_1^{\text{right}}, X_2^{\text{right}}]_e = -c^I_{JK}\, X^1_J X^2_K\, f_I\,. \tag{28}$$

Notice the extra minus sign arising from the Lie bracket structure. Here we used the well known Lie-group fact: *if one extends vectors in $\mathcal{G}$ right invariantly the structure coefficients in the Lie bracket appear with opposite sign.*

Eqs. (26) and (27) lead to the KAKS (Kirillov–Arnold–Kostant–Souriau) bracket in the dual Lie algebra $\mathcal{G}^*$ found independently by S. Lie [13].

The commutation relations for the forms ϵ_I, dm_I in $T^*_{u\cdot\epsilon}(T^*G)$ are given by

$$\{\epsilon_I, \epsilon_J\} = 0\,, \qquad \{dm_I, \epsilon_J\} = \delta_{IJ}\,, \qquad \{dm_I, dm_J\} = E_{IJ} = -m_K c^K_{IJ}\,. \tag{29}$$

The last commutation formula implies for $f, g : \mathcal{G}^* \to \mathbb{R}$, that at $\mu \in \mathcal{G}^*$,

$$\{f\,,\, g\}\,(\mu) = -\mu\left[\frac{\delta}{\delta f}, \frac{\delta}{\delta g}\right], \tag{30}$$

where $df(\mu) \in T^*_\mu \mathcal{G}^*$ is identified with $\frac{\delta}{\delta f}(\mu) \in \mathcal{G}$.

4.2. Principal bundles with connection

We use heretofore the following convention: capital roman letters I, J, K, etc., run from 1 to n. Lower case roman characters i, j, k run from 1 to s. Greek characters α, β, γ, etc., run from $s+1$ to n.

Let $\pi : Q^n \to S^s$ be a principal bundle with Lie group G^r, where $r = n - s$. For definiteness, we take G acting on the left. Fix a connection $\lambda = \lambda(q) : T_qQ \to \mathcal{G}$ defining a G-invariant distribution $\mathcal{E}$ of horizontal subspaces. Denote by $K(q) = d\lambda \circ \mathrm{Hor} : T_qQ \times T_qQ \to \mathcal{G}$ the curvature 2-form (which is, as well known, Ad-equivariant).

Choose a local frame $\overline{e}_i$ on S. For simplicity, we may assume that

$$\overline{e}_i = \partial/\partial s_i \tag{31}$$

are the coordinate vector fields of a chart $s : S \to \mathbb{R}^s$.

Let $e_i = h(\overline{e}_i)$ be the horizontal lift to Q. We complete to a moving frame on Q with vertical vectors e_α which we will specify in a moment. The dual basis will be denoted $\epsilon_i, \epsilon_\alpha$ and we write $p_q = m_i\epsilon_i + m_\alpha\epsilon_\alpha$. These are in a sense the "least moving" among all the moving frames adapted to this structure. We now describe what the $n \times n$ matrix $E = (E_{IJ})$ looks like in this setting.

i) The $s \times s$ block (E_{ij}). Decompose $[e_i, e_j] = h[\bar{e}_i, \bar{e}_j] + V[e_i, e_j] = V[e_i, e_j]$ into vertical and horizontal parts. The choice (31) is convenient, since $\bar{e}_i$ and $\bar{e}_j$ commute, $[e_i, e_j]$ is vertical. Hence

$$E_{ij} = -p_q[e_i, e_j] = -m_\alpha \epsilon_\alpha [e_i, e_j] \,. \tag{32}$$

Now by Cartan's rule,

$$K(e_i, e_j) = e_i \lambda(e_j) - e_j \lambda(e_i) - \lambda[e_i, e_j] = -\lambda[e_i, e_j] \in \mathcal{G} \,.$$

Thus we have shown that

$$[e_i, e_j]_q = -K(e_i, e_j) \cdot q \tag{33}$$

Moreover, let $J : T^*Q \to \mathcal{G}^*$ be the momentum mapping. We have

$$(J(p_q), K_q(e_i, e_j)) = p_q \left(K(e_i, e_j).q \right) = -p_q[e_i, e_j] \ \ (= E_{ij}) \,.$$

THEOREM 4. (The J. K. formula)

$$E_{ij} = (J(p_q), K_q(e_i, e_j)) \,. \tag{34}$$

This gives a nice description for this block, under the choice $[\bar{e}_i, \bar{e}_j] = 0$. Notice that the functions E_{ij} depend on s and the components m_α, but do not depend on g. This is because the Ad*-ambiguity of the momentum mapping J is cancelled by the Ad-ambiguity of the curvature K.

ii) The $r \times r$ block $(E_{\alpha\beta})$. Choose a basis X_α for $\mathcal{G}$. We take $e_\alpha(q) = X_\alpha \cdot q$ as the vertical distribution. Choosing a point q_o allows for the identification of the Lie group G with the fiber containing Gq_o, where id $\mapsto q_o$. Through the mapping $g \in G \mapsto gq_o \in Gq_o$, the vector field e_α is identified to a *right* (not left!) invariant vector field in G. Thus the commutation relations for the e_α are as in (28) so that $[e_\alpha, e_\beta] = -c^\gamma_{\alpha\beta}\, e_\gamma$ appears with a minus sign. Therefore

$$E_{\alpha\beta} = m_\gamma c^\gamma_{\alpha\beta} \,. \tag{35}$$

iii) The $s \times n$ block $(E_{i\alpha})$. The vectors $[e_i, e_\alpha]$ are vertical, but their values depend on the specific principal bundle one is working with. Given a section

$\sigma : U_S \to Q$ over the coordinate chart $s : U_S \to \mathbb{R}^m$ on S, we need to know the coefficients $b^\gamma_{i\alpha}$ in the expansion

$$[e_i, e_\alpha](\sigma(s)) = b^\gamma_{i\alpha}(s)\, e_\gamma\,.$$

Then

$$E_{i\alpha}(\sigma(s)) = -m_\gamma\, b^\gamma_{i\alpha}(s)\,. \tag{36}$$

At another point on the fiber, we need the adjoint representation $\mathrm{Ad}_g : \mathcal{G} \to \mathcal{G}$, $X \mapsto g_*^{-1} X g$, described by a matrix $(A_{\mu\alpha}(g))$ such that

$$\mathrm{Ad}_g(X_\alpha) = A_{\mu\alpha}(g) X_\mu\,. \tag{37}$$

Then

$$[e_i, e_\alpha](g \cdot \sigma(s)) = -m_\gamma b^\gamma_{i\mu}(s) A_{\mu\alpha}(g)\,. \tag{38}$$

5. Nonholonomic mechanics

Consider the Lagrange–d'Alembert equations

$$X_L^{d'A} : \frac{d}{dt}\partial L/\partial \dot q - \partial L/\partial q = \lambda A\,, \qquad A\dot q = 0\,, \tag{39}$$

with $q \in \mathbb{R}^n, \lambda \in \mathbb{R}^r$, $A(q)$ a $r \times n$ matrix. For the regularity assumptions see [11]. More intrinsically, the constraint equations define a $s = n - r$ dimensional distribution $\mathcal{E}$ of subspaces $E_q \subset T_qQ$. The constraint forces $\lambda A \in T_q^*Q$ belong to the annihilator $\mathcal{E}^\circ$, of $\mathcal{E}$, a distribution of r-dimensional subspaces $E_q^\circ \subset T_q^*Q$. Under the Legendre transformation $\mathrm{Leg} : TQ \to T^*Q$, $p = \frac{\partial L}{\partial \dot q}$, $L + H = p \cdot \dot q$ the Lagrange–d'Alembert system (39) of equations $(q, \dot q) \in \mathcal{E} \mapsto X_L^{d'A}(q, \dot q) \in T\mathcal{E}$ transforms into the vector field $(q,p) \in \mathrm{Leg}(\mathcal{E}) \mapsto X_H^{d'A}(q,p) \in T\mathrm{Leg}(\mathcal{E})$ given by the differential-algebraic system

$$\Omega(X_H^{d'A} + \lambda, \bullet) = -dH(\bullet)\,, \quad \bullet \in T(T^*Q)\,, \quad \lambda \in E^\circ\,, \quad \pi_* X_H^{d'A}(q,p) \in E_q\,, \tag{40}$$

where $\pi : T^*Q \to Q$ is the bundle projection. Here we identify the constraint forces (semibasic vectors) $\lambda(p,q) \in T_{p_q}(T_q^*Q) \subset T(T^*Q)$ as elements of $E_q^\circ \subset T^*Q$.

The ODEs (39) restricted to $(q, \dot q) \in \mathcal{E}$ must satisfy $X_L^{d'A}(q, \dot q) \in T\mathcal{E}$ (self-consistency requirement). In our view, self-consistency is precisely what traditional texts in mechanics use to *construct* the system of ODEs, "eliminating the multipliers" λ. This step involves differentiating the condition $A(q)\dot q = 0$[5].

[5] The symplectic approach seems to be merely an algebraic calculation, but this is not the case.

5.1. Equations of motion

Consider an *adapted frame* e_i, e_α for $\mathcal{E}$ (this means that $e_i(q) \in E_q$) and its dual coframe $\epsilon_i, \epsilon_\alpha$. Notice that we are not assuming that e_α are orthogonal to E_q with respect to a given metric[6]. Our approach emphasizes the Lie brackets of the frame vector fields, but many authors prefer to compute the almost Poisson bracket entirely within the bracket formalism using suitable projections. See e.g. [2]. We write in full the defining equation (40),

$$\Omega(v_j e_j^* + \dot{m}_J \partial/\partial m_J + \lambda_\alpha \partial/\partial m_\alpha\,, \quad A_I e_I^* + B_I \partial/\partial m_I) = -dH(A_I e_I^* + B_I\, \partial/\partial m_I)\,,$$

$$X_H^{d'A} = v_j e_j^* + \dot{m}_J \partial/\partial m_J\,, \quad \lambda = \lambda_\alpha \partial/\partial m_\alpha\,, \quad \bullet = A_I e_I^* + B_I \partial/\partial m_I\,. \qquad (41)$$

Here the superscript "d'A" stands for constrained Lagrange-d'Alembert, not to be confused with constrained variational type [1]. Using Theorem 2 we get

$$-v_k B_k + \lambda_\alpha A_\alpha + \dot{m}_J\, A_J + v_j E_{jI} A_I = -A_R dH(e_R^*) - B_S \frac{\partial H}{\partial m_S}\,. \qquad (42)$$

Equating the coefficients of A_R and B_S we obtain the equations for nonholonomic systems. First notice that in the left-hand side there are no terms with B_α, hence we are *forced* to work in the subset P of T^*Q given by $\frac{\partial H}{\partial m_\alpha} = 0$, $\alpha = s+1, \ldots, n$.

THEOREM 5. *An "Operational System" for nonholonomic systems:*

(i) *The condition*

$$\frac{\partial H}{\partial m_\alpha} = 0\,, \qquad \alpha = s+1, \ldots, n\,, \qquad (43)$$

is equivalent to $P = \mathrm{Leg}(\mathcal{E})$, *where* $\mathrm{Leg} : TQ \to T^*Q$ *is the Legendre transformation. Assume the hypothesis for the implicit function theorem (P intersects $\mathcal{E}^\perp$ transversally) so we can solve for the* $m_\alpha = m_\alpha(q, m_k)$ *in terms of the* $n+s$ *variables* q, m_k.

Differentiation is automatically built in the algebra since we differentiate the ϵ_I. Equivalently, the almost Poisson bracket approach, first introduced by van der Schaft & Maschke [12], also requires a differentiation, namely taking the Lie bracket of vector fields satisfying the constraint equations. The referee pointed out an interesting question: intrinsically speaking, differentiating $A(q) \cdot \dot{q} = 0$ uses the Levi-Civita connection of the metric, or is the calculation moving to a higher tangent bundle? For each constraint $\alpha_i = 0$, we could perhaps differentiate the identity $d(X^{d'A} \dashv \pi^* \alpha_i) \equiv 0$ to get $d\alpha_i(\pi_* X^{d'A}, \bullet) = \mathcal{L}_{\pi_* X^{d'A}} \alpha_i$.

[6]When H comes from a natural Lagrangian $L = T - V$, it seems natural to choose e_I orthonormal with respect to T, as proposed by Cartan [5]. However, in the presence of symmetries transversal to the constraints, it may be more interesting to choose the e_α as vector fields generated by the symmetries [8]. See Section 5.2. below.

(ii) *The dynamic equations are given by:*

$$v_i = \frac{\partial H}{\partial m_i}, \qquad \dot{m}_i + v_k\, E_{ki} = -dH_{|(q,m)}(e_i^*), \tag{44}$$

where for $m = (m_i, m_\alpha)$ *the* m_α *are as in* (i).

(iii) *The multipliers are explicitly given by*

$$\lambda_\alpha = -\dot{u}_\alpha - v_j E_{j\alpha} - dH_{|(q,m)}(e_\alpha^*). \tag{45}$$

The reader should not fear having difficulties in computing $dH_{|(q,m)}(e_i^*)$. Recall the earnest duality $\{e_I^*, \partial/\partial m_J\}$ to $\{\epsilon_K, dm_L\}$, so it suffices to write

$$dH = \alpha_I \epsilon_I + \beta_J dm_J, \tag{46}$$

so $dH(e_I^*) = \alpha_I$, $dH(\partial/\partial m_J) = \beta_J$.

5.2. Reduction

Identify a point of P with its coordinates (q, m_k). Therefore, in order to compute the $(n+s) \times (n+s)$ (almost)-Poisson matrix, with respect to the basis ϵ_I, dm_k it suffices to cut the last $r = n - s$ rows and columns of $[\Lambda]$ in (21). This gives

$$[\Lambda]_{\text{constrained}} = \begin{pmatrix} 0_{s\times s} & 0_{s\times r} & I_{s\times s} \\ 0_{r\times s} & 0_{r\times r} & 0_{r\times s} \\ -I_{s\times s} & 0_{s\times r} & E^c \end{pmatrix}, \tag{47}$$

where

$$E^c_{jk} = -p_q \cdot [e_j, e_k], \qquad j,k = 1,\ldots,s, \tag{48}$$

and $p_q \in P \subset T^*Q$ is the point with coordinates q, m_k, m_α satisfying

$$m_\alpha = m_\alpha(q, m_k). \tag{49}$$

Notice that the middle rows and columns vanish. In the presence of transversal symmetries yielding a principal bundle $G^r \hookrightarrow Q^n \rightarrow S^s$, we can "zip" (compress) the system down to an almost Poisson structure in T^*S. Let $H^*(q, m_i) = H(q, m_i, m_\alpha(q, m_i))$. Since $\partial H/\partial m_\alpha = 0$, we have $\partial H^*/\partial q = \partial H/\partial q$, $\partial H^*/\partial m_i = \partial H/\partial m_i$ so the right-hand side in Theorem 5 is preserved under reduction.

In many nonholonomic problems such as a rigid convex body rolling on a flat plane, the symmetry group (here $\mathbb{R}^2$) does indeed intersect the constraints transversally. Internal symmetries (that is, satisfying the constraints) will produce conserved quantities [1] and the quest for integrability of the reduced system.

We have observed in examples [8] that compressed systems are sometimes *conformally* symplectic. In the Chaplygin sphere, interestingly enough, the compressed system to $T^*SO(3)$ has an extra integral of motion arising from a conserved measure, but it is *not* conformally symplectic. We will report on this work elsewhere.

5.3. Final remarks

We believe that moving frames can be useful for studying a manifold endowed with a skew symmetric structure (symplectic, Poisson, Dirac, Jacobi) together with some competing structure (for instance, homogeneous or Kahler), for which the Darboux charts could be cumbersome[7].

ODEs for nonholonomic systems have been derived again and again, but the main question remains open: to construct a theory for nonholonomic systems, similar to the one that Hamilton and Jacobi created for holonomic systems. In future work we will present some ideas on the issues of symmetry, reduction and integrability. Here we just present two simple observations to conclude this paper[8].

It is common knowledge that constraints count double in holonomic mechanics. The Lagrangian vector field is a spray: a restriction on $\dot{q}$ affects its "twin brother" in $T(TQ)$. Constraints also count in double for nonholonomic systems (well, perhaps 1 3/4). The rank of the almost Poisson tensor is indeed $(2n-r)-r = 2n-2r$. Using the identification $\partial/\partial m_\alpha \equiv \epsilon_\alpha$ (a vertical vector), it follows that

$$\partial H/\partial m_\alpha = dH_{|(q,m)}(\epsilon_\alpha) = \epsilon_\alpha(q)(\partial H/\partial p) = \epsilon_\alpha(q)(\dot{q})\,. \tag{50}$$

Here we consider $\partial H/\partial p \in T_qQ \equiv (T_q^*Q)^* \equiv (T_{p_q}T^*Q)^*$. Therefore, condition (i) is a consequence of the constraint $\dot{q} \in E_q$. This condition "does it twice", in the construction of the reduced space P and in the projection to Q. The vanishing middle rows and columns in (47) means the almost Poisson bracket of ϵ_α with any differential $\xi \in T_{p_q}^*(T^*Q)$ is zero. We call ϵ_α an *almost Casimir*. As for an ordinary Casimir in Poisson geometry, this implies that $\epsilon_\alpha(X) = 0$ for any constrained vector field X, equivalent to the statement that $\pi_*(X) \in \mathcal{E}$. Any exact combination of the ϵ_α's will produce a *bona fide* Casimir function on P.

[7]Local symplectic geometry is considered to be trivial due to Darboux theorem. Global symplectic geometry is reputed to be difficult. Recently H. Hofer proposed introducing piecewise linear symplectic structures as a way to pass from local to global. Perhaps moving frames could be an alternative approach.

[8]These remarks are in line with the viewpoint that nonholonomic systems bear many similarities with holonomic systems, as pointed out by Prof. Śniatycki in this meeting.

Actually, these will be functions on Q, because the ϵ_α are *basic* differentials. Since we are interested in strictly nonholonomic systems, we may assume that no exact combinations exist[9].

We finish with a spiritual observation, which we hope proper, both in terms of mathematics and religion as well. Mathematicians use a universal hand waving gesture to represent a Riemannian manifold, through a moving frame attached to it. A similar gesture to represent a symplectic manifold is in order. We believe that such a gesture ("mudra") may be found in Buddhism[10].

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[9]This does not rule out "gauge conservation laws", see eg. [4], which appear whenever a combination of group action generators satisfies the constraints.

[10]Siddhartha's right hand explores the Earth, (a Lagrangian submanifold); the left hand explores the spiritual fibre (another Lagrangian submanifold). In so doing, the earthly hand acquires a spiritual component.