Weyl Quantization from Geometric Quantization

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Abstract. In [23] a nice looking formula is conjectured for a deformed product of functions on a symplectic manifold in case it concerns a hermitian symmetric space of non-compact type. We derive such a formula for simply connected symmetric symplectic spaces using ideas from geometric quantization and prequantization of symplectic groupoids. We compute the result explicitly for the natural 2-dimensional symplectic manifolds $\mathbf{R}^2$, $\mathbf{H}^2$, and $\mathbf{S}^2$. For $\mathbf{R}^2$ we obtain the well known Moyal-Weyl product. The other cases show that the original idea in [23] should be interpreted with care.

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Introduction

In [23] Weinstein discusses the quantization by groupoids program as a way to obtain an integral product which would deform the multiplication of the Poisson algebra of functions on a symplectic manifold M. Such a product would have the general form

$$(fg)(z) = \int_{M\times M} f(x)g(y)K(x,y,z)\,dx\,dy\ ,$$

with a kernel $K_{\hbar}$, depending on the deformation parameter $\hbar$, of the kind $K_{\hbar}(x,y,z) = \hbar^{-\dim M}\cdot\exp(iS(x,y,z)/\hbar)$, eventually multiplied by an "amplitude" $A(x,y,z)$. It is argued in [23] that for hermitian symmetric spaces the function $S(x,y,z)$ should be the symplectic area of a surface whose boundary is the geodesic triangle for which the points x, y, and z are the midpoints of its sides, generalizing what is known for $\mathbf{R}^{2n}$.

In this paper we will derive such a formula (formula (6) below) for simply connected symmetric symplectic spaces M by means of geometric quantization of the symplectic groupoid $M\times M$ and its prequantization as described in [24]. Our approach is inspired by the center-chord representation on euclidean spaces as described in [13]. We then apply this procedure to three simple 2-dimensional examples: the euclidean plane $\mathbf{R}^2$, the hyperbolic plane $\mathbf{H}^2\subset\mathbf{R}^3$ and the 2-sphere $\mathbf{S}^2\subset\mathbf{R}^3$. The first example, already worked out similarly in [6], gives us the well known integral product formula of Groenewold and Von Neumann for the Moyal-Weyl quantization of observables. In the hyperbolic plane we see that we have to interpret the amplitude function in a rather large sense: the phase function S is defined only on a proper subset of $\mathbf{H}^2\times\mathbf{H}^2\times\mathbf{H}^2$, forcing the amplitude function to be zero outside this (open) domain, and S blows up at the boundary of this domain. In the 2-sphere there is the additional complication that midpoints do not always determine a unique triangle (see also [15]).

Preliminaries

Let (M,ω) be a symplectic manifold and let $\hbar \in \mathbf{R}^+$ be a parameter. Let (Y,θ) be a prequantization of $(M,\omega/\hbar)$, meaning that $\pi : Y \to M$ is a principal $\mathbf{S}^1$-bundle equipped with a connection form θ whose curvature is $\omega/\hbar$ (which implies that the group of periods of ω is a discrete subgroup of $\mathbf{R}$). Using the identity representation of the circle $\mathbf{S}^1 \subset \mathbf{C}$ on $\mathbf{C}$, we let $L \to M$ be the associated complex line bundle over M with connection ∇ and compatible hermitian structure. It follows that we can identify Y with the subset of L of points of length 1 (with respect to the hermitian structure). We now assume that the curvature of ∇ also equals $\omega/\hbar$, which implies that $\omega/\hbar$ represents an integral cohomology class. This imposes a quantization condition on $\hbar$ in case ω is not exact.

Our purpose is to construct a map $\mathscr{F}(M) \times \mathscr{F}(M) \to \mathscr{F}(M)$ by means of geometric quantization of $M \times M$ as a symplectic groupoid, where $\mathscr{F}(M)$ stands for some space of functions on M. We will usually think of $\mathscr{F}(M)$ as the set of smooth functions on M, but analytical consideration (which will not be pursued in this paper) might decide otherwise. Our strategy will be to use a polarization such that the polarized sections of geometric quantization can be identified with functions on M (usually these sections form the Hilbert space, but here we will interpret them as observables). Using a groupoid structure on the prequantization, we construct the looked-for product. To make this work, we will have to restrict our attention to symplectic spaces with a complete affine connection for which geodesic inversion with respect to a point is a symplectomorphism, and whose first homology group is zero, bringing us in the category of simply connected symmetric symplectic spaces, which includes all simply connected hermitian symmetric spaces [3].

In our construction we will use extensively the results of [24], as well as its notation, but we will restrict to the barest minimum of terminology. For more of that, the interested reader is referred to [24] and the references therein (see also [17]).

Prequantization of the pair groupoid

The construction starts by giving the manifold $M \times M$ the symplectic structure $(\omega,-\omega)$. More precisely, if α and β denote the canonical projections $M \times M \to M$ onto the first and second factor, then the symplectic form on $M \times M$ is $\alpha^*\omega - \beta^*\omega$. The manifold $Y \times Y$ is in a natural way a principal $\mathbf{S}^1 \times \mathbf{S}^1$-bundle over $M \times M$. Taking the quotient of the diagonal action of $\mathbf{S}^1$ on $Y \times Y$, $\mathrm{e}^{i\phi} \cdot (y_1,y_2) = (\mathrm{e}^{i\phi} \cdot y_1, \mathrm{e}^{i\phi} \cdot y_2)$, we obtain a principal $\mathbf{S}^1$-bundle $[Y] = Y \times Y/\mathbf{S}^1 \to M \times M$. We will denote points in $[Y]$ as $[y_1,y_2]$ with $y_i \in Y$. The induced $\mathbf{S}^1$-action is taken to be $\mathrm{e}^{i\phi} \cdot [y_1,y_2] = [\mathrm{e}^{i\phi} \cdot y_1, y_2] = [y_1, \mathrm{e}^{-i\phi} \cdot y_2]$. Moreover, the 1-form $(\theta,-\theta)$ induces a connection form $[\theta] \equiv [\theta,-\theta]$ on $[Y]$, whose curvature is $(\omega/\hbar, -\omega/\hbar)$. We thus obtain a (particular) prequantization of $M \times M$. We let $[L] \to M \times M$ be the associated complex line bundle with connection and compatible hermitian structure. And as before we identify $[Y]$ with the subset of $[L]$ of points of length 1.

We define the diagonal section $\varepsilon_0 : M \to [Y]$ as $\varepsilon_0(m) = [y,y]$ with $y \in Y$ such that $\pi(y) = m$. This section is horizontal for the connection $[\theta]$. It then follows from [24, theorem 3.1, proposition 3.2] that there exists a unique groupoid structure on $[Y]$ with given properties. In our case this means that there exists a smooth map $\odot$ with values in $[Y]$ and defined on pairs $[x,y_1],[y_2,z] \in [Y]$ such that $\pi(y_1) = \pi(y_2)$. Using that $\mathbf{S}^1$ acts transitively on the fibres of $Y \to M$, and the diagonal $\mathbf{S}^1$ action on $Y \times Y$, this condition means that there exists a $z' \in Y$ such that $[y_2,z] = [y_1,z']$. With such a representation of the points, this "multiplication" $\odot$ is given by

$$[x,y] \odot [y,z] = [x,z] \ . \tag{1}$$

The central polarization

The next step in the geometric quantization procedure is the choice of a polarization on $M \times M$. We want a polarization that "mixes" both factors of M, but for generic M we know of no natural choice for such a polarization. We thus seek symplectic spaces for which we can define a rather natural mixing polarization. Here is the idea. For any complete affine connection ∇ on M we can define a smooth map:

$$F : TM \to M \times M \ \ ; \ \ F(m,v) = (\exp_m(-v), \exp_m(v)) \ ,$$

where $\exp_m : T_mM \to M$ denotes the geodesic flow at time $t = 1$, starting at $m \in M$ and in the direction of the tangent vector $v \in T_mM$. Since $\exp_m$ is a diffeomorphism in a neighborhood of $0 \in T_mM$, F is a diffeomorphism in a neighborhood of the zero section of TM. We define $U \subset TM$, as a maximal connected and symmetric (with respect to inversion in the fibres of the tangent bundle) open neighborhood of the zero section on which F *is* a diffeomorphism, and its image $V = F(U) \subset M \times M$. If the (complete) affine connection ∇ has no closed geodesics, then $U = TM$ and $V = M \times M$.

On TM we have a natural foliation $\mathscr{F}_v$ whose leaves are just the fibres T_mM of the tangent bundle. Our idea is that its image $\mathscr{P} = F_*\mathscr{F}_v$ should be a polarization for the restriction of the symplectic form $(\omega, -\omega)$ to V. An elementary computation shows that $\mathscr{P}$ is a polarization on V if and only if for each $m \in M$ the map $\exp_m(v) \mapsto \exp_m(-v)$ is a symplectomorphism on $\exp_m(U \cap T_mM) \subset M$. We thus require that the symplectic manifold M admits a complete affine connection for which geodesic inversion is a symplectomorphism. In this way, we arrive in the category of symmetric symplectic spaces [3], [17], which includes the category of hermitian symmetric spaces because the connection associated to the natural (complete) metric on a hermitian symmetric space satisfies this condition. When this condition is satisfied, we obtain a (real) polarization $\mathscr{P}$ on $V \subset M \times M$, which we will call the *central polarization*. Moreover, as is obvious from the definition of $\mathscr{P}$ via $\mathscr{F}_v$, the space of leaves $V/\mathscr{P}$ is naturally isomorphic to M, seen either as the diagonal in $M \times M$ or as the zero section in TM.

Central polarized sections

We now claim that there exists a section $s_0 : V \longrightarrow [Y]$ which is horizontal in the direction of $\mathscr{P}$ and which coincides with ε_0 on its domain of definition. The easiest way to construct this section is by pulling back all structures on $M \times M$ to TM by means of the map F. More precisely, we define Ω as the closed 2-form $F^*(\omega, -\omega)$ on TM and (B, Θ) as the principal $\mathbf{S}^1$-bundle with connection over TM obtained by pulling back the bundle $([Y], \theta)$. Obviously the curvature form of Θ is $\Omega/\hbar$. As argued above, Ω is identically zero on the fibres of TM, i.e., on the leaves of $\mathscr{F}_v$. The section ε_0 of $[Y]$ gets transformed to a section ε_0' of B above M seen as the zero section of TM. Since the fibres of TM are simply connected and since the curvature of Θ is identically zero on these fibres, we can extend the section ε_0' to a global section $\sigma : TM \to B$ which is horizontal when restricted to a leaf of $\mathscr{F}_v$. Restricting this section to U we obtain σ_0 and then pushing it to V by means of F we obtain our section s_0 as claimed.

More explicitly, let $(m_1, m_2) \in V \subset M \times M$ be arbitrary. We can define the curve $\gamma : [0,1] \to V \subset M \times M$ by $\gamma(t) = F(m, tv)$, with $TM \supset U \ni (m, v) = F^{-1}(m_1, m_2)$. More or less by construction, $s_0(m_1, m_2)$ is the end point of the horizontal lift of γ starting at $\varepsilon_0(m)$. But the two components $\gamma_1(t) = \exp_m(-tv)$ and $\gamma_2(t) = \exp_m(tv)$ of the curve γ form together the geodesic from m_1 to m_2 with m as midpoint. Choosing $\mu \in \pi^{-1}(m)$ arbitrary, we thus can define $\widetilde{\gamma}_i(t)$ as the horizontal lift of $\gamma_i(t)$ in Y starting at μ. Together they form a horizontal lift in Y above the geodesic between m_1 and m_2. By definition of the connection form on $[Y]$, the curve $\widetilde{\gamma}(t) = [\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)] \in [Y]$ is the horizontal lift of γ starting at $\varepsilon_0(m) = [\mu, \mu]$. It follows that $s_0(m_1, m_2) = [x, y]$ in which x and y are the end points of a horizontal curve above the geodesic (unique in V) between m_1 and m_2.

Definition 1 *We will call the sections of $[L]$ above V that are covariantly constant in the direction of $\mathscr{P}$ central polarized sections or $\mathscr{P}$-constant sections.*

Viewing $[Y]$ as a subset of $[L]$, the section s_0 constructed above is $\mathscr{P}$-constant. Moreover, it is a smooth nowhere vanishing section. It follows that central polarized sections $s : V \longrightarrow [L]$ are in 1-1 correspondence with functions f that are constant on the leaves of $\mathscr{P}$, i.e., with functions on $M = V/\mathscr{P}$. The identification is given by $s = f \cdot s_o$, or, more precisely, by $s(m_1, m_2) = f(m_{12}) \cdot s_0(m_1, m_2)$, where m_{12} is the midpoint of the geodesic between m_1 and m_2.

The product of sections

We now stop the geometric quantization program and we turn our attention to the groupoid structure on $[Y]$. We extend formula (1) to $[L]$ by the following prescription. Any $p \in [L]$ can be written in a unique way as $p = \lambda[x, y]$ with $\lambda \in [0, \infty)$ and $[x, y] \in [Y] \subset [L]$. Now, for $p_i = \lambda_i[x_i, y_i]$ such that $\pi(y_1) = \pi(x_2)$ we define

$$p_1 \odot p_2 = \lambda_1 \lambda_2 [x_1, y_1] \odot [x_2, y_2] \; .$$

With this extended quasi-groupoid structure (quasi because now not every element has an inverse), we construct a product on sections of $[L]$. If s^1 and s^2 are two sections of $[L]$

(not necessarily above V, not necessarily $\mathscr{P}$-constant), we define a new section $s^1 \circledcirc s^2$ of $[L]$ by

$$(s^1 \circledcirc s^2)(m_1, m_3) = \int_M s^1(m_1, m_2) \odot s^2(m_2, m_3)\, dm_2 \ .$$

In this formula the measure dm_2 is the Liouville measure on M associated to the symplectic form ω. The integration makes sense because all groupoid products $s^1(m_1, m_2) \odot s^2(m_2, m_3)$ lie in the same fibre of $[L]$: the one above (m_1, m_3). Of course there is no guarantee that this integral converges, but we will not deal with these delicate analytical issues here.

A particular case

We now, for the moment, restrict our attention to the case in which the metric g has no closed geodesics, i.e., the case in which F is a diffeomorphism from TM onto $M \times M$. In that case $\mathscr{P}$-constant sections of $[L]$ are globally defined sections. For two $\mathscr{P}$-constant sections $s^i = f_i \cdot s_0$, $i = 1, 2$ with $f_i \in C^\infty_\hbar(M)$ we thus get the formula

$$(s^1 \circledcirc s^2)(m_1, m_3) = \int_M f_1(m_{12}) f_2(m_{23}) s_0(m_1, m_2) \odot s_0(m_2, m_3)\, dm_2 \ , \qquad (2)$$

in which m_{jk} denotes the midpoint of the geodesic between m_j and m_k. Since s_0 is nowhere vanishing, there must be a constant λ such that $s_0(m_1, m_2) \odot s_0(m_2, m_3) = \lambda s_0(m_1, m_3)$. In order to determine this constant we argue as follows. We choose x_1, x_2, x_3, and x_3' such that $s_0(m_1, m_2) = [x_1, x_2]$, $s_0(m_2, m_3) = [x_2, x_3]$, and $s_0(m_1, m_3) = [x_1, x_3']$. Note that we may take the same x_1 and x_2 beacuse of the equivalence relation defining the points in $[Y]$. It follows from formula (1) that $s_0(m_1, m_2) \odot s_0(m_2, m_3)$ equals $[x_1, x_3]$. But we know that x_1 and x_2 are the endpoints of a horizontal lift above the geodesic between m_1 and m_2, and similarly for the pairs x_2, x_3 and x_1, x_3'. We thus have a geodesic triangle $m_3 m_2 m_1$ and a horizontal lift starting at x_3 above m_3, passing through x_2 and x_1 and coming to x_3', again above m_3. It follows that $x_3' = \lambda x_3$ with $\lambda \in \mathbf{S}^1$ the holonomy (with respect to the principal $\mathbf{S}^1$-bundle $[Y]$) of the geodesic triangle $m_3 m_2 m_1$. In particular we have $[x_1, x_3] = \lambda [x_1, x_3']$. Now if $\Delta(m_3 m_2 m_1)$ is any 2-chain whose boundary is the geodesic triangle $m_3 m_2 m_1$, then $\lambda = \exp(i \int_{\Delta(m_3 m_2 m_1)} \omega / \hbar)$. The result does not depend upon the choice for Δ because the curvature form $\omega/\hbar$ represents an integral cohomology class.

We are thus led to introduce the phase function $\widetilde{S}(m_3, m_2, m_1) = \int_{\Delta(m_3 m_2 m_1)} \omega$ representing the symplectic area of the surface $\Delta(m_3 m_2 m_1)$ whose boundary is the geodesic triangle with corners at m_3, m_2, and m_1. Actually $\widetilde{S}$ is in general multiple valued because there is (in dimensions higher than 2) no unique such 2-chain Δ, but this indeterminacy disappears when taking the exponential. On the other hand, in order to be sure that such a 2-chain exists for all geodesic triangles, we further restrict our attention to spaces M without homology in dimension 1. This excludes for instance the 2-torus, but all simply connected hermitian symmetric spaces satisfy this condition, and thus in particular the hermitian symmetric spaces of compact and non-compact type.

Substituting these results in formula (2) we obtain $(f_1 \cdot s_0 \circledcirc f_2 \cdot s_0)(m_1,m_3) = g(m_1,m_3) \cdot s_0(m_1,m_3)$, where g is given by

$$g(m_1,m_3) = \int_M f_1(m_{12})\, f_2(m_{23}) \mathrm{e}^{i\widetilde{S}(m_3,m_2,m_1)/\hbar} dm_2 \ . \tag{3}$$

If we forget the trivializing section s_0, we thus have associated to two functions f_1, f_2 on M a new function g on $M \times M$. In general, the product $s^1 \circledcirc s^2$ of two $\mathscr{P}$-constant sections will not be $\mathscr{P}$-constant. In terms of the function g this means that, in general, the function $g : M \times M \to \mathbf{C}$ is not constant on the leaves of $\mathscr{P}$, i.e., of the form $g(m_1,m_3) = \widehat{g}(m_{13})$ for some function $\widehat{g} : M \to \mathbf{C}$ with m_{13} the midpoint of the geodesic between m_1 and m_3.

The skewed product of functions on M

In order to get a central polarized section, i.e., in order to associate to two functions f_1 and f_2 on M a new function $f_1 \star f_2$ on M (not on $M \times M$), we integrate (average) formula (3) over the leaves of $\mathscr{P}$. This is easily done in terms of the fibres of TM and we get

$$\begin{aligned}(f_1 \star f_2)(m) &= \int_{T_mM} dv\, g(m_1,m_3) = \int_{T_mM} dv\, g(F(m,v)) \\ &= \int_{T_mM} dv \int_M dm_2\ f_1(m_{12})\, f_2(m_{23}) \mathrm{e}^{i\widetilde{S}(m_3,m_2,m_1)/\hbar} \ ,\end{aligned} \tag{4}$$

with $(m_1,m_3) = F(m,v)$ and m_{jk} the midpoint on the geodesic between m_j and m_k.

It remains to be decided what measure dv to use on T_mM for our averaging procedure, but there exists a rather canonical way to obtain one. Using that F is a global diffeomorphism (we are still in that case), $F^*(\omega,-\omega)$ is a symplectic form on TM, and thus we have its Liouville volume form $d\mu_{TM}(m,v)$ on TM. On the other hand, the zero section of TM is diffeomorphic to the symplectic manifold (M,ω), and thus on the zero section of TM we have its Liouville volume form $d\mu_M(m)$. It follows that there exists a unique volume form $dv_m(v) \equiv dv$ on each fibre T_mM such that $d\mu_M(m) \wedge dv_m(v) = d\mu_{TM}(m,v)$. In the sequel it will be this choice for the measure on T_mM that we will use in our *averaging procedure*.

Definition 2 *The skewed product of two functions f_1 and f_2 on M is given by (4).*

Definition 3 *The composition of central polarized sections $< s^1 \circledcirc s^2 >$ corresponds to the skewed product of functions via the identification of central polarized sections with functions on M: if $s^i = f_i \cdot s_0$, $i = 1,2$ are two central polarized sections, then $< s^1 \circledcirc s^2 > \stackrel{\text{def}}{=} (f_1 \star f_2) \cdot s_0$ is the product central polarized section; it is an averaged version of the product $s^1 \circledcirc s^2$.*

We call this product the *skewed product* to emphasize the distinction of its construction to some more well known noncommutative products of functions on symplectic manifolds, the *star product* defined in the context of deformation quantization (see [1]) and the *twisted product* defined via symbol mapping homomorphism (see [22], for instance), although the latter is often also called star product and these two are often confused as if the same.

In order to write the skewed product in a nicer way, we look at the map $\Psi_m : (v, m_2) \mapsto (m_{12}, m_{23})$ from $T_m M \times M$ to $M \times M$. We conjecture that this map is injective; it certainly needs not be surjective as can be seen in the case of the hyperbolic plane. If we denote by dm_{12} the Liouville measure on the first factor of $M \times M$ and by dm_{23} the Liouville measure on the second factor, then there exists a positive function A_m on $W_m = \Psi_m(T_m M \times M) \subset M \times M$ such that $\Psi_m^*(A_m\, dm_{12}\, dm_{23}) = dv_m\, dm_2$. Associated to W_m we define the set $W \subset M^3$ as $W = \{(m_{12}, m_{23}, m) \in M^3 \mid (m_{12}, m_{23}) \in W_m\}$. We then can interpret the family of functions A_m as a single function $A : W \to [0, \infty)$ by $A(m_{12}, m_{23}, m) = A_m(m_{12}, m_{23})$.

In order to better understand the amplitude function A, we define the map $G : M^3 \to W \subset M^3$ by the following sequence of maps:

$$(m_3, m_2, m_1) \mapsto (F^{-1}(m_1, m_3), m_2) \equiv (m, v, m_2) \mapsto (\Psi_m(v, m_2), m) \equiv (m_{12}, m_{23}, m) .$$

If we interpret M^3 as the description of the space of all geodesic triangles by their three corners (m_3, m_2, m_1), the map G can be seen as the "coordinate change" to the description of these triangles by the midpoints of their sides $(m_{12}, m_{23}, m \equiv m_{13})$. The assumption that the maps Ψ_m are injective translates as the statement that G is a bijection from M^3 to W. The map G and the function A are then related by

$$(G^{-1})^*(dm_1\, dm_2\, dm_3) = A \cdot dm_{12}\, dm_{23}\, dm_{13} . \tag{5}$$

Still under the assumption that $G : M^3 \to W$ is bijective, we define the function S on W by $S = \widetilde{S} \circ G^{-1}$. The function S can thus be described as the symplectic area of a surface Δ whose boundary is the geodesic triangle with given midpoints for its sides.

Theorem 1 *Let M be a symmetric symplectic space without closed geodesics and denote by A the symplectic jacobian (5) of the map $G^{-1} : W \to M^3$, which relates the three vertices (m_3, m_2, m_1) of a triangle to the three midpoints of its sides (m', m'', m). Let $S(m', m'', m)$ be the symplectic area of a geodesic triangle determined by its midpoints and denote by W_m the slice $W \cap (M \times M \times \{m\})$. Then the skewed product of functions on M associated to the composition of central polarized sections $M \times M \to [L]$ is given by:*

$$(f_1 \star f_2)(m) \;=\; \iint_{W_m} f_1(m')\, f_2(m'')\, \mathrm{e}^{iS(m', m'', m)/\hbar} A(m', m'', m)\, dm'\, dm'' . \tag{6}$$

Except for the important restriction of the integration to W_m instead of $M \times M$, this is the kind of product as conjectured in [23].

The general case

We have derived formula (6) under the assumption that F is a global diffeomorphism from TM to $M \times M$. If this is not the case, we were led to introduce the subsets $U \subset TM$

and $V = F(U) \subset M \times M$, and the section s_o defined only above V. It follows that the integration procedure which led us to formula (3) can only be performed for those values of m_2 such that (m_1, m_2) and (m_2, m_3) both lie in V.

The next step of averaging over the leaves of $\mathscr{P}$ should also be done with care. These leaves are only defined in V (elsewhere $\mathscr{P}$ is not defined), which means in terms of TM that we have to integrate, not over the whole tangent space T_mM, but only over the part in U, i.e., over $T_mM \cap U$. On the other hand, the argument which led to the measure dv remains valid: the pull-back by F of the Liouville measure on V to U gives us a measure on U. The zero section still carries its natural Liouville measure, and thus there exists a natural measure dv_m on $T_mM \cap U$ such that it completes the natural Liouville measure on the zero section to the pull back of the Liouville measure on V. We conclude that in the general case, formula (4) can still present an integral product of functions, provided we restrict integration to the appropriate subset of $T_mM \times M$.

In the general case, the map Ψ_m need not be injective, not even on the relevant subset $(T_mM \cap U) \times M$ as described above, as can be seen in the example of the 2-sphere. However, inspired by the example of the 2-sphere, we conjecture that there still exists a positive function A_m on $W_m = \Psi_m((T_mM \cap U) \times M)$ such that $\Psi_m^*(A_m\, dm_{12}\, dm_{23}) = dv_m\, dm_2$. We also conjecture that Ψ_m is injective outside a closed subset of measure zero in $(T_mM \cap U) \times M$ (see further in [17] proposition 6.1). This means that we can copy the arguments leading to formula (6), and that this formula is also valid in the general case, but with the new subset W_m.

Example I: The Euclidean plane $\mathbf{R}^2$

Let $M = \mathbf{R}^2$ be the Euclidean plane with the symplectic form $\omega = dp \wedge dq = d(p\,dq)$. The (unique) prequantization is the bundle $Y = M \times \mathbf{S}^1$ with connection form $\hbar\theta = p\,dq + d\varphi$. The map F, a global diffeomorphism, is given as $F(p,q;v_p,v_q) = (p - v_p, q - v_q; p + v_p, q + v_q)$. A horizontal lift of the curve $(p + tv_p, q + tv_q)$ is given by $(q + tv_q, p + tv_p, exp(\frac{i}{\hbar}(pt + \frac{1}{2}t^2 v_p)v_q))$. A simple calculation yields $s_o(p_1,q_1;p_2,q_2) = [(p_1,q_1;1),(p_2,q_2;\exp(\frac{i}{2\hbar}(p_1+p_2)(q_2-q_1))]$, where we used the equivalence relation on $[\ ,\ ]$ to put the first phase equal to 1. From this and formula (1) the phase factor λ in $s_o(m_1,m_2) \odot s_o(m_2,m_3) = \lambda s_o(m_1,m_3)$ is given by

$$\lambda = \exp(\tfrac{i}{2\hbar}\{(p_1+p_2)(q_1-q_2) + (p_2+p_3)(q_2-q_3) + (p_3+p_1)(q_3-q_1)\})\ .$$

A trivial calculation shows that this is indeed $\exp(i\widetilde{S}(p_3,q_3;p_2,q_2;p_1,q_1)/\hbar)$ with $\widetilde{S}$ the symplectic area (oriented with respect to the volume form $dp \wedge dq$) of the triangle with corners at (p_3,q_3), (p_2,q_2), and (p_1,q_1). In this example, the change of coordinates $(v,m_2) \mapsto (m_{12},m_{23})$ is a linear bijection with Jacobian $\frac{1}{4}$, which implies that the amplitude function $A = \frac{1}{4}$ is constant. Moreover, in the Euclidean plane, the area $\widetilde{S}(p_3,q_3;p_2,q_2;p_1,q_1)$ is four times the area of the triangle determined by its midpoints, i.e., $S(p,q;p_{12},q_{12};p_{23},q_{23}) = 4\widetilde{S}(p,q;p_{12},q_{12};p_{23},q_{23})$. In this way we obtain the usual formula of Von Neumann and Groenewold that defines the Moyal-Weyl quantization of the Euclidean plane (see [6]).

Example 2: The hyperbolic plane $\mathbf{H}^2$

Our next example is the hyperbolic plane $\mathbf{H}^2$ which we interpret as one sheet of the 2-sheeted hyperboloid in $\mathbf{R}^3$ determined by the equations $z^2 - x^2 - y^2 = 1$ and $z > 0$. We introduce the Lorentzian metric $\langle \ | \ \rangle_L$ by the formula $\langle (x,y,z)|(x',y',z')\rangle_L = zz' - xx' - yy'$.

This metric induces a surface element, which we take as symplectic form. An elementary but tedious calculation shows that the oriented hyperbolic area of a triangle determined by its three corners $a,b,c \in \mathbf{H}^2 \subset \mathbf{R}^3$ is given by the formula

$$\widetilde{S}(a,b,c) = 2\mathrm{Arg}\Big(1 + \langle a|b\rangle_L + \langle b|c\rangle_L + \langle c|a\rangle_L + i\mathrm{Det}(abc)\Big) \ ,$$

where Arg denotes the argument of a complex number; it lies in the interval $(-\pi,\pi)$. This formula is derived in [9] and [20] in the context of relativistic addition of velocities.

The next steps are to express the area of a hyperbolic triangle as a function of its midpoints and to determine the change of coordinates $(v,m_2) \mapsto (m_{12},m_{23})$. A straightforward calculation shows that if $a,b,c \in \mathbf{H}^2 \subset \mathbf{R}^3$ are the corners of a hyperbolic triangle, and if $\alpha,\beta,\gamma \in \mathbf{H}^2 \subset \mathbf{R}^3$ denote the midpoints of the three sides, then the area of the triangle is given by the simple formula (see [17] or [19] for two independent derivations)

$$S(\alpha,\beta,\gamma) = 2\mathrm{Arg}\Big(\sqrt{1-\mathrm{Det}(\alpha\beta\gamma)^2} + i\mathrm{Det}(\alpha\beta\gamma)\Big) = 2\arcsin(\mathrm{Det}(\alpha\beta\gamma)) \ . \tag{7}$$

The same analysis shows that the map $(a,b,c) \mapsto (\alpha,\beta,\gamma)$ is injective onto the triples (α,β,γ) satisfying $\mathrm{Det}(\alpha\beta\gamma)^2 < 1$, justifying the formula for S. It follows immediately that the subsets W_α are given as

$$W_\alpha = \{(\beta,\gamma) \in \mathbf{H}^2 \times \mathbf{H}^2 \mid \mathrm{Det}(\alpha\beta\gamma)^2 < 1\} \ . \tag{8}$$

A lengthier straightforward computation shows that the amplitude function is given by

$$A(\alpha,\beta,\gamma) = 16\langle\alpha|\beta\rangle_L \cdot \langle\beta|\gamma\rangle_L \cdot \langle\gamma|\alpha\rangle_L \cdot \Big(1 - \mathrm{Det}(\alpha\beta\gamma)^2\Big)^{-5/2} \ . \tag{9}$$

The fact that this amplitude function diverges on the boundary of W_α shows that we correctly restricted integration to this subset and that it is optimal.

Corollary 1 *The skewed product of two functions on the hyperbolic plane* $\mathbf{H}^2$ *at a point* $\alpha \in \mathbf{H}^2$ *is given by integration over a* proper *subset* $W_\alpha \subset \mathbf{H}^2 \times \mathbf{H}^2$ *determined by (8).*

Example 3: The sphere $\mathbf{S}^2$

In the last example we consider the compact hermitian symmetric space $\mathbf{S}^2$ seen as the unit sphere in $\mathbf{R}^3$, i.e., determined by the equation $z^2 + x^2 + y^2 = 1$. We equip $\mathbf{R}^3$ with the Euclidean metric $\langle \ | \ \rangle_E$ given by $\langle (x,y,z)|(x',y',z')\rangle_E = zz' + xx' + yy'$.

As for the hyperbolic plane, we take the induced surface element as symplectic form. And again, an elementary but tedious calculation shows that the oriented spherical area of a triangle determined by its three corners $a,b,c \in \mathbf{S}^2 \subset \mathbf{R}^3$ is given by the formula

$$\widetilde{S}(a,b,c) = 2\mathrm{Arg}\Big(1 + \langle a|b\rangle_E + \langle b|c\rangle_E + \langle c|a\rangle_E + i\mathrm{Det}(abc)\Big) , \tag{10}$$

i.e., by exactly the same formula as in the hyperbolic case, except that we use the Euclidean metric instead of the Lorentzian one. However, this formula needs more explanation than its hyperbolic counter part, because on $\mathbf{S}^2$ there are several triangles with the same three corners. The area given by formula (10) is the area of the triangle whose three corners are a, b, and c and whose three sides all have length less than π.

Elementary geometry shows that the subset $U \subset T\mathbf{S}^2$ is given by those tangent vectors that have length less than $\pi/2$. In fact, if $v \in T_m\mathbf{S}^2$ has length $\pi/2$, the two points $\exp_m(-v)$ and $\exp_m(v)$ are antipodal, and thus there is a circle of pairs (m,v) having these antipodal points as image under F. It follows that the image $V = F(U)$ is the set of pairs (m_1,m_2) such that $m_1 \neq -m_2$. And indeed for any two non-antipodal points there is a unique geodesic with length less than π joining them. The integration over m_2 in formula (3) has to be done over all those m_2 such that the two pairs (m_1,m_2) and (m_2,m_3) belong to V. Since in the definition of V we only exclude antipodal points, this means that we have to leave out a set of measure zero in the integration over m_2. In other words, we can maintain formula (3) as it stands. The factor $\mathrm{e}^{i\widetilde{S}(m_3,m_2,m_1)/\hbar}$ in the integration over m_2 in (3) is defined except on a set of measure zero (when m_2 is antipodal to either m_1 or m_3).

The integration over $v \in T_mM$ should not be done over the whole of T_mM but only over $T_mM \cap U$, i.e., over tangent vectors of length less than $\pi/2$. This corresponds exactly to integrating over the leaves of $\mathscr{P}$ because two (pairs of) points in $V \subset \mathbf{S}^2 \times \mathbf{S}^2$ lie on the same leaf of $\mathscr{P}$ if and only if they have the same midpoint on the geodesic segment joining them. Since we avoid antipodal pairs, there exists a unique geodesic segment of length less than π joining (m_1,m_2), on which the midpoint is given by the normalized average $(m_1+m_2)\cdot(\langle m_1+m_2|m_1+m_2\rangle_E)^{-1/2} \in \mathbf{S}^2$. Thus, the space of leaves is characterized by $\mathbf{S}^2$, which is the space of midpoints, and the distance of such a midpoint to one of its endpoints is less than $\pi/2$, justifying the restriction to integrate only over tangent vectors of length less than $\pi/2$. It means that we only consider triangles whose sides are all shorter than π.

It remains to express the phase function $\widetilde{S}$ in terms of midpoints and to compute the amplitude function A. Contrary to the hyperbolic case, there always exists a geodesic triangle with given midpoints $\alpha,\beta,\gamma \in \mathbf{S}^2$. More precisely, if $a,b,c \in \mathbf{S}^2 \subset \mathbf{R}^3$ are the corners of a spherical triangle, and if $\alpha,\beta,\gamma \in \mathbf{S}^2 \subset \mathbf{R}^3$ denote the midpoints of the three sides, then the oriented area S of the triangle is given as (see [17], [19])

$$S(\alpha,\beta,\gamma) = 2\mathrm{Arg}\Big(\eta\sqrt{1-\mathrm{Det}(\alpha\beta\gamma)^2} + i\mathrm{Det}(\alpha\beta\gamma)\Big) , \tag{11}$$

where η is a sign: the same as the majority of signs among the three scalar products $\langle\alpha|\beta\rangle_E$, $\langle\beta|\gamma\rangle_E$, and $\langle\gamma|\alpha\rangle_E$ (provided they are all non zero). We see that it is (up to the factor η) the same formula as in the hyperbolic case. Unlike the hyperbolic case, we do not have a restriction on the midpoints, a fact which is corroborated by the fact that for points on the unit sphere, the determinant $\mathrm{Det}(\alpha\beta\gamma)^2$ is always less than or equal to 1.

However, the calculations leading to the formula for S show that, if all three sides of a triangle have length less than π, then all three scalar products $\langle\alpha|\beta\rangle_E$, $\langle\beta|\gamma\rangle_E$, and $\langle\gamma|\alpha\rangle_E$ have the same sign, where the sign should be interpreted as a function on $\mathbf{R}$ defined as being $+1$ for positive values, -1 for negative values, and 0 for zero. Thus, the set W_α is

$$W_\alpha = \{(\beta,\gamma) \in \mathbf{S}^2 \times \mathbf{S}^2 \mid \mathrm{sign}\langle\beta|\gamma\rangle_E = \mathrm{sign}\langle\alpha|\beta\rangle_E = \mathrm{sign}\langle\alpha|\gamma\rangle_E\} \,. \tag{12}$$

Moreover, the calculations also show that if all three inner products are zero, then there is an infinity of triangles having the given points as midpoints (roughly a set parametrized by a point on $\mathbf{S}^2$). But this set has measure zero in W_α and hence can be neglected in the integration. Note that even though the triangle itself is not uniquely determined by its midpoints, its area is (see also [17] proposition 6.1).

Since W_α is only a quarter of $\mathbf{S}^2 \times \mathbf{S}^2$ (with respect to the natural measure), we must treat the restriction of the integration to W_α in formula (6) seriously. If the three inner products do not all have the same sign, there still exists a triangle *abc* (unique if no inner product is zero) but at least one of its sides will be longer than or equal to π (conditions similar to (12) appear in the other cases). Computing the amplitude function, we find

$$A(\alpha,\beta,\gamma) = 16\left|\langle\alpha|\beta\rangle_E \cdot \langle\beta|\gamma\rangle_E \cdot \langle\gamma|\alpha\rangle_E\right| \cdot \Big(1-\mathrm{Det}(\alpha\beta\gamma)^2\Big)^{-5/2} \,. \tag{13}$$

Summary and comparisons

Formula (6) defines a product of functions on a symplectic manifold, whose form was conjectured in [23], in the spirit of the central (Weyl) representation of quantum observables [13] and strict deformation quantization [14]. We have derived this skewed product for simply connected symplectic spaces which admit a complete affine connection for which geodesic inversion is a symplectomorphism, using only basic ideas from geometric quantization and groupoids. The main ingredients of our construction are the prequantization of the pair groupoid, the central polarization, the product of sections using the groupoid structure and finally the averaging procedure.

Further investigations on the properties of this product is work in progress (see also [15]), but we remark that stationary phase evaluation of the skewed product of two oscillatory functions brings in the composition of central generating functions of canonical relations, as defined in [17] (see also [10], as well as [18] for an independent partial proof of Theorem 6.1 in [17]). Also remark that, on $\mathbf{R}^{2n}$, this connection between the skewed product of oscillatory functions and the composition of central generating functions is a very important feature of semiclassical analysis [13].

We end this section by briefly comparing this work with a few others in the literature.

The well known formal deformation quantization [1], as developed by Fedosov [5] among others (in particular Moreno & Ortega-Navarro [11]), differs from our approach from the start by considering, not generic $\hbar$-dependent functions on M, but the ring of polinomials in $\hbar$ with coefficients in $C^\infty(M)$. Important questions of whether or how such formal products converge are already second to whether or how some important functions (e.g. oscillatory) can be appropriately treated in this context (see [16], [12]).

In the approach to quantization by means of pseudodifferential operators and symbol mapping, also not as general in scope (either p-localized symbols, or finite dimension functional spaces), the work of Unterberger & Unterberger [21], for $\mathbf{H}^2$, and the work of Varilly & Gracia-Bondia [22], for $\mathbf{S}^2$, are close to the object of this paper and constitute excellent treatments of these fundamental examples in a context of Weyl quantization.

For the integral product defined by Karasev [7] based on Berezin's quantization [2], the functions to be multiplied stand in bijection to (anti)holomorphic functions on a Kahler manifold M. In his more recent collaboration with Osborn [8], an approach closer to this paper was developed for functions on cotangent bundles: the product obtained for functions on $T^*\mathbf{H}^2$ is defined on a subset of $T^*\mathbf{H}^2 \times T^*\mathbf{H}^2$ that is the natural extension of (8).

Finally, the work of Bieliavsky on solvable symmetric symplectic spaces [4] is also close to the object of this paper, in these more specific cases (solvable), and defines an integral product (with asymmetric kernel) which is very close to the skewed product. A comparison between the two approaches shall be reported elsewhere.

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