Chapter 1

Regular surfaces

In these notes the term differentiable means differentiable of class C^∞.

1.1 Regular surfaces

In few words, a regular surface in $\mathbb{R}^3$ is a subset that is locally homeomorphic to an open subset of $\mathbb{R}^2$.

Definition 1.1.1. A subset $M \subset \mathbb{R}^3$ is called a *regular surface* if, for each point $p \in M$, there exists an open set $V \subset \mathbb{R}^3$, with $p \in V$, and a homeomorphism $\varphi : U \to M \cap V$ defined on an open set $U \subset \mathbb{R}^2$, such that

(a) φ is differentiable,

(b) For each point $x \in U$, the differential $\mathrm{d}\varphi(x) : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

The mapping φ is called a *parametrization* of M around p, and the subset $M \cap V$ is called a *coordinate neighborhood* of M. This means that M is endowed with the induced topology of $\mathbb{R}^3$, and therefore any regular surface is, in particular, a topological subspace of $\mathbb{R}^3$.

The condition that φ is differentiable means that if we write

$$\varphi(x_1, x_2) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2), \varphi_3(x_1, x_2)),$$

then the coordinate functions $\varphi_1, \varphi_2, \varphi_3$ have continuous partial derivatives of all orders in the open set U.

The condition that $\mathrm{d}\varphi(x) : \mathbb{R}^2 \to \mathbb{R}^3$ is a injective linear map is equivalent to any of the following conditions:

(a) The set $\{\mathrm{d}\varphi(x) \cdot e_i : 1 \leq i \leq 2\}$ is linearly independent, where $\{e_1, e_2\}$ denotes the canonical basis of $\mathbb{R}^2$.

(b) The Jacobian matrix $\mathrm{d}\varphi(x)$ has rank two at any point $x \in U$.

Example 1.1.2. Any two-dimensional vector space $E \subset \mathbb{R}^3$ is a regular surface. In fact, consider a linear isomorphism $T \colon E \to \mathbb{R}^2$, and endow E with the unique topology (induced of $\mathbb{R}^3$) that makes T a homeomorphism. Since any linear mapping into $\mathbb{R}^2$ is differentiable, it follows that T is a diffeomorphism, and thus T is a global parametrization of E.

Example 1.1.3. Let us show that the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is a regular surface. Fix a point $p \in \mathbb{S}^2$ other than the north pole $N = (0, 0, 1)$ and consider the stereographic projection $\pi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$. We already know that π_N is a homeomorphism, whose inverse is the map $\varphi : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{N\}$ given by

$$\varphi(x) = \left(\frac{2x_1}{\|x\|^2 + 1}, \frac{2x_2}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1} \right),$$

for every point $x = (x_1, x_2) \in \mathbb{R}^2$. Since each coordinate function of φ is differentiable, it follows that φ is also differentiable. It is straightforward to check that $d\varphi(x)$ has rank two at any point $x \in \mathbb{R}^2$. Finally, if $p = N$, just consider the stereographic projection π_S relative to the south pole $S \in \mathbb{S}^2$.

Example 1.1.4. The graph of a differentiable function $f : U \to \mathbb{R}$, defined in an open set $U \subset \mathbb{R}^2$, is a regular surface. In fact, denoting by $\mathrm{Gr}(f)$ the graph of f, let us show that the map $\varphi : U \to \mathbb{R}^3$ given by

$$\varphi(x) = (x, f(x)),$$

is a global parametrization of $\mathrm{Gr}(f)$. Since f is differentiable, the same holds for φ. Each point $(x, f(x)) \in \mathrm{Gr}(f)$ is the image under φ of the unique point $x \in U$, and φ is therefore injective. Moreover, the restriction to $\mathrm{Gr}(f)$ of the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$ is a inverse to φ, and this shows that φ^{-1} is also continuous. It follows that φ is a homeomorphism. Finally, it is easy to see that $\mathrm{d}\varphi(x)$ has rank two at any point $x \in U$.

The following result provides a local converse of Example 1.1.4. More precisely, any regular surface is locally the graph of a differentiable function.

Proposition 1.1.5. Given a regular surface $M \subset \mathbb{R}^3$ and a point $p \in M$, there exist an open set $U \subset \mathbb{R}^2$, an open set $V \subset \mathbb{R}^3$ with $p \in V$, and a differentiable function $g : U \to \mathbb{R}$ such that $M \cap V = \mathrm{Gr}(g)$.

Proof. Fix a point $p \in M$ and consider a parametrization $\varphi : U \to \varphi(U)$ of M, with $p = \varphi(x)$. Since $E = \mathrm{d}\varphi(x)(\mathbb{R}^2)$ is a two-dimensional vetor subspace of $\mathbb{R}^3$, there exists an orthogonal decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ such that the projection $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ maps E isomorphically onto $\mathbb{R}^2$, and define the map

$$\eta = \pi \circ \varphi : U \to \mathbb{R}^2.$$

Since $\mathrm{d}\eta(q) = \pi \circ \mathrm{d}\varphi(q)$ is a linear isomorphism, it follows from the inverse function theorem that there exist a open set $W \subset \mathbb{R}^2$, with $q \in W \subset U$, such that $\eta|_W : W \to \eta(W) = Z$ is a diffeomorphism. Define

$$\xi = (\eta|_W)^{-1} : Z \to W \quad \text{and} \quad \psi = \varphi \circ \xi.$$

It follows that ψ is also a parametrization of M and

$$\pi \circ \psi = \pi \circ (\varphi \circ \xi) = \eta \circ \xi = id.$$

According to the orthogonal decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$, it follows from the above equality that the first coordinate of $\psi(x)$ is x. Let us denote by $g(x)$ the second one. Thus,

$$\psi(Z) = \varphi(W) = \{(x, g(x)) : x \in W\}$$

for some differentiable function $g : W \to \mathbb{R}$. Since φ is a open map, one has

$$\varphi(W) = M \cap V = \mathrm{Gr}(g),$$

for some open set $V \subset \mathbb{R}^3$, with $p \in V$. □

Let us look a simple application of Proposition 1.1.5.

Example 1.1.6. Let us consider the *one-sheeted cone* $M \subset \mathbb{R}^3$ given by

$$M = \{(x, y, z) : x^2 + y^2 = z^2, z \geq 0\}.$$

We will show that M is not a regular surface. If M were a regular surface then, by virtue of Proposition 1.1.5, M would be locally a graph of a differential function around $(0, 0, 0)$. More precisely, there exist open sets $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^3$, with $0 \in V$, and a differentiable function $g : U \to \mathbb{R}$ such that $M \cap V = \mathrm{Gr}(g)$. Note that, according to a decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$, the only possibility for $M \cap V$ to be a graph is for the second factor to be the axis-z. Thus, it follows that $g = f|_U$, where $f(x, y) = \sqrt{x^2 + y^2}$. However, f is not differentiable at $(0, 0)$.

Let $f : V \to \mathbb{R}$ be a differentiable function, defined in an open set $V \subset \mathbb{R}^3$. We say that a point $p \in V$ is a *regular point* of f if the direrential $\mathrm{d}f(p)$ is surjective, that is, $\mathrm{d}f(p) \neq 0$. A point $c \in \mathbb{R}$ is called a *regular value* of f if the inverse image $f^{-1}(c)$ contains only regular points of f. Notice that any point $c \notin f(V)$ is trivially a regular value of f.

Proposition 1.1.7. Let $f : V \to \mathbb{R}$ be a differentiable function, defined in an open set $V \subset \mathbb{R}^3$, and $c \in \mathbb{R}$ be a regular value of f. If $f^{-1}(c) \neq \emptyset$, then $M = f^{-1}(c)$ is a regular surface.

Proof. By virtue of Example 1.1.4, it suffices to prove that M is locally graph of some differentiable function. Given a point $p \in M$, with $p = (x_0, y_0, z_0)$, we can assume that $\frac{\partial f}{\partial z}(p) \neq 0$. Therefore, it follows from the implicit function theorem that there exist an open set $W = U \times I$, where U is an open set of $\mathbb{R}^2$ with $(x_0, y_0) \in U$, and I is an open interval with $z_0 \in I$, and a differentiable function $g : U \to \mathbb{R}$ such that

$$f((x, y), g(x, y)) = c,$$

for every $(x, y) \in U$. This proves that $M \cap W = \mathrm{Gr}(g)$. □

Example 1.1.8. The unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ can be described as the inverse image $f^{-1}(1)$ of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) = \|x\|^2 = \langle x, x\rangle,$$

for every $x \in \mathbb{R}^3$. Notice that f is differentiable and, for any point $p \in \mathbb{R}^3$ and any vector $v \in \mathbb{R}^3$, we obtain

$$\mathrm{d}f(p) \cdot v = 2\langle p, v\rangle.$$

This implies that $0 \in \mathbb{R}^3$ is the unique critical point of f. Since $f(0) = 0 \neq 1$, we conclude that 1 is regular value of f. Therefore, the sphere $\mathbb{S}^2$ is a regular surface as we have already seen.

Remark 1.1.9. The inverse image $f^{-1}(c)$ can be a regular surface without c being a regular value of f. For instance, consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = z^2$. Note that f is differentiable and $f^{-1}(0)$ is the plane-xy, which is a regular surface in $\mathbb{R}^3$. However, the point $0 \in \mathbb{R}$ is not a regular value of f, because $\mathrm{d}f(x, y, 0) = 0$, for every $(x, y, 0) \in f^{-1}(0)$.

The following result is a converse of Proposition 1.1.7.

Theorem 1.1.10. *Any regular surface M in $\mathbb{R}^3$ is locally the inverse image of a regular value. More precisely, given a point $p \in M$, there exist an open set $V \subset \mathbb{R}^3$ with $p \in V$, and a differentiable function $f\colon V \to \mathbb{R}$ such that $M \cap V = f^{-1}(0)$, where $0 \in \mathbb{R}$ is a regular value of f.*

Proof. It follows from Proposition 1.1.5 that there exists an open set $V \subset \mathbb{R}^3$, with $p \in V$, such that $M \cap V = \mathrm{Gr}(g)$, where $g\colon U \to \mathbb{R}$ is a differentiable function defined in an open set $U \subset \mathbb{R}^2$. Define a function $f\colon V \to \mathbb{R}$ by $f(x, y) = y - g(x)$. By construction, one has

$$M \cap V = \mathrm{Gr}(g) = f^{-1}(0).$$

It suffices to prove that $\mathrm{d}f(x, y)$ is surjective at any point $(x, y) \in f^{-1}(0)$. In fact, given $(x, y) \in f^{-1}(0)$ and $(u, v) \in \mathbb{R}^3$, we obtain:

$$\begin{aligned} \mathrm{d}f(x, y) \cdot (u, v) &= \mathrm{d}f(x, y) \cdot (u, 0) + \mathrm{d}f(x, y) \cdot (0, v) \\ &= \mathrm{Id}(0) - \mathrm{d}g(x) \cdot u + \mathrm{Id}(v) - \mathrm{d}g(x) \cdot 0 \\ &= v - \mathrm{d}g(x) \cdot u. \end{aligned}$$

Therefore, given $v \in \mathbb{R}$, one has $\mathrm{d}f(x, y) \cdot (0, v) = v$, and this proves that 0 is a regular value of f. □

1.2 Differentiable mappings between surfaces

In this section we will define what it means for a map $f\colon M \to N$, between two regular surfaces M and N, to be differentiable at a point $p \in M$.

Definition 1.2.1. A map $f : M \to N$, between the regular surfaces M and N, is said to be *differentiable* at a point $p \in M$ if there exist parametrizations $\varphi : U \to \varphi(U)$ of M and $\psi : V \to \psi(V)$ of N, with $f(\varphi(U)) \subset \psi(V)$ and $p = \varphi(x)$, such that the map

$$\psi^{-1} \circ f \circ \varphi : U \to V \tag{1.1}$$

is differentiable at $x \in U$.

The map given in (1.1) is called a *representation* of f in terms of the parametrizations φ and ψ. We have to show that this definition does not depend on the choice of parametrizations. In fact, consider parametrizations $\varphi'\colon U' \to \varphi'(U')$ of M and $\psi'\colon V' \to \psi'(V')$ of N, with $p \in \varphi'(U')$ and $f(\varphi'(U')) \subset \psi'(V')$. In the intersection $\varphi'^{-1}(\varphi(U) \cap \varphi'(U'))$, one has

$$\psi'^{-1} \circ f \circ \varphi' = (\psi'^{-1} \circ \psi) \circ (\psi^{-1} \circ f \circ \varphi) \circ (\varphi^{-1} \circ \varphi').$$

Therefore, the differentiability of f will be well-defined if we prove that $\psi'^{-1} \circ \psi$ and $\varphi^{-1} \circ \varphi'$ are differentiable. But this will be a consequence of the following lemma.

Let $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ be two parametrizations of a regular surface M such that $\varphi(U) \cap \psi(V) \neq \emptyset$. The map

$$\psi^{-1} \circ \varphi : \varphi^{-1}(W) \to \psi^{-1}(W), \tag{1.2}$$

is called the *change of coordinates* between the parametrizations φ and ψ, where $W = \varphi(U) \cap \psi(V)$.

Lemma 1.2.2. The change of coordinates (1.2) is a diffeomorphism.

Proof. Fix a point $p \in \varphi(U) \cap \psi(V)$, with $p = \varphi(q)$. Since $d\varphi(q)$ is injective, and writing

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

we can assume without loss of generality that $\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$. Define a map $\xi : U \times \mathbb{R} \to \mathbb{R}^3$ by

$$\xi(u, v, w) = (x(u, v), y(u, v), z(u, v) + w).$$

ξ is clearly differentiable and $\xi|_{U\times\{0\}} = \varphi$, thus

$$\det(\mathrm{d}\xi(q, 0)) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{pmatrix} = \frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0.$$

Therefore, it follows from inverse function theorem that there exists an open set $K \subset \mathbb{R}^3$, with $(q, 0) \in K$, such that $\xi|_K : K \to \xi(K)$ is a diffeomorphism. Note that $Z = \xi(K)$ is an open set of $\mathbb{R}^3$, with $p \in Z$. Since $\xi|_{U\times\{0\}} = \varphi$, one has

$$\xi^{-1}|_{\varphi(U)\cap Z} = \varphi^{-1}|_{\varphi(U)\cap Z}.$$

On the other hand, since $\varphi(U) \cap Z$ is an open set of M and ψ is a homeomorphism, it follow that $\psi^{-1}(\varphi(U) \cap Z)$ is an open set of V. Therefore,

$$\varphi^{-1} \circ \psi|_{\psi^{-1}(\varphi(U)\cap Z)} = \xi^{-1} \circ \psi|_{\psi^{-1}(\varphi(U)\cap Z)}$$

is differentiable as a composition of differentiable maps. Analogously, we can show that $\psi^{-1} \circ \varphi$ is differentiable, and thus it is a diffeomorphism. □

Let us explore some consequences.

Corollary 1.2.3. Let M, N be regular surfaces, and assume that $M \subset V$, where V is an open set of $\mathbb{R}^3$, and that $f\colon V \to \mathbb{R}^3$ is a differentiable map such that $f(M) \subset N$. Then the restriction $f|_M\colon M \to N$ is a differentiable map.

Proof. Given a point $p \in M$, consider parametrizations $\varphi\colon U \to \varphi(U)$ of M an $\psi\colon V \to \psi(V)$ of N, with $p \in \varphi(U)$ and $f(\varphi(U)) \subset \psi(V)$. Then, the map

$$\psi^{-1} \circ f \circ \varphi\colon U \to V$$

is differentiable. □

Example 1.2.4. The map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (ax, by, cz)$, where a, b, c are positive real numbers, is clearly differentiable. The restriction of f to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is differentiable. In fact, $f|_{\mathbb{S}^2}$ is a differentiable map of the sphere $\mathbb{S}^2$ into the ellipsoid $\mathcal{E}$.

Remark 1.2.5. In the case of a map $f\colon M \to \mathbb{R}^2$, of a regular surface M into $\mathbb{R}^2$, the Definition 1.2.1 takes a rather simpler form. Namely, in this case, f is differentiable at $p \in M$ if there exists a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$, such that

$$f \circ \varphi\colon U \to \mathbb{R}^2$$

is differentiable at $x \in U$. In fact, just consider ψ equal to the identity in Definition 1.2.1.

Corollary 1.2.6. If $\varphi\colon U \to \varphi(U)$ is a parametrization of a regular surface M, then $\varphi^{-1}\colon \varphi(U) \to \mathbb{R}^2$ is also differentiable.

Proof. Given a point $p \in \varphi(U)$, consider the parametrization $\varphi\colon U \to \varphi(U)$ of M. One has $p \in \varphi(U)$ and the representation of φ^{-1} in terms of φ is just the identity map, which is differentiable. □

Two regular surfaces M and N are *diffeomorphic* if there exists a bijective differentiable map $f\colon M \to N$, whose inverse $f^{-1}\colon N \to M$ is also differentiable. In this case, f is called a *diffeomorphism* from M to N. In particular, it follows from Corollary 1.2.6 that, if $\varphi\colon U \to \varphi(U)$ is a parametrization of a regular surface M, then U and $\varphi(U)$ are diffeomorphic.

Finally, we will now give a definition for a differentiable function on a regular surface.

Definition 1.2.7. A function $f\colon M \to \mathbb{R}$, defined on a regular surface M, is said to be *differentiable* at a point $p \in M$ if there exists a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$, such that the composition

$$f \circ \varphi\colon U \to \mathbb{R}$$

is differentiable at $x \in U$.

It follows from Lemma 1.2.2 that the Definition 1.2.7 does not depend on the choice of the parametrization φ. In fact, if $\psi\colon V \to \psi(V)$ is another parametrization of M, with $p = \psi(y)$, then

$$f \circ \psi = (f \circ \varphi) \circ (\varphi^{-1} \circ \psi)$$

is also differentiable.

Corollary 1.2.8. If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, then the restriction of f to any regular surface M is a differentiable function on M.

Proof. For any point $p \in M$ and any parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$, the function $f \circ \varphi\colon U \to \mathbb{R}$ is differentiable at $x \in U$. □

Example 1.2.9. Given a regular surface M and a unit vector $v \in \mathbb{R}^3$, consider the height function $f\colon M \to \mathbb{R}$ relative to v, given by $f(p) = \langle p, v\rangle$ for every $p \in M$. It follows immediatly from Corollary 1.2.8 that f is differentiable.

1.3 The tangent plane

In this section we will define the tangent plane to a regular surface M at a point $p \in M$. Before stating the concepts we will need some terminology.

A *parametrized differentiable curve* in $\mathbb{R}^3$ is just a differentiable map $\alpha\colon I \to \mathbb{R}^3$ defined in an open interval $I \subset \mathbb{R}$. The term differentiable means that α is a correspondence which maps each instant $t \in I$ into a point $\alpha(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ in such way that the functions $x(t), y(t), z(t)$ are differentiable. The variable t is called the *parameter* of the curve. The vector $\alpha'(t) = (x'(t), y'(t), z'(t)) \in \mathbb{R}^3$ is called the *tangent vector* of α at t. The image set $\alpha(I) \subset \mathbb{R}^3$ is called the *trace* of α.

Let M be a regular surface. A differentiable curve $\alpha\colon I \to M$ is simply a differentiable curve $\alpha\colon I \to \mathbb{R}^3$ such that $\alpha(I) \subset M$, that is, $\alpha(t) \in M$ for every $t \in I$. Fix a point $p \in M$. A vector $v \in \mathbb{R}^3$ is called a *tangent vector* to M at p if there exists a differentiable curve $\alpha\colon (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. The *tangent plane* to M at p is the collection of all tangent vectors to M at p, and it will be denoted by T_pM.

Proposition 1.3.1. For any parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$, one has

$$T_pM = d\varphi(x)(\mathbb{R}^2).$$

Proof. Any vector in the image of $d\varphi(x)$ is of the form $d\varphi(x)\cdot v$, for some $v \in \mathbb{R}^2$ and therefore is the tangent vector at 0 of the differentiable curve $\alpha(t) = \varphi(x+tv)$. Conversely, let $v \in T_pM$, with $v = \alpha'(0)$, where $\alpha\colon (-\epsilon, \epsilon) \to M$ is a differentiable curve, with $\alpha(0) = p$. By virtue of Corollary 1.2.6, the curve $\gamma = \varphi^{-1} \circ \alpha\colon (-\epsilon, \epsilon) \to U$ is differentiable, with $\beta(0) = x$. Since $\alpha = \varphi \circ \beta$, it follows from the chain rule that

$$v = \alpha'(0) = d\varphi(x) \cdot \gamma'(0)$$

lies in the image of $d\varphi(x)$. □

It follows directly from Proposition 1.3.1 that the tangent plane T_pM is a two-dimensional vector subspace of $\mathbb{R}^3$, and it does not depend on the parametrization φ. Moreover, the choice of a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(q)$, determines a basis $\{\varphi_u(p), \varphi_v(p)\}$ of T_pM, called the *basis associated* to φ. Here the notations φ_u, φ_v mean

$$\varphi_u(p) = \frac{\partial \varphi}{\partial u}(q) = d\varphi(q) \cdot e_1 \quad \text{and} \quad \varphi_v(p) = \frac{\partial \varphi}{\partial v}(q) = d\varphi(q) \cdot e_2$$

Let us see how to determine the coordinates of a vector $v \in T_pM$ in the basis $\{\varphi_u(p), \varphi_v(p)\}$ associated to a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(q)$.

Example 1.3.2. Let M be a regular surface given as the inverse image under a differentiable function $f\colon \mathbb{R}^3 \to \mathbb{R}$ of a regular value, namely $M = f^{-1}(c)$. We claim that

$$T_pM = \ker \mathrm{d}f(p)$$

for any $p \in M$. In fact, let $w \in T_pM$, with $w = \alpha'(0)$, where $\alpha\colon (-\epsilon, \epsilon) \to M$ is a diferentiable curve, with $\alpha(0) = p$. Then, $\beta(t) = (f \circ \alpha)(t)$ is a constant curve along $(-\epsilon, \epsilon)$. By the chain rule we obtain

$$0 = \beta'(0) = \mathrm{d}f(p) \cdot w,$$

and this proves the inclusion $T_pM \subset \ker \mathrm{d}f(p)$ and hence the equality by dimensional reasons.

Given a differentiable map $f\colon M \to N$ and a point $p \in M$, we want to define the differential of f at p, denoted by $\mathrm{d}f(p)$, and being a linear map from T_pM to $T_{f(p)}N$. More precisely, for each vector $w \in T_pM$, define

$$\mathrm{d}f(p) \cdot w = (f \circ \alpha)'(0), \tag{1.3}$$

where $\alpha\colon (-\epsilon, \epsilon) \to M$ is a differentiable curve with $\alpha(0) = p$ and $\alpha'(0) = w$.

Proposition 1.3.3. The map $df(p)\colon T_pM \to T_{f(p)}N$ given in (1.3) is well-defined and is linear, and it will be called the *differential* of f at $p \in M$.

Proof. Firstly, we have to check that $df(p) \cdot w$ does not depend on the choice of curve α. Let $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ parametrizations of M and N, respectivaly, with $p = \varphi(q)$ and $f(\varphi(U)) \subset \psi(V)$. Writing

$$\varphi = \varphi(u, v) \quad \text{and} \quad \psi = \psi(z, w),$$

suppose that f is expressed in these coordinates by

$$f(u, v) = (f_1(u, v), f_2(u, v))$$

and that α is expressed by

$$\alpha(t) = (u(t), v(t)).$$

Thus, the curve $\beta = f \circ \alpha$ can be write as

$$\beta(t) = \big(f_1(u(t), v(t)), f_2(u(t), v(t))\big),$$

and the expression of $\beta'(0)$ in the basis $\{\psi_z, \psi_w\}$ is

$$\beta'(0) = \left(\frac{\partial f_1}{\partial u}u'(0) + \frac{\partial f_1}{\partial v}v'(0), \frac{\partial f_2}{\partial u}u'(0) + \frac{\partial f_2}{\partial v}v'(0)\right). \tag{1.4}$$

The relation (1.4) shows that $\beta'(0)$ depends only on the map f and the coordinates $(u'(0), v'(0))$ of w in the basis $\{\varphi_u, \varphi_v\}$. Therefore, $\beta'(0)$ is independent of α. Moreover, it also follows from (1.4) that

$$\beta'(0) = \mathrm{d}f(p) \cdot w = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}. \tag{1.5}$$

This shows that $\mathrm{d}f(p)$ is a linear map from T_pM to $T_{f(p)}N$, whose matrix in the basis $\{\varphi_u, \varphi_v\}$ of T_pM and $\{\psi_z, \psi_w\}$ of $T_{f(p)}N$ is just the matrix given in (1.5). □

Analogously, given a differentiable function $f\colon M \to \mathbb{R}$, we can define the differential of f at $p \in M$ as a linear map $df(p)\colon T_pM \to \mathbb{R}$ by

$$\mathrm{d}f(p) \cdot v = (f \circ \alpha)'(0),$$

where $\alpha\colon (-\epsilon, \epsilon) \to M$ is a differentiable curve with $\alpha(0) = p$ and $\alpha'(0) = v$.

Example 1.3.4. Given a unit vector $v \in \mathbb{R}^3$, consider the height function $f\colon M \to \mathbb{R}$ given by $f(p) = \langle v, p\rangle$, for every $p \in M$. Fix a point $p \in M$ and let $w \in T_pM$. To compute the differential $df(p) \cdot w$, choose a differentiable curve $\alpha\colon (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = w$. Then

$$\begin{aligned} df(p) \cdot w &= (f \circ \alpha)'(0) = \frac{d}{dt} f(\alpha(t))(0) = \frac{d}{dt}\langle v, \alpha(t)\rangle|_{t=0} \\ &= \langle v, \alpha'(0)\rangle = \langle v, w\rangle. \end{aligned}$$

Example 1.3.5. Let $R_\theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle θ about the axis-z. The rotation $\mathbb{R}_\theta$ is linear, therefore it is differentiable. Moreover, R_θ restricted to the unit sphere $\mathbb{S}^2$ is a differentiable map of $\mathbb{S}^2$. Thus, fixed a point $p \in \mathbb{S}^2$ and given $v \in T_p\mathbb{S}^2$, we obtain

$$dR_\theta(p) \cdot v = R_\theta(v),$$

because R_θ is linear. Note that R_θ leaves the north pole N fixed, and $dR_\theta(N)$ is just a rotation of angle θ in the plane T_NM.

1.4 Orientable surfaces

Intuitively, orientable surfaces are those for which ir is possible to define a clockwise consistently. To ilustrate the underlying idea, we consider two familiar surfaces: a cylinder and a Mobius band. We can distinguish between a cylinder and a Mobius band by noticing that every cylinder has an inside and an outside, and we can paint one blue and other yellow, for example. But if we tray to paint a Mobius band in two colors, we fail because it has just one side.

Let E be a finite-dimensional real vector space. We say that two bases $\mathcal{E}$ and $\mathcal{F}$ *define the same orientation* in E if the transition matrix from $\mathcal{E}$ to $\mathcal{F}$ has positive determinant. In this case, we write $\mathcal{E} \equiv \mathcal{F}$. This property defines an equivalence relation on the set of all bases of E, and each equivalence class according to this relation is called an *orientation* for E. Moreover, the relation $\equiv$ has exactly two equivalence class, that is, the vector space E admits two orientations.

A vector space E together with a choice of orientation $\mathcal{O}$ is called an *oriented vector space*. Once an orientation $\mathcal{O}$ for E is fixed, the other one is called the *opposite orientation*, and it will be denoted by $-\mathcal{O}$. The bases in the orientation $\mathcal{O}$ will be called *positive*, while the others will be called *negative*. A linear isomorphism $T\colon E \to F$ between two oriented vector spaces is called *positive* if maps positive bases of E into positive bases of F.

Example 1.4.1. The Euclidean space $\mathbb{R}^n$ will be considered oriented requiring that the canonical basis of $\mathbb{R}^n$ be positive. Therefore, a linear isomorphism $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is positive if and only if $\det(T) > 0$.

Now we will extend the notion of orientability to each tangent space of a regular surface M. We say that two parametrizations $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ of M are *compatible* if either $\varphi(U) \cap \psi(V) = \emptyset$ or the change of coordinates

$$\psi^{-1} \circ \varphi : \varphi^{-1}(W) \to \psi^{-1}(W),$$

has positive jacobian determinant everywhere on $\varphi^{-1}(W)$.

Remark 1.4.2. If $\varphi(U) \cap \psi(V) \neq \emptyset$, the change of coordinates $\psi^{-1} \circ \varphi$ has jacobian determinant different from zero on $\varphi^{-1}(\varphi(U) \cap \psi(V))$. Since determinant is a continuous function, its sign is constant in each connected component of the open set $\varphi^{-1}(\varphi(U) \cap \psi(V) \subset \mathbb{R}^2$.

Definition 1.4.3. A regular surface M is called *orientable* if there exists a cover $\mathcal{A}$ of M consisting of coordinate neighborhoods such that any two parameterizations of $\mathcal{A}$ are compatible.

The choice of such a cover is called an *orientation* of M, and in this case we say that M is *oriented*. If it is not possible to make such a choice, the surface M is called *nonorientable*.

Example 1.4.4. The plane $\mathbb{R}^2$ is an orientable surface, because the identity map is a global compatible parametrization of $\mathbb{R}^2$. The orientation given by such parametrization is called the *canonical orientation* of $\mathbb{R}^2$.

Example 1.4.5. A regular surface which is the graph of a differentiable function is an orientable surface. More generally, all surfaces which can be covered by one coordinate neighborhood are trivially orientable.

Proposition 1.4.6. An orientation on a regular surface M determines an orientation on each tangent plane of M.

Proof. Let $\mathcal{A}$ be an orientation on M. Given a point $p \in M$, consider a parametrization $\varphi \in \mathcal{A}$, with $p = \varphi(x)$, and define an orientation $\mathcal{O}_p$ on T_pM requiring the basis $\{d\varphi(x) \cdot e_1, d\varphi(x) \cdot e_2\}$ be positive. If ψ is another parametrization in the cover $\mathcal{A}$, with $p = \psi(y)$, we obtain:

$$\mathrm{d}\psi(y) = \mathrm{d}(\varphi \circ \varphi^{-1} \circ \psi)(y) = \mathrm{d}\varphi(x) \circ \mathrm{d}(\varphi^{-1} \circ \psi)(y).$$

The isomorphism $d(\varphi^{-1} \circ \psi)(y)$ preserve orientation, because φ and ψ are compatible, e $d\varphi(x)$ preserves orientation by hypothesis. Therefore, the set $\{d\psi(y) \cdot e_1, d\psi(y) \cdot e_2\}$ is also a positive basis of T_pM. $\square$

Before giving a geometric interpretation of the idea of orientability of a regular surface in $\mathbb{R}^3$, we need a few definitions.

Given a regular surface M, the inner product of $\mathbb{R}^3$ naturally induces an inner product in each tangent plane T_pM of M. More precisely, given $p \in M$ and $v, w \in T_pM$, we define $\langle v, w \rangle$ to be the inner product of v and w as vectors of $\mathbb{R}^3$.

The inner product defined above allows us to consider the notion of orthogonality. More precisely, we say that a vector $\eta \in \mathbb{R}^3$ is *orthogonal* to a regular surface M at a point $p \in M$ if $\langle \eta, v \rangle = 0$, for every $v \in T_pM$. Globally, a *normal vector field* to a regular surface M is a map $\eta \colon M \to \mathbb{R}^3$ such that $\eta(p)$ is orthogonal to T_pM, for every $p \in M$.

Theorem 1.4.7. *A regular surface $M \subset \mathbb{R}^3$ is orientable if and only if there exists a differentiable unit normal vector field $N \colon M \to \mathbb{R}^3$ on M.*

Proof. Let $\mathcal{A}$ be an orientation of M, that is, a cover of M by compatible coordinate neighborhoods. Fix a point $p \in M$ with $p = \varphi(u, v)$, where $\varphi \colon U \to \varphi(U)$ is a positive parametrization of M. Now, define a map $N \colon \varphi(U) \to \mathbb{R}^3$ by

$$N(q) = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|}(q), \tag{1.6}$$

for every $q \in \varphi(U)$, where $\times$ denotes the vector product in $\mathbb{R}^3$. Thus we obtain a differentiable map, orthogonal to M in every point of $\varphi(U)$. If $\psi \colon V \to \psi(V)$ is another positive parametrization of M with $p = \psi(z, w)$, let us denote by $h = \varphi^{-1} \circ \psi$ the change of coordinates, with $(u, v) = h(z, w)$. Thus $\psi = \varphi \circ h$ and by setting $q = (z, w)$, we obtain

$$\begin{aligned} \psi_z(q) &= \varphi_u(h(q))\frac{\partial u}{\partial z}(q) + \varphi_v(h(q))\frac{\partial v}{\partial z}(q) \\ \psi_w(q) &= \varphi_u(h(q))\frac{\partial u}{\partial w}(q) + \varphi_v(h(q))\frac{\partial v}{\partial w}(q) \end{aligned}.$$

It follows that

$$\psi_z \times \psi_w = \det(dh(q)) \cdot \varphi_u \times \varphi_v,$$

and from which we conclude that the normals associated to φ and ψ coincide, since $\det(dh(q)) > 0$. Therefore, if p belongs to the intersection of two coordinate neighborhoods, the corresponding normals coincide, and thus we obtain an unit normal differentiable vector field $N\colon M \to \mathbb{R}^3$. Conversely, given a point $p \in M$, define an orientation in T_pM as follows: a basis $\{v_1, v_2\}$ of T_pM is positive if and only if $\{v_1, v_2, N(p)\}$ is a positive basis of $\mathbb{R}^3$. Given a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$ and U connected, and changing the sign of φ if necessary, we can assume that the set

$$\{d\varphi(x) \cdot e_1, d\varphi(x) \cdot e_2, N(\varphi(x))\}$$

is a positive basis of $\mathbb{R}^3$, for every $x \in U$. Thus, for each point $p \in M$, we can choose a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p \in \varphi(U)$, such that $d\varphi(x)\colon \mathbb{R}^2 \to T_{\varphi(x)}M$ is a positive linear isomorphism, for every $x \in U$. Denote by $\mathcal{A}$ the cover of M by such parametrizations. If $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are two parametrizations in $\mathcal{A}$, with $\varphi(U) \cap \psi(V) \neq \emptyset$, then $\psi^{-1} \circ \varphi$ has positive jacobian determinant everywhere, since $d(\psi^{-1} \circ \varphi)(x)$ is the composite of two positive linear isomorphisms. □

It follows from the proof of Theorem 1.4.7 that, given a point $p \in M$ and a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p \in \varphi(U)$, we can always consider an unit normal differentiable vector field N in a neighborhood of p and given by (1.6). Thus, any regular surface M is always locally orientable.

Let us now look at some examples of orientable surfaces.

Example 1.4.8. Let $\mathcal{P}$ be the plane plane through the point $p \in \mathbb{R}^3$ which contains the orthonormal vectors $w_1, w_2 \in \mathbb{R}^3$. Thus, a parametrization of $\mathcal{P}$ is given by

$$\varphi(u, v) = p + uw_1 + vw_2,$$

with $(u, v) \in \mathbb{R}^2$. In this case one has $\varphi_u = w_1$ and $\varphi_v = w_2$, therefore $N = w_1 \times w_2$ is an unit normal vector field along $\mathcal{P}$.

Example 1.4.9. Consider the right cylinder $\mathcal{C}$ over the circle $x^2 + y^2 = 1$. Then $\mathcal{C}$ admits a parametrization $\varphi\colon U \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (\cos u \sin u, v),$$

where

$$U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 2\pi \text{ and } v \in \mathbb{R}\}.$$

In this case, we obtain

$$\varphi_u = (-\sin u, \cos u, 0) \quad \text{and} \quad \varphi_v = (0,0,1),$$

which implies that

$$N = \varphi_u \times \varphi_v = (\cos u, \sin u, 0)$$

is an unit normal vector field to the cylinder $\mathcal{C}$.

Example 1.4.10. The tangent plane to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ at a point $p \in \mathbb{S}^2$ is given by

$$T_p\mathbb{S}^2 = p^{\perp},$$

where $p^{\perp} = \{v \in \mathbb{R}^3 : \langle v, p\rangle = 0\}$. In fact, fix a vector $v \in T_p\mathbb{S}^2$ with $v = \alpha'(0)$, where $\alpha\colon (-\epsilon, \epsilon) \to \mathbb{S}^2$ is a differentiable curve, with $\alpha(0) = p$. Since $\alpha(t) \in \mathbb{S}^2$ for every $t \in (-\epsilon,\epsilon)$, one has $\|\alpha(t)\| = 1$, for every $t \in (-\epsilon, \epsilon)$. This implies that

$$2\langle \alpha'(t), \alpha(t)\rangle = 0,$$

for every $t \in (-\epsilon,\epsilon)$. For $t = 0$, we obtain $\langle v, p\rangle = 0$. This shows that $T_p\mathbb{S}^2 \subset p^{\perp}$ and hence the equality by dimensional reasons. Therefore, the position vector field $N\colon \mathbb{S}^2 \to \mathbb{R}^3$, $N(p) = p$, is an unit normal vector field to the sphere, and this shows that $\mathbb{S}^2$ is orientable.

Example 1.4.11. Let M be a regular surface given as the inverse image under a differentiable function $f\colon \mathbb{R}^3 \to \mathbb{R}$ of a regular value $c \in \mathbb{R}$, that is, $M = f^{-1}(c)$. Fix a point $p \in M$, with $p = (x_0, y_0, z_0)$, and consider a differentiable parametrized curve $\alpha\colon (-\epsilon,\epsilon) \to M$, with $\alpha(0) = p$ and

$$\alpha(t) = (x(t), y(t), z(t)).$$

Since $\alpha(t) \in M$ for every $t \in (-\epsilon, \epsilon)$, we obtain

$$f(\alpha(t)) = c, \tag{1.7}$$

for every $t \in (-\epsilon,\epsilon)$. By differentiating both sides of (1.7) with respect to t, we see that at $t = 0$

$$\begin{aligned} 0 &= \frac{d}{dt} f(\alpha(t))(0) = \frac{\partial f}{\partial x}(p)\cdot\frac{dx}{dt}(0) + \frac{\partial f}{\partial y}(p)\cdot\frac{dy}{dt}(0) + \frac{\partial f}{\partial z}(p)\cdot\frac{dz}{dt}(0) \\ &= \left\langle \left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p)\right), \left(\frac{dx}{dt}(0), \frac{dy}{dt}(0), \frac{dz}{dt}(0)\right)\right\rangle \\ &= \langle \operatorname{grad} f(p), \alpha'(0)\rangle = \langle \operatorname{grad} f(p), v\rangle. \end{aligned}$$

This shows that the gradient vector of f at $p \in M$ is orthogonal to T_pM. Therefore, the map

$$N(p) = \frac{\text{grad} f}{\|\text{grad} f\|}(p)$$

is an unit normal vector field to M, and thus M is orientable.

Bibliography

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