Clifford algebras and Lie groups

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Contents

CHAPTER 1

Symmetric bilinear forms

In this chapter, $\mathbb{K}$ will denote any field of characteristic $\neq 2$. We are mainly interested in the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and sometimes specialize to those two cases.

1. Basic definitions

Suppose V is a finite-dimensional vector space over $\mathbb{K}$. For any bilinear form $B\colon V \times V \to \mathbb{K}$, define a linear map

$$B^\flat\colon V \to V^*,\ \ v \mapsto B(v, \cdot).$$

The bilinear form B is called *symmetric* if it satisfies $B(v_1, v_2) = B(v_2, v_1)$ for all $v_1, v_2 \in V$. Since $\dim V < \infty$ this is equivalent to $(B^\flat)^* = B^\flat$. Since we assume $\mathrm{char}(\mathbb{K}) \neq 2$, the symmetric bilinear form B is uniquely determined by the associated quadratic form, $Q_B(v) = B(v, v)$ by the polarization identity,

$$B(v, w) = \tfrac{1}{2}\big(Q_B(v + w) - Q_B(v) - Q_B(w)\big).$$

The *kernel* (also called *radical*) of B is the subspace

$$\ker(B) = \{v \in V|\ B(v, v_1) = 0 \text{ for all } v_1 \in V\},$$

i.e. the kernel of the linear map $B^\flat$. The bilinear form B is called *non-degenerate* if $\ker(B) = 0$, i.e. if and only if $B^\flat$ is an isomorphism. For any subspace $F \subset V$, the *orthogonal* or *perpendicular* subspace is defined as

$$F^\perp = \{v \in V|\ B(v, v_1) = 0 \text{ for all } v_1 \in F\}.$$

Note that $\ker(B) = V^\perp$. More generally, for any subspace $F \subset V$ the restriction of B to F has kernel $\ker(B|_{F\times F}) = F \cap F^\perp$.

Proposition 1.1. *If B is non-degenerate, then*

$$\dim F + \dim F^\perp = \dim V \tag{1}$$

for all $F \subset V$. Furthermore,

$$(F^\perp)^\perp = F,\quad (F_1 \cap F_2)^\perp = F_1^\perp + F_2^\perp,\quad (F_1 + F_2)^\perp = F_1^\perp \cap F_2^\perp$$

for all $F, F_1, F_2 \subset V$.

Proof. Observe that the image of $F^\perp$ under the isomorphism $B^\flat\colon V \to V^*$ is the annihilator of F in V^*,

$$B^\flat\colon F^\perp \xrightarrow{\cong} \mathrm{ann}(F).$$

The dimension formula now follows from $\dim F + \dim \mathrm{ann}(F) = \dim V$, and the other statements are easy consequences by dimension count. ☐

A vector space V together with a non-degenerate symmetric bilinear form B will often be referred to as a *quadratic vector space.* [1]

2. Isotropic and coisotropic subspaces

DEFINITION 2.1. Let V be a vector space with a possibly degenerate symmetric bilinear form B. A subspace $F \subset V$ is called

- *non-degenerate* or *quadratic* if $F \cap F^\perp = 0$,
- *isotropic* if $F \subset F^\perp$,
- *co-isotropic* if $F^\perp \subset F$,
- *Lagrangian* if $F = F^\perp$.

Observe that $F \subset V$ is isotropic if and only if $B|_{F\times F} = 0$, and F is non-degenerate if and only if $B|_{F\times F}$ is non-degenerate.

LEMMA 2.2. *Suppose B is non-degenerate. Then*

(1) *F is quadratic if and only if $F^\perp$ is quadratic, if and only if $V = F \oplus F^\perp$.*
(2) *F is isotropic if and only if $F^\perp$ is co-isotropic.*
(3) *For any isotropic subspace F, $\dim F \le \dim V/2$, with equality if and only if F is Lagrangian.*

PROOF. Clear from $(F^\perp)^\perp = F$ and the dimension formula $\dim V = \dim F + \dim F^\perp$. □

Vectors v with $B(v, v) = 0$ are called *isotropic vectors.* By the polarization identity, all vectors of a subspace F are isotropic if and only if B vanishes on F, if and only if F is isotropic. Note on the other hand that the set of isotropic vectors is usually not a subspace.

An isotropic subspace is called *maximal isotropic* if it is not properly contained in a another isotropic subspace.

LEMMA 2.3. *An isotropic subspace F is maximal isotropic if and only if*

$$F = \{v \in F^\perp |\ B(v, v) = 0\}.$$

PROOF. Suppose F is isotropic, and $v \in V$. Then $F + \mathrm{span}(v)$ is isotropic if and only if $v \in F^\perp$ and $B(v, v) = 0$. Thus, F is maximal isotropic if and only if it equals the set of all $v \in F^\perp$ with $B(v, v) = 0$. □

PROPOSITION 2.4. *Suppose (V, B) is a quadratic vector space.*

(a) *If F is a maximal isotropic subspace, and F' an isotropic subspace, then*
$$F \cap F' = F^\perp \cap F'.$$
(b) *If F, F' are maximal isotropic, then*
$$F \cap F' = 0 \Leftrightarrow V = F^\perp \oplus F'.$$
(c) *If F, F' are isotropic, then*
$$V = F^\perp \oplus F' \Leftrightarrow V = F \oplus (F')^\perp.$$
Furthermore, these conditions imply $\dim F = \dim F'$.

[1] This terminology is misleading if one is dealing with super-symmetric bilinear forms on super-vector spaces. Indeed, such forms are usually *not* determined by the associated quadratic form. For instance, a symplectic vector space may be viewed as a purely odd vector space with symmetric bilinear form, but the associated quadratic form is zero.

(d) *For any isotropic* $F \subset V$ *there exists an isotropic subspace* F' *such that* $F^\perp \oplus F' = V$.

PROOF. Since elements $v \in F'$ are isotropic, (a) is immediate from Lemma 2.3. If F, F' are maximal isotropic, it follows from (a) that

$$F \cap F' = 0 \Leftrightarrow F^\perp \cap F' = 0 \Leftrightarrow F \cap (F')^\perp = 0 \Leftrightarrow F^\perp + F' = V.$$

From the second and fourth version of this equality we find that in fact the sum $F^\perp \oplus F' = V$ is direct. This proves (b). To prove (c), observe that $V = F^\perp \oplus F'$ gives $\dim F = \dim F'$ by dimension count. If F, F' are isotropic, then

$$V = F^\perp \oplus F' \Leftrightarrow 0 = F \cap (F')^\perp \Rightarrow V = F \oplus (F')^\perp.$$

(the last sum is all of V, for dimensional reasons). Switching the roles of F, F' one obtains the reverse implication. To prove (d), let F be an isotropic subspace. We begin by choosing *any* complement W to $F^\perp$. (Thus, $\dim F = \dim W$.) There exists a unique linear map $S\colon W \to F$ with the property,

$$B(S(w_1), w_2) = B(w_1, w_2), \quad w_1, w_1 \in W.$$

The map S is obtained by composing the linear map $(B|_W)^\flat\colon W \to W^*$ with the inverse to the map

$$F \to W^*, \; v \mapsto B(v, \cdot).$$

(This last map is 1-1 since for any non-zero $w \in W \cong V/F^\perp$, there exists $v \in F$ with $B(v, w) \neq 0$. Hence, it is an isomorphism by dimension count.) Using S let

$$F' = \{w - \tfrac{1}{2}S(w) |\; w \in W\}.$$

Then F' is still a complement to $F^\perp$, and since

$$B(w - \tfrac{1}{2}S(w), w_1 - \tfrac{1}{2}S(w_1)) = B(w, w_1) - \tfrac{1}{2}\big(B(w, S(w_1)) + B(S(w), w_1)\big) = 0$$

it follows that F' is isotropic. □

PROPOSITION 2.5. *For any two maximal isotropic subspaces* F, F' *of a quadratic vector space, the kernel of* $B|_{F+F'}$ *equals* $F \cap F'$, *and the images of* F, F' *in the quadratic vector space* $(F+F')/(F\cap F')$ *are Lagrangian. In particular, if* $F \cap F' = 0$ *then* $F \oplus F'$ *is quadratic.*

PROOF. Using Property (a) from the last Proposition, we find $(F+F')\cap F^\perp = F + (F' \cap F^\perp) = F + (F' \cap F) = F$. Hence,

$$\begin{aligned}(F + F') \cap (F + F')^\perp &= (F + F') \cap F^\perp \cap (F')^\perp \\ &= F \cap (F')^\perp \\ &= F \cap F'.\end{aligned}$$

This calculation shows that $\ker(B|_{F+F'}) = F \cap F'$. Hence B descends to a non-degenerate bilinear form on the quotient,

$$W = (F + F')/(F \cap F').$$

Let $\pi\colon F + F' \to W$ denote the projection. Then

$$\pi^{-1}(\pi(F)^\perp) = (F + F') \cap F^\perp = F$$

(see above), and hence

$$\pi(F)^\perp = \pi(F).$$

□

An immediate consequence of this Proposition is that any two maximal isotropic subspaces have the same dimension: $\dim \pi(F) = \dim \pi(F')$ since $\pi(F), \pi(F')$ are Lagrangian, and consequently $\dim F = \dim F'$. Thus, if F is maximal isotropic, then any isotropic complement to $F^\perp$ is again maximal isotropic, for dimensional reasons.

DEFINITION 2.6. The *index* of a non-degenerate symmetric bilinear form B is the dimension of a maximal isotropic subspace. The bilinear form is called *split* if $\dim V = 2\,\mathrm{index}(B)$.

Thus, B is split if and only if there are Lagrangian subspaces. For a general quadratic vector space (V, B), pick two transverse maximal isotropic subspaces F, F'. Then $V_1 = F \oplus F'$ is quadratic, with a split bilinear form, and $V = V_1 \oplus V_2$ is an orthogonal direct sum where the bilinear form on $V_2 = V_1^\perp$ has index 0. Split bilinear forms are easily classified:

PROPOSITION 2.7. *Let (V, B) be a quadratic vector space with a split bilinear form. Then there exists a basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ of V in which the bilinear form is given as follows:*

$$B(e_i, e_j) = 0,\ B(e_i, f_j) = \delta_{ij},\ B(f_i, f_j) = 0. \tag{2}$$

PROOF. Choose a pair of complementary Lagrangian subspaces, F, F'. Since B defines a non-degenerate pairing between F and F', it defines an isomorphism, $F' \cong F^*$. Choose a basis $e_1, \ldots, e_k$, and let $f_1, \ldots, f_k$ be the dual basis of F' under this identification. Then $B(e_i, f_j) = \delta_{ij}$ by definition of dual basis, and $B(e_i, e_j) = B(f_i, f_j) = 0$ since F, F' are Lagrangian. $\square$

A general symmetric bilinear form B on V can always be 'diagonalized', in the following sense. A basis $E_1, \ldots, E_n$ of V is called an *orthogonal basis* if $B(E_i, E_j) = 0$ for $i \neq j$.

PROPOSITION 2.8. *For any symmetric bilinear form B on V there is an orthogonal basis. Furthermore, any linearly independent set of pairwise orthogonal vectors $E_1, \ldots, E_k$ with $B(E_i, E_i) \neq 0$ for $1 \leq i \leq k$ can be extended to an orthogonal basis.*

The proof is a straightforward induction.

Our basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ for a quadratic vector space (V, B) with split bilinear form is not orthogonal. However, it may be replaced by an orthogonal basis

$$E_i = e_i + \tfrac{1}{2} f_i,\ \ \tilde{E}_i = e_i - \tfrac{1}{2} f_i.$$

In the new basis, the bilinear form reads,

$$B(E_i, E_j) = \delta_{ij},\ B(E_i, \tilde{E}_j) = 0,\ B(\tilde{E}_i, \tilde{E}_j) = -\delta_{ij}. \tag{3}$$

Consider the case $\mathbb{K} = \mathbb{R}$. Denote by $\mathbb{R}^{n,m}$ the space $\mathbb{R}^{n+m}$ with basis $E_1, \ldots, E_n$, $\tilde{E}_1, \ldots, \tilde{E}_m$ and bilinear form B given by the formulas (3). This has index $\mathrm{index}(B) = \min(n, m)$, the span of $E_i + \tilde{E}_i$ for $i = 1, \ldots, \min(n, m)$ is a maximal isotropic subspace.[2]

PROPOSITION 2.9. *Let (V, B) be a quadratic vector space over $\mathbb{R}$. Then there are unique integers n, m with $n + m = \dim V$, such that there exists an isomorphism of $V \to \mathbb{R}^{n,m}$ preserving bilinear forms.*

[2] Note that this differs mildly from the notion of index in Morse theory, which is defined to be m, i.e. the number of minus signs.

This is proved by choosing a suitable orthogonal basis; details are left as an exercise. For $\mathbb{K} = \mathbb{C}$, the classification is even easier:

PROPOSITION 2.10. *Let (V, B) be a quadratic vector space over $\mathbb{R}$. Then there exists an isomorphism $V \to \mathbb{C}^n$ preserving bilinear forms. (Here $\mathbb{C}^n$ carries the standard bilinear form). The index of B equals $n/2$ if n is even, and $(n-1)/2$ if n is odd.*

PROOF. Let $E_1', \ldots, E_n'$ be an orthogonal basis of V. Choose $\lambda_i \in \mathbb{C}$ with $\lambda_i^2 = B(E_i', E_i')$ and set $E_i = \lambda_i^{-1} E_i'$. The span of

$$E_1 + \sqrt{-1}E_2,\ E_3 + \sqrt{-1}E_4, \cdots, E_{2k-1} + \sqrt{-1}E_{2k}$$

is a maximal isotropic subspace; here $k = n/2$ if n is even, and $k = (n-1)/2$ if n is odd. $\square$

3. The orthogonal group $\mathrm{O}(V)$

Let V be a vector space with a symmetric bilinear form B. The *orthogonal group* $\mathrm{O}(V)$ is the group

$$\mathrm{O}(V) = \{A \in \mathrm{GL}(V) |\ \ B(Av, Aw) = B(v, w) \text{ for all } v, w \in V\}.$$

The subgroup of orthogonal transformations of determinant 1 is denoted $\mathrm{SO}(V)$, and is called the *special orthogonal group*. For the case $V = \mathbb{K}^n$, with bilinear form $B(E_i, E_j) = \delta_{ij}$, we write $\mathrm{O}(n, \mathbb{K})$ and $\mathrm{SO}(n, \mathbb{K})$.

If $V = \mathbb{K}^{n,m}$ (defined similar to $\mathbb{R}^{n,m}$) we will write $\mathrm{O}(n, m; \mathbb{K})$ and $\mathrm{SO}(n, m; \mathbb{K})$. Note

$$\mathrm{O}(n, \mathbb{K}) \times \mathrm{O}(m, \mathbb{K}) \subset \mathrm{O}(n, m; \mathbb{K}).$$

If (V, B) is a quadratic vector space with split bilinear form, denote by $\mathrm{Lag}(V)$ the set of Lagrangian subspaces. Recall that any such V is isomorphic to $\mathbb{K}^{n,n}$ where $\dim V = 2n$. For $\mathbb{K} = \mathbb{R}$ we have the following result.

THEOREM 3.1. *Let $V = \mathbb{R}^{n,n}$ with the standard basis satisfying* (3). *Then*

$$L_0 = \mathrm{span}\{E_1 + \tilde{E}_1, \ldots, E_n + \tilde{E}_n\}$$

is a Lagrangian subspace of V, and any other Lagrangian subspace $L \in \mathrm{Lag}(V)$ is obtained from L_0 by a unique orthogonal transformation in the subgroup $\mathrm{O}(n, \mathbb{R}) \times \{1\} \subset \mathrm{O}(n, n; \mathbb{R})$. That is,

$$\mathrm{Lag}(V) \cong \mathrm{O}(n, \mathbb{R}).$$

PROOF. Let V_+ be the span of the E_i's and V_- the span of the $\tilde{E}_i$'s. For any $A \in \mathrm{O}(n, \mathbb{R}) = \mathrm{O}(V_+)$, the calculation

$$B(A(E_i) + \tilde{E}_i, A(E_j) + \tilde{E}_j) = B(A(E_i), A(E_j)) - \delta_{ij} = 0,$$

shows that the image of L_0 under A is Lagrangian. Conversely, let $L \in \mathrm{Lag}(V)$ be an arbitrary Lagrangian subspace. Then $L \cap V_+ = \{0\} = L \cap V_-$, since L is Lagrangian. Hence, L is the graph of an invertible linear map $V_- \to V_+$. It hence admits a basis $A(E_1) + \tilde{E}_1, \ldots, A(E_n) + \tilde{E}_n$ for a unique linear transformation $A \in \mathrm{GL}(V_+)$. The fact that L is Lagrangian gives

$$0 = B(A(E_i) + \tilde{E}_i, A(E_j) + \tilde{E}_j) = B(A(E_i), A(E_j)) - \delta_{ij},$$

hence $A \in \mathrm{O}(n, \mathbb{R})$. $\square$

REMARK 3.2. There is a similar result in symplectic geometry, for a real vector space V with a non-degenerate *skew*-symmetric linear form. Any such V is identified with $\mathbb{C}^n = \mathbb{R}^{2n}$ with the standard symplectic form, $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$ is a Lagrangian subspace, and the action of $\mathrm{U}(n) \subset \mathrm{Sp}(V, \omega)$ on L_0 identifies

$$\mathrm{Lag}(V) \cong \mathrm{U}(n)/\,\mathrm{O}(n)$$

The analogue of $\mathrm{U}(n) \subset \mathrm{Sp}(V, \omega)$ in our case is the subgroup $\mathrm{O}(n) \times \mathrm{O}(n) \subset \mathrm{O}(n, n)$), and the result simplifies since $\mathrm{O}(n) \times \mathrm{O}(n)/\,\mathrm{O}(n) \cong \mathrm{O}(n)$.

Theorem 3.1 does not, as it stands, generalize to other fields. Indeed, the group $\mathrm{O}(n) \times \mathrm{O}(n)$ takes L_0 to a Lagrangian subspace transverse to V_+, V_-. However, there may be other Lagrangian subspaces: E.g. if $\mathbb{K} = \mathbb{C}$ and $n = 2$, the span of $E_1 + \sqrt{-1}E_2$ and $\tilde{E}_1 + \sqrt{-1}\tilde{E}_2$ is a Lagrangian subspace not transverse to $V_\pm$. Nonetheless, there is a good description of the space Lag in the complex case.

Let $V = \mathbb{C}^{2m}$, viewed as the complexification of $\mathbb{R}^{2m}$. Recall that an orthogonal complex structure on $\mathbb{R}^{2m}$ is an automorphism $J \in \mathcal{O}(2m)$ with $J^2 = -I$. Then J has eigenvalues $\pm\sqrt{-1}$, and the corresponding eigenspaces are complex conjugate:

$$F = \ker(J - \sqrt{-1}I), \quad \bar{F} = \ker(J + \sqrt{-1}I),$$

and $\mathbb{C}^n = F \oplus \overline{F}$. In fact, F and $\bar{F}$ are Lagrangian: If $v \in F$ then

$$B(v, v) = B(Jv, Jv) = B(\sqrt{-1}v, \sqrt{-1}v) = -B(v, v).$$

Conversely, given a Lagrangian subspace F we may recover J, as follows: Given $w \in \mathbb{R}^{2n}$, we may uniquely write $w = v + \overline{v}$ where $v \in F$. Thus $w = 2\,\mathrm{Re}(v)$. Define a linear map J by $Jw := -2\,\mathrm{Im}(v)$. Then $v = w - \sqrt{-1}Jw$. Since F is Lagrangian, we have

$$0 = B(v, v) = B(w - \sqrt{-1}Jw, w - \sqrt{-1}Jw) = B(w, w) - B(Jw, Jw) - 2\sqrt{-1}B(w, Jw),$$

which shows that $J \in \mathrm{O}(V)$ and that $B(w, Jw) = 0$ for all w. Multiplying the definition of J by $\sqrt{-1}$, we get

$$\sqrt{-1}v = \sqrt{-1}w + Jw$$

which shows that $J(Jw) = -w$. Hence J is an orthogonal complex structure. It follows that $\mathrm{Lag}(\mathbb{C}^{2m}$ can be identified with the space $\mathcal{J}(2m)$ of orthogonal complex structures on $\mathbb{R}^{2m}$. The transitive action of $\mathcal{O}(2m)$ translates into the standard action on $\mathcal{J}(2m)$, with stabilizer at J_0 (the standard complex structure, corresponding to $\mathbb{R}^{2m} \cong \mathbb{C}^m$) equal to the unitary group $\mathrm{U}(m)$. We conclude

THEOREM 3.3. *The space of Lagrangian subspaces of $V = \mathbb{C}^{2m}$ is naturally isomorphic to the homogeneous space of complex structures on $\mathbb{R}^{2m}$:*

$$\mathrm{Lag}(V) \cong \mathrm{O}(2m)/\,\mathrm{U}(m).$$

In particular, it is a compact space with two connected components.

Let us return to quadratic vector spaces over arbitrary fields.

PROPOSITION 3.4. *Suppose (V, B) is a quadratic vector space with split bilinear form, and let F, F' be transverse Lagrangian subspaces. The subgroup of $\mathrm{O}(V)$ preserving the splitting $V = F \oplus F'$ is isomorphic to $\mathrm{GL}(F)$.*

PROOF. Recall that B identifies $F' \cong F^*$. Given $B \in \mathrm{GL}(F)$, let $B' = (B^{-1})^* \in \mathrm{GL}(F')$. Then the transformation $A = B \oplus B'$ of $F \oplus F'$ is orthogonal. Conversely, it is easy to see that any orthogonal transformation preserving the decomposition $F \oplus F'$ must have this form. □

COROLLARY 3.5. *Let (V, B) be a quadratic vector space, and $F \subset V$ isotropic. Then any general linear transformation $h \in \mathrm{GL}(F)$ extends to an orthogonal transformation $g \in \mathrm{O}(V)$, with the property that the induced transformations of $F^\perp/F$ and $V/F^\perp \cong F^*$ are the trivial transformation and $(h^{-1})^*$, respectively.*

Note that since g preserves F, it is automatic that g preserves $F^\perp$.

PROOF. Pick an isotropic complement F' to $F^\perp$, and apply the Proposition to the quadratic vector space $V_1 = F \oplus F'$. Extend to an orthogonal transformation of V, equal to I on $V_2 = V_1^\perp$. □

PROPOSITION 3.6. *Suppose (V, B) is a quadratic vector space. The orthogonal group $\mathrm{O}(V)$ acts transitively on the space of maximal isotropic subspaces, and more generally on the space of isotropic subspaces of a given dimension.*

PROOF. Suppose first that F, F' are transverse Lagrangian subspaces. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be the associated basis of V. Then the linear map given by

$$e_i \mapsto f_i, \;\; f_i \mapsto -e_i$$

is orthogonal, and takes F to F'. In the general case, let V_1 be a complement to $F \cap F'$ inside $F + F'$. Then V_1 is quadratic, and $V_1 \cap F$ and $V_1 \cap F'$ are transverse Lagrangian subspaces. By the above, there is an orthogonal transformation of V_1 taking $V_1 \cap F$ to $V_1 \cap F'$. Its extension to an orthogonal transformation of V, equal to I on $V_1^\perp$, has the desired properties.

More generally, suppose F, F' are two isotropic subspaces of equal dimension, and F_1, F_1' are maximal isotropic subspaces containing F, F' respectively. Since there exists an orthogonal transformation taking F_1' to F_1, we may assume $F_1' = F_1$. Now pick any $B \in \mathrm{GL}(F_1)$ taking F to F'. By the previous proposition, B extends to an orthogonal transformation of V. □

For any subspace $S \subset V$, let $\mathrm{O}(V)_S$ denote the subgroup fixing each vector of S. The following result will be needed in our proof of the Cartan-Dieudonné theorem in the following section.

PROPOSITION 3.7. *Suppose (V, B) is a quadratic vector space, and $F \subset V$ is an isotropic subspace.*

(1) *There is a canonical group isomorphism*

$$\mathrm{O}(V)_{F^\perp} \cong \wedge^2(F)$$

where we think of the Abelian group $\wedge^2(F)$ as the space of linear maps $D\colon F^* \to F$ such that $D^* = -D$. For $A \in \mathrm{O}(V)_{F^\perp}$, the range of the corresponding D equals the range of $A - I$. In particular, the rank $n(A)$ of the map $A - I$ is even.

(2) *If F is* maximal *isotropic, and $A \in \mathrm{O}(V)$, then the following are equivalent:*

$$\begin{aligned}(A-I)^2 = 0 &\Leftrightarrow \mathrm{im}(A-I) \text{ is isotropic} \\ &\Leftrightarrow A \text{ is conjugate to an element of } \mathrm{O}(V)_{F^\perp}.\end{aligned}$$

(3) *Two elements* $A_1, A_2 \in \mathrm{O}(V)$ *with* $(A_i - I)^2 = 0$ *are conjugate if and only if* $n(A_1) = n(A_2)$.

PROOF. A simple fact use throughout this proof is that for any $A \in \mathrm{O}(V)$,

$$\operatorname{im}(A - I) = \ker(A^t - I)^\perp = \ker((-A^t)(A - I))^\perp = \ker(A - I)^\perp.$$

If F is isotropic,

$$A \in \mathrm{O}(V)_{F^\perp} \Leftrightarrow \ker(A - I) \supset F^\perp \Leftrightarrow \operatorname{im}(A - I) \subset F.$$

It follows that $\tilde{D} := A - I$ induces a linear map,

$$D\colon V/F^\perp \to F.$$

The bilinear form B identifies $V/F^\perp \cong F^*$, so that D can also be viewed as a linear map $D\colon F^* \to F$. By definition

$$\langle \alpha_1, D(\alpha_2)\rangle = B(x_1, \tilde{D}x_2)$$

if $\alpha_i \in F^* \cong V/F^\perp$ are the images of $x_i \in V$. From

$$\begin{aligned} B(x_1, x_2) &= B(Ax_1, Ax_2) = B((I + \tilde{D})x_1, (I + \tilde{D})x_2) \\ &= B(x_1, x_2) + B(x_1, \tilde{D}x_2) + B(\tilde{D}x_1, x_2) \end{aligned}$$

we obtain $B(x_1, \tilde{D}x_2) + B(\tilde{D}x_1, x_2) = 0$, which shows $\tilde{D}^t = -\tilde{D}$ and hence $D^* = -D$. Conversely, if D is skew-adjoint, lift D to a linear map $\tilde{D}\colon V \to F$ (equal to 0 on $F^\perp$) and set $A = I + \tilde{D}$. Then the above calculation (read in reverse) shows that A is orthogonal. Note finally that if $A_i = I + \tilde{D}_i \in \mathrm{O}(V)_{F^\perp}$ for $i = 1, 2$, then $\tilde{D}_1\tilde{D}_2 = 0$, and therefore

$$A_1A_2 = (I + \tilde{D}_1)(I + \tilde{D}_2) = I + \tilde{D}_1 + \tilde{D}_2.$$

Hence, the map $\mathrm{O}(V)_{F^\perp} \to \wedge^2(F),\ A \mapsto D$ is a group isomorphism. This proves (1). Consider next (2), where now F is maximal isotropic:

$$\begin{aligned} (A - I)^2 = 0 &\Leftrightarrow \operatorname{im}(A - I) \subset \ker(A - I) \equiv \operatorname{im}(A - I)^\perp \\ &\Leftrightarrow \exists g \in \mathrm{O}(V):\ g \cdot \operatorname{im}(A - I) \subset F \\ &\Leftrightarrow \exists g \in \mathrm{O}(V):\ \operatorname{im}(gAg^{-1} - I) \subset F \\ &\Leftrightarrow \exists g \in \mathrm{O}(V):\ gAg^{-1} \in \mathrm{O}(V)_{F^\perp}. \end{aligned}$$

Here we have used that by Proposition 3.6, any isotropic subspace is conjugate to a subspace of the given maximal isotropic subspace F.

To prove (3), fix a maximal isotropic subspace F. We may assume, conjugating by elements of $\mathrm{O}(V)$ if necessary, that $\operatorname{im}(A_i - I) \subset F$ for $i = 1, 2$. Now, if $D\colon F^* \to F$ is a fixed skew-adjoint linear map of rank l (necessarily even), there exists a basis[3] $e_1, \ldots, e_k$ of F, with dual basis $f_1, \ldots, f_k$ of F^*, such that

$$D(f_1) = e_2,\ D(f_2) = -e_1, \ldots, D(f_{l-1}) = e_l,\ D(f_l) = -e_{l-1},$$

[3] D can be interpreted as a rank l skew-symmetric bilinear form ω on F^*. There exists a basis $f_1, \ldots, f_k$ of F^* such that

$$\omega(f_1, f_2) = -\omega(f_2, f_1) = 1,\ \ldots, \omega(f_{l-1}, f_l) = -\omega(f_l, f_{l-1}) = 1,$$

and all other pairings between basis vectors equal to zero. In terms of the dual basis e_i of F, this means $D(f_1) = e_2,\ D(f_2) = -e_1$ etc.

and $D(f_j) = 0$ for $l < j \leq k$. (This follows from the standard normal form theorems for skew-symmetric bilinear forms.) If we apply this result to the skew-adjoint maps $D_i \colon F^* \to F$ corresponding to A_i, we see that D_1, D_2 are conjugate by some element $h \in \mathrm{GL}(F)$. Since F is isotropic, Corollary 3.5 shows that h extends to an orthogonal transformation $g \in \mathrm{O}(V)$, with the property that the induced transformation of $F^\perp/F$ and $F^* = V/F^\perp$ are 1 resp. $(h^{-1})^*$. This g then satisfies $g\tilde{D}_1 g^{-1} = \tilde{D}_2$, and hence $gA_1g^{-1} = A_2$. □

Let $F \subset V$ be an isotropic subspace, and consider the direct sum $W = F \oplus F^*$, with bilinear form given by the pairing. That is, F, F^* are Lagrangian subspaces of $W = F \oplus F^*$, and for $v \in F$ and $\alpha \in F^*$

$$B_W(\alpha, v) = \alpha(v).$$

Then there is a canonical isomorphism,

$$\mathrm{O}(V)_{F^\perp} \cong \mathrm{O}(F \oplus F^*)_F. \tag{4}$$

since both spaces are canonically isomorphic to the space $\wedge^2(F)$ of skew-adjoint linear maps.

4. The E.Cartan-Dieudonné Theorem

Throughout this Section, we assume that (V, B) is a quadratic vector space. Suppose $v \in V$ is a non-isotropic vector, that is, $B(v, v) \neq 0$. Then v determines a unique transformation $R_v \in \mathrm{O}(V)$ fixing the orthogonal complement $\mathrm{span}(v)^\perp$ and taking v to $-v$. That is,

$$\ker(R_v - I) = \mathrm{span}(v)^\perp, \quad R_v(v) = -v$$

The *reflection* R_v is given by the explicit formula,

$$R_v(w) = w - 2\frac{B(v, w)}{B(v, v)} v,$$

as one checks by considering the cases $w = v$ and $B(v, w) = 0$. Some fairly obvious properties of reflections are,

(1) $\det(R_v) = -1$,
(2) $R_v^2 = I$,
(3) if $v_1, v_2 \in V$ are non-isotropic, and $B(v_1, v_2) = 0$, then $R_{v_1} R_{v_2} = R_{v_2} R_{v_1}$,
(4) $A R_v A^{-1} = R_{Av}$ for all $A \in \mathrm{O}(V)$.

For any $A \in \mathrm{O}(V)$, let $l(A)$ denote the smallest number l such that $A = R_{v_1} \cdots R_{v_l}$ where $v_i \in V$ are non-isotropic. (We set $l(I) = 0$.) It is by no means obvious that this is always possible, but we have:

THEOREM 4.1 (E.Cartan-Dieudonné). *Any orthogonal transformation $A \in \mathrm{O}(V)$ can be written as a product of $l(A) \leq \dim V$ reflections.*

The proof of this Theorem will require some preparations. For any $A \in \mathrm{O}(V)$ let $n(A)$ denote the codimension of the space of A-fixed vectors, or equivalently

$$n(A) = \dim(\mathrm{im}(A - I)).$$

LEMMA 4.2. *For any $A \in \mathrm{O}(V)$ and any non-isotropic $v \in V$,*

$$\mathrm{im}(R_v A - I) + \mathrm{span}(v) = \mathrm{im}(A - I) + \mathrm{span}(v).$$

In particular, the numbers $n(R_v A)$ and $n(A)$ differ by at most 1.

PROOF. Taking orthogonal complements, the Lemma is equivalent to

$$\ker(R_v A - I) \cap \ker(R_v - I) = \ker(A - I) \cap \ker(R_v - I).$$

But this is just the obvious fact that if $w \in V$ with $R_v w = w$, then $Aw = w$ if and only if $R_v A w = w$. □

By induction, the Lemma implies $\mathrm{im}(R_{v_1} \cdots R_{v_l} - I) \subset \mathrm{span}(v_1, \ldots, v_l)$, and hence gives a lower bound

$$n(A) \leq l(A)$$

for any $A \in \mathrm{O}(V)$. Clearly, $n(A) = 0 \Leftrightarrow A = I$. We could hence prove the Cartan-Dieudonné theorem by induction, if we could always find a non-isotropic $v \in V$ such that $n(R_v A) = n(A) - 1$. Unfortunately, as it turns out, such an element does not exist if $\mathrm{im}(A - I)$ is isotropic. Hence we will first discuss that case separately.

PROPOSITION 4.3. *Suppose* $\mathrm{im}(A - I)$ *is isotropic. Then* $l(A) \leq 2n(A) \leq \dim V$.

PROOF. Let $F = \mathrm{im}(A-I)$, and let $F' \subset V$ be an isotropic complement to $F^\perp$. Then $V_1 = F \oplus F'$ is non-degenerate with a split bilinear form, i.e. $V_1 \cong F \oplus F^*$. Since $F^\perp = \ker(A - I)$, we have $A \in \mathrm{O}(V)_{F^\perp}$. Since A fixes $V_1^\perp \subset F^\perp$ pointwise, we can think of A as an orthogonal transformation of V_1.

This reduces the problem to the case $V = F \oplus F^*$ with $A \in \mathrm{O}(V)_F$. Let $D : F^* \to F$ be the skew-adjoint linear map corresponding to A, so that $A = I + D$. The assumption $\ker(A - I) = F$ means that $D\colon F^* \to F$ has zero kernel, i.e. it is invertible. In particular $\dim F$ must be *even*. By bringing D into normal form, it finally suffices to consider the case $\dim F = 2$, with basis e_1, e_2 and dual basis f_1, f_2, and D given by

$$Df_1 = e_2,\ Df_2 = -e_1.$$

In this case $n(A) = 2$, and we have to show that A is a product of 4 reflections. Observe that the square A^2 has a similar block form, but with D replaced by $\tilde{D} = 2D$. The base change

$$\tilde{e}_1 = 2e_1, \tilde{e}_2 = e_2,\ \tilde{f}_1 = \tfrac{1}{2} f_1,\ \tilde{f}_2 = f_2.$$

takes $\tilde{D}$ back to D. (Note that nothing happens in the $e_2 - f_2$ plane.) Let $g \in \mathrm{O}(V)$ be the orthogonal transformation given by $\tilde{e}_i \mapsto e_i,\ \tilde{f}_i \mapsto f_i$, so that

$$A^2 = gAg^{-1}.$$

One easily verifies that $g = R_{v_1} R_{v_2}$, where

$$v_1 = e_1 - f_1,\ v_2 = e_1 - \tfrac{1}{2} f_1.$$

Indeed, using

$$R_{v_1} e_1 = f_1, \quad R_{v_2} e_1 = \tfrac{1}{2} f_1$$

(while R_{v_1}, R_{v_2} fix e_2, f_2), one obtains

$$R_{v_1} R_{v_2}(\tilde{e}_1) = R_{v_1} R_{v_2}(2e_1) = R_{v_1}(f_1) = e_1,$$

$$R_{v_1} R_{v_2}(\tilde{f}_1) = R_{v_1} R_{v_2}(\tfrac{1}{2} f_1) = R_{v_1}(e_1) = f_1.$$

Hence

$$A = gAg^{-1}A^{-1} = R_{v_1} R_{v_2} R_{A(v_2)} R_{A(v_1)}$$

is a product of 4 reflections, as required. □

LEMMA 4.4. *Let* $A \in \mathrm{O}(V)$.

(1) *Suppose* $w \in V$ *is an element such that* $v = (A - I)w$ *is non-isotropic. Then* $R_v Aw = w$.
(2) *If* $\mathrm{im}(A - I)$ *is non-isotropic, there exists a non-isotropic vector* $w \in V$ *such that* $(A - I)w$ *is non-isotropic.*

PROOF. (1) Suppose first that $w \in V$ is an arbitrary element, and $v = (A - I)w$. Then

$$B(v, v) = -2B(v, w) \tag{5}$$

by the calculation,

$$\begin{aligned} B(Aw - w, Aw - w) &= 2B(w, w) - 2B(Aw, w) \\ &= -2B(Aw - w, w). \end{aligned}$$

If v is non-isotropic this implies,

$$R_v Aw = R_v(v + w) = R_v(w) - v = w - 2\frac{B(v, w)}{B(v, v)}v - v = w.$$

(2) Suppose that for every non-isotropic $w \in V$, the vector $v = (A - I)w$ is isotropic. We have to show that $\mathrm{im}(A - I)$ is isotropic, in other words that $v = (A - I)w$ is isotropic for *all* w (not just the non-isotropic ones). [4] Given an isotropic element $w \in V$, pick any non-isotropic element $w_1 \in \mathrm{span}(w)^\perp$, and let $v = (A - I)w$ and $v_1 = (A - I)w_1$. Then $w + w_1$ and $w - w_1$ are also non-isotropic. By assumption, this implies that $v_1,\ v + v_1,\ v - v_1$ are all isotropic. The resulting equations give that v is isotropic.

□

We are finally in position to prove the general case.

PROOF OF THE E.CARTAN-DIEUDONNÉ THEOREM 4.1. Proposition 4.3 settles the case that $\mathrm{im}(A - I)$ is isotropic. We may hence assume that $\mathrm{im}(A - I)$ is non-isotropic. By the Lemma, there exists a non-isotropic element $w \in V$ such that $v = (A - I)w$ is non-isotropic. Furthermore, $A_1 = R_v A$ fixes w, and hence restricts to an orthogonal transformation of $V_1 = \mathrm{span}(w)^\perp$. By induction, $l(A_1) \le \dim V - 1$. Hence $l(A) = l(R_v A_1) \le \dim V$. □

There is a similar result for the group $\mathrm{SO}(V)$ of special orthogonal transformations. An element $A \in \mathrm{SO}(V)$ is called a *2-plane rotation* if there exists a 2-dimensional non-degenerate subspace S, such that A fixes $S^\perp$. (Hence A is determined by its restriction to S.)

THEOREM 4.5. *Any* $A \in \mathrm{SO}(V)$ *is a product of a finite number* $r(A)$ *of 2-plane rotations, with* $2r(A) \le \dim V$.

PROOF. Using the E.Cartan-Dieudonné theorem, it suffices to show that any product of two reflections $A = R_{v_1} R_{v_2}$ with $B(v_i, v_i) \neq 0$ can be written as a product of two (or less) 2-plane rotations. This is obvious if v_1, v_2 are proportional (in which case A is the identity) or if the space $S \subset V$ spanned by v_1, v_2 is non-degenerate (in which case A *is* a 2-plane rotation). Suppose that this is not the case, so that v_1, v_2 are linearly independent, but the space S spanned by v_1, v_2 is

[4] If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, this follows immediately once it is known that non-isotropic elements are dense in V.

degenerate. It suffices to find $y \in V$ such that the span of v_1, v and the span of v_2, v are both non-degenerate, since then $R_{v_1}R_{v_2} = (R_{v_1}R_v)(R_vR_{v_2})$ is a product of two 2-plane rotations. Let $y = \lambda_1 v_1 + \lambda_2 v_2$ be a generator of the (1-dimensional) kernel of $B|_{S\times S}$. Rescaling v_1, v_2, we may assume $y = (v_1 - v_2)/2$. Set $w = (v_1 + v_2)/2$. Then $B(w, w) \neq 0$, since otherwise the restriction $B|_{S\times S}$ would be zero. We have

$$v_1 = w + y, \;\; v_2 = w - y.$$

The orthogonal space $\mathrm{span}(w)^\perp$ is non-degenerate, and contains the isotropic vector y. Hence we can find $z \in V$ with

$$B(z, w) = 0, B(z, z) = 0, B(z, y) = 1.$$

Put $v = y + \lambda z$. (with $\lambda \neq 0$ to be determined). Then $B(v, v) = 2\lambda$. Suppose $\mu_1 v_1 + \mu_2 v$ is orthogonal to both v_1 and v. Then

$$\begin{aligned} 0 &= B(\mu_1(w + y) + \mu_2(y + \lambda z), w + y) = \mu_1 B(w, w) + \mu_2\lambda, \\ 0 &= B(\mu_1(w + y) + \mu_2(y + \lambda z), y + \lambda z) = \mu_1\lambda + 2\mu_2\lambda. \end{aligned}$$

Thus, as long as $\lambda \neq 2B(w, w)$ this system has no non-trivial solutions, and the space spanned by v_1, v is non-degenerate. A similar argument applies to the span of v_2, v. □

5. $\mathrm{O}(n, m; \mathbb{R})$ and $\mathrm{O}(n, \mathbb{C})$

In this Section, we will describe in some more detail the topology of the orthogonal groups, for the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

We begin with $\mathbb{K} = \mathbb{R}$. Let (V, B) be a quadratic vector space. Being a closed subgroups of $\mathrm{GL}(V)$, the orthogonal group $\mathrm{O}(V)$ is a Lie group. Recall that for a Lie subgroup G of $\mathrm{GL}(V)$, the corresponding Lie algebra $\mathfrak{g}$ is the subspace of all $\xi \in \mathrm{End}(V)$ with the property $\exp(t\xi) \in G$ for all $t \in \mathbb{K}$ (using the exponential map of matrices).

Proposition 5.1. *The Lie algebra $\mathfrak{o}(V)$ of the orthogonal group $\mathrm{O}(V)$ consists of all $A \in \mathrm{End}(V)$ such that $B(Av, w) + B(v, Aw) = 0$ for all $v, w \in V$. In particular,* $\dim \mathfrak{o}(V) = N(N-1)/2$ *where* $N = \dim V$.

Proof. Suppose $A \in \mathfrak{o}(V)$. Then $\exp(tA) \in \mathrm{O}(V)$ for all t. Taking the t-derivative of $B(\exp(tA)v, \exp(tA)w) = B(v, w)$ we obtain $B(Av, w) + B(v, Aw) = 0$ for all $v, w \in V$. Conversely, given $A \in \mathfrak{gl}(V, \mathbb{R})$ suppose $B(Av, w) + B(v, Aw) = 0$ for all $v, w \in V$. Then

$$\begin{aligned} B(\exp(tA)v, \exp(tA)w) &= \sum_{k,l=0}^{\infty} \frac{t^{k+l}}{k!l!} B(A^k v, A^l w) \\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{t^k}{i!(k-i)!} B(A^i v, A^{k-i} w) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i=0}^{\infty} \binom{k}{i} B(A^i v, A^{k-i} w) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} B(v, A^k w) \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} \\ &= B(v, w) \end{aligned}$$

since $\sum_{i=0}^{k}(-1)^i\binom{k}{i} = (1+(-1))^k$ equals 0 for $k=0$, and 1 for $k=0$. The property $B(Av,w)+B(v,Aw)=0$ says that the composition $B^\flat \circ A\colon V \to V^*$ is skew-adjoint. Since the space of skew-adjoint linear maps $V \to V^*$ has dimension $N(N-1)/2$, the proof is complete. □

Recall that any quadratic vector space (V,B) over $\mathbb{R}$ is isomorphic to one of the examples $\mathbb{R}^{n,m}$. The corresponding orthogonal group is denoted $\mathrm{O}(n,m)$, and its Lie algebra $\mathfrak{o}(n,m)$. The special orthogonal group will be denoted $\mathrm{SO}(n,m)$, and its identity component $\mathrm{SO}_0(n,m)$. The dimension of $\mathrm{O}(n,m)$ coincides with the dimension of its Lie algebra, $N(N-1)/2$ where $N=n+m$. If $m=0$ we will write $\mathrm{O}(n)=\mathrm{O}(n,0)$ and $\mathrm{SO}(n)=\mathrm{SO}(n,0)$.

Proposition 5.2. *For each $n \geq 1$, the Lie group $\mathrm{SO}(n)$ is compact and connected. For $n\geq 3$ the fundamental group is $\pi_1(\mathrm{SO}(n))=\mathbb{Z}_2$. In low dimensions,*

$$\mathrm{SO}(1)=\{1\},\ \mathrm{SO}(2)\cong S^1,\ \mathrm{SO}(3)\cong \mathbb{R}P(3),\ \mathrm{SO}(4)=S^3\times S^3/\sim$$

as manifolds, where in the last case $\sim$ denotes the equivalence relation $(x,y)\sim(-x,-y)$.

Proof. Since $\mathrm{SO}(n)$ is a closed, bounded subset of $\mathrm{GL}(n)$, it is compact. The defining action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ restricts to a transitive action on the unit sphere S^{n-1}, with stabilizer at $(0,\ldots,0,1)$ equal to $\mathrm{SO}(n-1)$. Hence, for $n\geq 2$ the Lie group $\mathrm{SO}(n)$ is the total space of a principal fiber bundle over S^{n-1}, with fiber $\mathrm{SO}(n-1)$. This shows by induction that $\mathrm{SO}(n)$ is connected. Also, since S^{n-1} is 2-connected for $n>3$, the inclusion of the fiber $\mathrm{SO}(n-1)\to\mathrm{SO}(n)$ defines an isomorphism of fundamental groups for $n>3$. That is, for all $n\geq 3$ the inclusion $\mathrm{SO}(3)\hookrightarrow\mathrm{SO}(n)$ (as the upper left corner) induces an isomorphism of fundamental groups, $\pi_1(\mathrm{SO}(n))=\pi_1(\mathrm{SO}(3))$ for $n\geq 3$. From the isomorphism $\mathrm{SO}(3)=\mathbb{R}P(3)=S^3/\sim$ proved below, it is immediate that $\pi_1(\mathrm{SO}(3))=\mathbb{Z}_2$.

The diffeomorphism $\mathrm{SO}(2)\cong S^1$ is obvious. To describe $\mathrm{SO}(3)$ and $\mathrm{SO}(4)$, consider the algebra of quaternions $\mathbb{H}\cong\mathbb{C}^2\cong\mathbb{R}^4$,

$$\mathbb{H}=\left\{\begin{pmatrix} z & w\\ -\overline{w} & \overline{z}\end{pmatrix},\ z,w\in\mathbb{C}\right\}.$$

The algebra $\mathbb{H}$ carries an involutive anti-automorphism, given by conjugate transpose:

$$A=\begin{pmatrix} z & w\\ -\overline{w} & \overline{z}\end{pmatrix}\mapsto A^*=\begin{pmatrix} \overline{z} & -w\\ \overline{w} & z\end{pmatrix}$$

Note that $A^*A=AA^*=(|z|^2+|w|^2)I$. Define a symmetric $\mathbb{R}$-bilinear form on $\mathbb{H}$ by

$$B(A_1,A_2)=\tfrac{1}{2}\operatorname{tr}(A_1^*A_2).$$

In terms of the parameters z,w, this is just the standard scalar product on $\mathbb{C}^2=\mathbb{R}^4$:

$$B\left(\begin{pmatrix} z & w\\ -\overline{w} & \overline{z}\end{pmatrix},\begin{pmatrix} z & w\\ -\overline{w} & \overline{z}\end{pmatrix}\right)=|z|^2+|w|^2.$$

The unit sphere $S^3\subset\mathbb{H}$, characterized by $|z|^2+|w|^2=1$ is the group $\mathrm{SU}(2)=\{A|\ A^*=A^{-1},\ \det(A)=1\}$. Define an action of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ on $\mathbb{H}$ by

$$(A_1,A_2)\cdot A=A_1AA_2^{-1}.$$

This action preserves the bilinear form on $\mathbb{H}\cong\mathbb{R}^4$, and hence defines a homomorphism $\mathrm{SU}(2)\times\mathrm{SU}(2)\to\mathrm{SO}(4)$. The kernel of this homomorphism is the finite subgroup $\{\pm(I, I)\} \cong \mathbb{Z}_2$. (Indeed, $A_1 A A_2^{-1} = A$ for all A implies in particular $A_1 = A A_2 A^{-1}$ for all invertible A. But this is only possible if $A_1 = A_2 = \pm I$.) Since $\dim \mathrm{SO}(4) = 6 = 2 \dim \mathrm{SU}(2)$, and since $\mathrm{SO}(4)$ is connected, this homomorphism must be onto. Thus $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(I, I)\}$.

Similarly, identify $\mathbb{R}^3$ with the space of tracefree (i.e. pure) quaternions, $\{A \in \mathbb{H} |\ \mathrm{tr}(A) = 0\}$. In terms of z, w, this is just the condition $\mathrm{Re}(z) = 0$. The conjugation action of $\mathrm{SU}(2)$ on $\mathbb{H}$ preserves the tracefree quaternions, and also preserves the inner product on $\mathbb{R}^3$. Hence we obtain a group homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. The kernel of this homomorphism is $\mathbb{Z}_2 \cong \{\pm I\} \subset \mathrm{SU}(2)$. Since $\mathrm{SO}(3)$ is connected and $\dim \mathrm{SO}(3) = 3 = \dim \mathrm{SU}(2)$, it follows that $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm I\} = S^3/\sim = \mathbb{R}P(3)$. $\square$

To study the more general groups $\mathrm{SO}(n, m)$ and $\mathrm{O}(n, m)$, we recall the polar decomposition of matrices. Let

$$\mathrm{Sym}(k) = \{A |\ A^T = A\} \subset \mathfrak{gl}(k, \mathbb{R})$$

be the space of real symmetric $k \times k$-matrices, and $\mathrm{Sym}^+(k)$ its subspace of positive definite matrices. As is well-known, the exponential map $\exp\colon \mathfrak{gl}(\mathfrak{k}, \mathbb{R}) \to \mathrm{GL}(k, \mathbb{R})$ for matrices restricts to a diffeomorphism,

$$\exp\colon \mathrm{Sym}(k) \to \mathrm{Sym}^+(k),$$

with inverse $\log\colon \mathrm{Sym}^+(k) \to \mathrm{Sym}(k)$. Furthermore, the map

$$\mathrm{O}(k) \times \mathrm{Sym}(k) \to \mathrm{GL}(k, \mathbb{R}),\ (O, X) \mapsto O e^X$$

is a diffeomorphism. The inverse map

$$\mathrm{GL}(k, \mathbb{R}) \to \mathrm{O}(k) \times \mathrm{Sym}(k),\ \mapsto (A|A|^{-1}, \log |A|),$$

where $|A| = (A^T A)^{1/2}$, is called the *polar decomposition* for $\mathrm{GL}(k, \mathbb{R})$. We will need the following simple observation:

LEMMA 5.3. *Suppose $X \in \mathrm{Sym}(k)$ is non-zero. Then the closed subgroup of $\mathrm{GL}(k, \mathbb{R})$ generated by e^X is non-compact.*

PROOF. It is enough to show that the subset $\{e^{nX},\ n \in \mathbb{Z}\}$ is unbounded. A set of matrices is bounded if and only if the eigenvalues are uniformly bounded. But is $\lambda \in \mathbb{R}$ is a non-zero eigenvalue of X, the corresponding set of eigenvalues $e^{n\lambda}$ of e^{nX} is clearly not bounded (since $\lambda \neq 0$). $\square$

This shows that $\mathrm{O}(k)$ is a maximal compact subgroup of $\mathrm{GL}(k, \mathbb{R})$. The polar decomposition for $\mathrm{GL}(k, \mathbb{R})$ restricts to a polar decomposition for any closed subgroup G that is invariant under the involution $A \mapsto A^T$. Let

$$K = G \cap \mathrm{O}(k, \mathbb{R}),\ P = G \cap \mathrm{Sym}^+(k),\ \mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(k).$$

The diffeomorphism $\exp\colon \mathrm{Sym}(k) \to \mathrm{Sym}^+(k)$ restricts to a diffeomorphism $\exp\colon \mathfrak{p} \to P$, with inverse the restriction of log. Hence the polar decomposition for $\mathrm{GL}(k, \mathbb{R})$ restricts to a diffeomorphism

$$K \times \mathfrak{p} \to G$$

called the polar decomposition of G. (It is a special case of a *Cartan decomposition.*) Using Lemma 5.3, we see that K is a maximal compact subgroup of G. Since $\mathfrak{p}$ is just a vector spaces, the algebraic topology of G coincides with that of its maximal compact subgroup K.

We will now apply these considerations to $G = \mathrm{O}(n,m)$. Write elements $A \in \mathrm{GL}(n+m,\mathbb{R})$ in block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6}$$

Then the bilinear form on $\mathbb{R}^{n,m}$ is related to the bilinear form (dot product) of $\mathbb{R}^{n+m}$ by $B(v,w) = (Jv) \cdot w$ where

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$$

The condition $A \in \mathrm{O}(n,m)$ is equivalent to $A^T J A = J$, which translates into the set of equations,

$$a^T a = I + c^T c, \quad d^T d = I + b^T b, \quad a^T b = c^T d. \tag{7}$$

Similarly, a matrix $X \in \mathrm{Mat}(n+m,\mathbb{R})$, written in block form

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{8}$$

lies in the Lie algebra $\mathfrak{o}(n,m)$ if and only if $X^T J + JX = 0$, which translates into

$$\alpha^T = -\alpha, \ \beta^T = \gamma, \ \delta^T = -\delta. \tag{9}$$

Since $\mathrm{O}(n,m)$ is invariant under the transposition map for $\mathrm{GL}(n+m,\mathbb{R})$, the polar decomposition applies and yields,

Proposition 5.4. *Relative to the polar decomposition of* $\mathrm{GL}(n+m,\mathbb{R})$, *the maximal subgroups of*

$$G = \mathrm{O}(n,m), \ \mathrm{SO}(n,m), \ \mathrm{SO}_0(n,m),$$

are, respectively,

$$K = \mathrm{O}(n) \times \mathrm{O}(m), \ \mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m)), \ \mathrm{SO}(n) \times \mathrm{SO}(m).$$

In all of these cases, the space $\mathfrak{p}$ *in the Cartan decomposition is given by matrices of the form*

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ x^T & 0 \end{pmatrix} \right\}$$

where x *is an arbitrary* $n \times m$*-matrix, and the space* $P = \exp(\mathfrak{p})$ *consists of matrices*

$$P = \left\{ \begin{pmatrix} (I + bb^T)^{1/2} & b \\ b^T & (I + b^T b)^{1/2} \end{pmatrix} \right\}$$

where b *ranges over all* $n \times m$*-matrices.*

Proof. The condition $A \in \mathrm{O}(n+m)$ gives the equations,

$$a^T a + c^T c = I, \ b^T b + d^T d = I, \ a^T b + c^T d = 0.$$

This is compatible with the (7) for $\mathrm{O}(n,m)$ if and only if $b = c = 0$, and the resulting conditions $a^T a = I, \ d^T d = 0, \ b = c = 0$ give $\mathrm{O}(n) \times \mathrm{O}(m)$. Similarly, a matrix $X \in \mathfrak{o}(n,m)$, given by the conditions (9), is symmetric if and only if the diagonal entries vanish. The discussion for $\mathfrak{p}$ is similar. $\square$

Corollary 5.5. *Unless* $n = 0$ *or* $m = 0$ *the group* $\mathrm{O}(n,m)$ *has four connected components and* $\mathrm{SO}(n,m)$ *has two connected components.*

In the above coordinates, it is not quite obvious how the matrix b appearing in the description of P is related to the matrix x in the description of $\mathfrak{p}$.

PROPOSITION 5.6. *One has*

$$\log \begin{pmatrix} (I+bb^T)^{1/2} & b \\ b^T & (I+b^Tb)^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & x \\ x^T & 0 \end{pmatrix}$$

where x and b are related as follows,

$$b = \frac{\sinh(xx^T)}{xx^T} x, \quad x = \frac{\operatorname{arsinh}((bb^T)^{1/2})}{(bb^T)^{1/2}} b. \tag{10}$$

Note that xx^T (resp. bb^T) need not be invertible. The quotient $\frac{\sinh(xx^T)}{xx^T}$ should be interpreted as $f(xx^T)$ where $f(z)$ is the entire holomorphic function $\frac{\sinh z}{z}$, and $f(xxT$ is given in terms of the spectral theorem or equivalently in terms of the power series expansion of f.

PROOF. Let $X = \begin{pmatrix} 0 & x \\ x^T & 0 \end{pmatrix}$. By induction on k,

$$X^{2k} = \begin{pmatrix} (xx^T)^k & 0 \\ 0 & (x^Tx)^k \end{pmatrix}, \; X^{2k+1} = \begin{pmatrix} 0 & (xx^T)^k x \\ x(x^Tx)^k & 0 \end{pmatrix}.$$

This gives

$$\exp(X) = \begin{pmatrix} \cosh(xx^T) & \frac{\sinh(xx^T)}{xx^T} x \\ x\frac{\sinh(x^Tx)}{x^Tx} & \cosh(x^Tx) \end{pmatrix},$$

which is exactly the form of elements in P with $b = \frac{\sinh(xx^T)}{xx^T} x$. The equation $\cosh(xx^T) = (1+bb^T)^{1/2}$ gives $\sinh(xx^T) = (bb^t)^{1/2}$. Plugging this into the formula for b, we obtain the second equation in (10). $\square$

For later reference, we mention one more simple fact about the orthogonal and special orthogonal groups:

PROPOSITION 5.7. *For all n, m, the center of the group* O(n, m) *is $\mathbb{Z}_2 = \{I, -I\}$. The center of* SO(n, m) *is $\{I, -I\}$ for $n + m \geq 3$ even, and is trivial for $n + m \geq 3$ odd. The center of the identity component* SO$_0(n, m)$ *is $I, -I$ if $n, m \geq 2$ are both even, and is trivial if n or m are odd.*

PROOF. For all $N \geq 1$ let D$(N) \cong (\mathbb{Z}_2)^N$ denote the group of diagonal matrices with entries ± 1 down the diagonal. Let SD$(N) \subset$ D(N) be the subgroup of matrices of determinant 1, i.e. with an even number of -1's. By elementary linear algebra, the matrices commuting with all members of $D(N)$ are exactly the diagonal matrices. The same is true for the centralizer of SD(N), provided $N \geq 3$.

Consider the center of O(n, m) where $n + m = N$. Since $D(N) \subset$ O(n, m), the center must consist of diagonal matrices. For all $1 \leq i < j \leq n$, and for $n+1 \leq i < j \leq n+m$, there exists an orthogonal matrix $A \in$ O(n, m) such that conjugation of a diagonal matrix by A interchanges the i'th and j'th diagonal entry. (E.g., take A to be the matrix of the transformation exchanging the basis vectors e_i and e_j and fixing all other basis vectors.) Hence, elements of the center must consist of block diagonal matrices where the only possible blocks are $\pm I_n$, $\pm I_m$ (where I_n, I_m denote the $n \times n$, $m \times m$ identity matrices). But the matrices commuting with diag$(I_n, -I_m)$ are the block diagonal matrices. Hence, it follows that the center of O(n, m) consists of $\pm I$.

A similar discussion applies to $\mathrm{SO}(n,m)$, provided $n+m \geq 3$. Note that $-I$ is contained in the maximal compact subgroup $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m))$, and that it is contained in $\mathrm{SO}(n) \times \mathrm{SO}(m)$ if and only if n, m are both even. From this one easily obtains the description of the center of $\mathrm{SO}_0(n,m)$. $\square$

The discussion above carries over to $\mathbb{K} = \mathbb{C}$, with only minor modifications. It is enough to consider the case $V = \mathbb{C}^n$, with the standard symmetric bilinear form. Again, our starting point is the polar decomposition, but now for complex matrices. Let $\mathrm{Herm}(n) = \{A|\ A^* = A\}$ be the space of Hermitian $n \times n$ matrices, and $\mathrm{Herm}^+(n)$ the subset of positive definite matrices. The exponential map gives a diffeomorphism

$$\mathrm{Herm}(n) \to \mathrm{Herm}^+(n),\ \ X \mapsto e^X.$$

This is used to show that the map

$$\mathrm{U}(n) \times \mathrm{Herm}(n) \to \mathrm{GL}(n,\mathbb{C}), \quad (U, X) \mapsto Ue^X$$

is a diffeomorphism; the inverse map takes A to (Ae^{-X}, X) with $X = \frac{1}{2}\log(A^*A)$. The polar decomposition of $\mathrm{GL}(n,\mathbb{C})$ gives rise to polar decompositions of any closed subgroups $G \subset \mathrm{GL}(n,\mathbb{C})$ that is invariant under the involution $*$. In particular, this applies to $\mathrm{O}(n,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{C})$. Indeed, if $A \in \mathrm{O}(n,\mathbb{C})$, the matrix A^*A lies in $\mathrm{O}(n,\mathbb{C}) \cap \mathrm{Herm}(n)$, and hence its logarithm $X = \frac{1}{2}\log(A^*A)$ lies in $\mathfrak{o}(n,\mathbb{C}) \cap \mathrm{Herm}(n)$. But clearly,

$$\mathrm{O}(n,\mathbb{C}) \cap \mathrm{U}(n) = \mathrm{O}(n,\mathbb{R}),$$
$$\mathrm{SO}(n,\mathbb{C}) \cap \mathrm{U}(n) = \mathrm{SO}(n,\mathbb{R})$$

while

$$\mathfrak{o}(n,\mathbb{C}) \cap \mathrm{Herm}(n) = \sqrt{-1}\mathfrak{o}(m,\mathbb{R}).$$

Hence, the maps $(U, X) \mapsto Ue^X$ restrict to polar decompositions

$$\mathrm{O}(n,\mathbb{R}) \times \sqrt{-1}\mathfrak{o}(n,\mathbb{R}) \to \mathrm{O}(n,\mathbb{C}),$$
$$\mathrm{SO}(n,\mathbb{R}) \times \sqrt{-1}\mathfrak{o}(n,\mathbb{R}) \to \mathrm{SO}(n,\mathbb{C}),$$

which shows that the algebraic topology of the complex orthogonal and special orthogonal group coincides with that of its real counterparts. Arguing as in the real case, the center of $\mathrm{O}(n,\mathbb{C})$ is given by $\{+I, -I\}$ while the center of $\mathrm{SO}(n,\mathbb{C})$ is trivial for n odd and $\{+I, -I\}$ for n even, provided $n \geq 3$. (Indeed, the same argument works for $\mathrm{O}(n,m,\mathbb{K})$ for any field $\mathbb{K}$ of characteristic $\neq 2$.)

CHAPTER 2

Clifford algebras

1. Preliminaries on exterior algebras

For any vector space V over a given field $\mathbb{K}$, let $T(V) = \bigoplus_{k\in\mathbb{Z}} T^k(V)$ be the tensor algebra, with $T^k(V) = V \otimes \cdots \otimes V$ the k-fold tensor product. The quotient of $T(V)$ by the two-sided ideal $\mathcal{I}(V)$ generated by all $v\otimes w + w\otimes v$ is the exterior algebra, denoted $\wedge(V)$. The product in $\wedge(V)$ is usually denoted $\alpha_1 \wedge \alpha_2$, although we will frequently omit the wedge sign and just write $\alpha_1\alpha_2$. Put differently, $\wedge(V)$ is the associative algebra linearly generated by V, subject to the relations $vw+wv = 0$. The exterior algebra has the following universal property: If $\mathcal{A}$ is an algebra, and $\phi\colon V \to \mathcal{A}$ a linear map with $\phi(v)\phi(w) + \phi(w)\phi(v) = 0$ for all $v, w \in V$, then ϕ extends uniquely to an algebra homomorphism $\wedge(V) \to \mathcal{A}$. Since $\mathcal{I}(V)$ is a *graded* ideal, the exterior algebra inherits a grading

$$\wedge(V) = \bigoplus_{k\in\mathbb{Z}} \wedge^k(V).$$

Clearly, $\wedge^0(V) = \mathbb{K}$ and $\wedge^1(V) = V$ so that we can think of V as a subspace of $\wedge(V)$. We briefly list the main constructions and properties for exterior algebras:

Basis. If $\dim V = n < \infty$, with basis e_i, the space $\wedge^k(V)$ has basis

$$e_I = e_{i_1}\cdots e_{i_k}$$

for all ordered subsets $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ (If $k = 0$, we put $e_\emptyset = 1$.) In particular, we see that $\dim \wedge^k(V) = \binom{n}{k}$, and

$$\dim \wedge(V) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Letting $e^i \in V^*$ denote the dual basis to the basis e_i considered above, we define a dual basis to e_I to be the basis $e^I = e^{i_1}\cdots e^{i_k} \in \wedge(V^*)$.

Graded commutativity. The exterior algebra is *commutative* (in the graded sense). That is, for $\alpha_1 \in \wedge^{k_1}(V)$ and $\alpha_2 \in \wedge^{k_2}(V)$,

$$[\alpha_1, \alpha_2] := \alpha_1\alpha_2 + (-1)^{k_1k_2}\alpha_2\alpha_1 = 0.$$

Functoriality. Any linear map $L\colon V \to W$ extends uniquely to an algebra homomorphism $\wedge(L)\colon \wedge(V) \to \wedge(W)$. One has $\wedge(L_1 \circ L_2) = \wedge(L_1) \circ \wedge(L_2)$. In particular, one hence obtains a group homomorphism [1]

$$\mathrm{Aut}(V) \to \mathrm{Aut}_{\mathrm{alg}}(\wedge(V)),\ L \mapsto \wedge(L)$$

For instance, the involution $v \mapsto -v$ induces the *parity operator* $\Pi \in \mathrm{Aut}_{\mathrm{alg}}(\wedge(V))$, equal to $(-1)^k$ on $\wedge^k(V)$.

If $g \in \mathrm{GL}(V) \subset \mathrm{End}(V)$ we will sometimes simply write g rather than $\wedge(g)$. (On the other hand, it is risky to adopt this convention for arbitrary maps, since for example $\wedge(0)$ is not the zero map.)

Direct sums. Suppose V_1, V_2 are two vector spaces, and define $\wedge(V_1) \otimes \wedge(V_2)$ as the tensor product of graded algebras. [2] The map

$$V_1 \oplus V_2 \to \wedge(V_1) \otimes \wedge(V_2),\ v_1 \oplus v_2 \mapsto v_1 \otimes 1 + 1 \otimes v_2$$

extends uniquely (by the universal property) to a homomorphism of graded algebras,

$$\wedge(V_1 \oplus V_2) \to \wedge(V_1) \otimes \wedge(V_2).$$

It is not hard to check that this map is an isomorphism.

The anti-automorphism. An anti-automorphism of an algebra $\mathcal{A}$ is an invertible linear map $\phi\colon \mathcal{A} \to \mathcal{A}$ with the property $\phi(ab) = \phi(b)\phi(a)$ for all $a, b \in \mathcal{A}$. Put differently, if $\mathcal{A}^{\mathrm{op}}$ is $\mathcal{A}$ with the opposite algebra structure $a \cdot b := ba$, an anti-automorphism is an algebra isomorphism $\mathcal{A} \to \mathcal{A}^{\mathrm{op}}$.

The exterior algebra carries a unique anti-automorphism that is equal to the identity on $V \subset \wedge(V)$. It is called the *transposition*, $(v_1 \cdots v_k)^t = v_k \cdots v_1$. One easily checks that

$$\phi^t = (-1)^{k(k-1)/2}\phi,\quad \phi \in \wedge^k(V).$$

Duals. Any $\alpha \in V^*$ defines a linear map $\iota_\alpha \equiv \iota(\alpha)\colon \wedge^k(V) \to \wedge^{k-1}(V)$, called *contraction*, with the properties $\iota_\alpha(v) = \langle \alpha, v\rangle$ for $v \in V$ and the derivation property

$$\iota_\alpha(\phi_1 \wedge \phi_2) = \iota_\alpha(\phi_1) \wedge \phi_2 + (-1)^{|\phi_1|}\phi_1 \wedge \iota_\alpha \phi_2$$

The contraction operators satisfy $\iota_{\alpha_1}\iota_{\alpha_2} + \iota_{\alpha_2}\iota_{\alpha_1} = 0$ and hence by the universal property extend to an algebra homomorphism,

$$\iota\colon \wedge(V^*) \to \mathrm{End}(\wedge(V)).$$

Define a bilinear pairing

$$\wedge(V^*) \otimes \wedge(V) \to \mathbb{K},\ \psi \otimes \phi \mapsto \langle \psi, \phi\rangle = (\iota_{\psi^t}(\phi))_{[0]},$$

[1] If $\mathcal{A}$ is any algebra, we denote by $\mathrm{End}(\mathcal{A})$ (resp. $\mathrm{Aut}(\mathcal{A})$) the vector space homomorphisms (res. automorphisms) $\mathcal{A} \to \mathcal{A}$, while $\mathrm{End}_{\mathrm{alg}}(\mathcal{A})$ (resp. $\mathrm{Aut}_{\mathrm{alg}}(V)$) denotes the set of algebra homomorphisms (resp. group of algebra automorphisms).

[2] If $\mathcal{A} = \bigoplus \mathcal{A}^k$ and $B = \bigoplus B^k$ are graded algebras, then the graded tensor product is the ordinary tensor product of vector spaces $\mathcal{A} \otimes B$, with grading $|a \otimes b| = |a| + |b|$, and multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2 \otimes b_1 b_2).$$

where the subscript [0] denotes the component in $\wedge^0(V) = \mathbb{K}$. (The inclusion of the transpose in the definition of the pairing is a matter of convention.) Concretely, for $\alpha_i \in V^*$ and $v_i \in V$,

$$\langle \alpha_1 \cdots \alpha_k, v_1 \cdots v_k \rangle = \det(\langle \alpha_i, v_j \rangle).$$

The pairing defines a 1-1 linear map

$$\wedge(V^*) \to \wedge(V)^*,$$

which is an isomorphism if $\dim V < \infty$.

Let $\phi \in \wedge(V)$, $\psi \in \wedge(V^*)$, $\alpha \in V^*$. Then

$$\langle \psi, \iota_\alpha(\phi) \rangle = (\iota_{\psi^t} \iota_\alpha(\phi))_{[0]} = (\iota_{(\alpha \wedge \psi)^t}(\phi))_{[0]} = \langle \alpha \wedge \psi, \phi \rangle.$$

It follows that the map ι_α on $\wedge(V)$ is dual to the map $\epsilon_\alpha = \alpha\wedge$ of exterior multiplication on $\wedge(V^*)$. Similarly, for any $v \in V$ the operator $\epsilon(v)$ of exterior multiplication is dual to the operator ι_v of contraction on $\wedge(V^*)$.

Bilinear forms. Any bilinear form $b : V \otimes W \to \mathbb{K}$ extends to a bilinear form $\wedge(V) \otimes \wedge(W) \to \mathbb{K}$, by the formula,

$$b(v_1 \cdots v_k, w_1 \cdots w_k) = \det(b(v_i, w_j)).$$

(The extension vanishes on $\wedge^k(V) \otimes \wedge^l(W)$ if $k \neq l$.) Equivalently, let $A_b \colon V \to W^*$, $v \mapsto b(v, \cdot)$ be the linear map determined by b, and consider its extension to the exterior algebra. Then we can interpret the extension of b as a composition of two maps,

$$\wedge(V) \otimes \wedge(W) \to \wedge(W^*) \otimes \wedge(W) \to \mathbb{K}$$

Observe that if b is a (non-degenerate) symmetric bilinear form on V, then the extension of b is again (non-degenerate) symmetric. For $\mathbb{K} = \mathbb{R}$, if b is positive definite on V then the extension to $\wedge(V)$ is positive definite. (This motivates, in hindsight, our choice of signs for the pairing between the exterior algebras of a vector space and its dual space.)

Derivations. Suppose $\mathcal{A}$ is a $\mathbb{Z}$-graded algebra. An endomorphism $D \in \mathrm{End}^k(\mathcal{A})$ of degree $|D| = k$ (i.e. $D(\mathcal{A}^\bullet) \subset \mathcal{A}^{\bullet + k}$) is called a derivation of degree k if

$$D(xy) = D(x)y + (-1)^{|x||D|} x D(y).$$

We denote[3] $\mathrm{Der}^k(\mathcal{A})$ the space of derivations of degree k. Some basis properties of derivations are

(1) Any $D \in \mathrm{Der}^k(\mathcal{A})$ vanishes on 1. This is immediate from the definition, applied to $x = y = 1$.
(2) Derivations are determined by their values on algebra generators.
(3) If D_1, D_2 are derivations of degree k_1, k_2 their (graded) commutator

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1$$

is a derivation of degree $k_1 + k_2$. It follows that the space $\bigoplus_k \mathrm{Der}^k(\mathcal{A})$ of derivations is a $\mathbb{Z}$-graded Lie algebra, with bracket the commutator.

[3]One similarly defines derivations of a $\mathbb{Z}_2$-graded algebra. Note however that if $\mathcal{A}$ is graded, the space $\mathrm{Der}(\mathcal{A})$ of $\mathbb{Z}_2$-graded derivations may be strictly larger then the space $\bigoplus_k \mathrm{Der}^k(\mathcal{A})$ of derivations of finite $\mathbb{Z}$-degree.

(4) If $\mathcal{A}$ is commutative, then $\bigoplus_{k\in\mathbb{Z}} \mathrm{Der}^k(\mathcal{A})$ is a left-module under the action of the $\mathbb{Z}$-graded algebra $\bigoplus_{k\in\mathbb{Z}} \mathrm{End}^k(\mathcal{A})$.

Let us now consider the space $\mathrm{Der}(\wedge(V))$ of derivations of the exterior algebra $\wedge(V)$. If $\alpha \in V^*$ then $\iota(\alpha) \in \mathrm{Der}^{-1}(\wedge(V))$. Let

$$\epsilon\colon \wedge(V) \to \mathrm{End}(\wedge(V))$$

denote the algebra homomorphism given by exterior multiplication, $\epsilon(\phi)\psi = \phi \wedge \psi$.

Lemma 1.1. *The linear map*

$$\wedge(V) \otimes V^* \to \mathrm{Der}(\wedge(V)),\ \phi = \sum_a \phi_a \otimes e^a \mapsto D_\phi = \epsilon(\phi_a)\iota(e^a), \tag{11}$$

is an isomorphism of graded vector spaces. It is an isomorphism of graded Lie algebras, provided the Lie bracket on the left hand side is defined as a semi-direct product, $\wedge(V) \rtimes V^*$, where V^* is viewed as a commutative Lie algebra in degree -1, and $\wedge(V)$ is viewed as a graded V^*-module via contraction.

Proof. The map (11) is 1-1, since ϕ_a is recovered from the derivation D_ϕ as $\phi_a = D_\phi(e_a)$. Conversely, if D is any derivation and ϕ is defined by $\phi_a = D(e_a)$, then $D = D_\phi$ since both derivations agree on generators e_a. The description of the Lie bracket is equivalent to

$$[D_{\phi_1}, D_{\phi_2}] = D_{[\phi_1\phi_2]}$$

where

$$[\phi_1, \phi_2]_a = \sum_b \left(\phi_{1,b}\, \iota(e^b)\phi_{2,a} - (-1)^{|\phi_1||\phi_2|}\, \phi_{2,b}\, \iota(e^b)\phi_{1,a}\right).$$

But this is easily verified, by applying the commutator to $e_a \in V \subset \wedge(V)$. $\square$

Example 1.2. Any endomorphism $L \in \mathrm{End}(V)$ extends uniquely to a derivation $D_L \in \mathrm{Der}^0(\wedge(V))$. In terms of a basis,

$$D_L = \sum_a \epsilon(S(e_a))\, \iota(e^a),$$

Note that the map $L \mapsto D_L$ is a Lie algebra homomorphism.

Endomorphisms. The following Lemma will play a role when we discuss the Spinor representation of Clifford algebras.

Theorem 1.3. *The linear map*

$$\wedge(V \oplus V^*) \cong \wedge(V) \otimes \wedge(V^*) \to \mathrm{End}(\wedge(V)),\quad \sum_i \phi_i \otimes \psi_j \mapsto \sum_i \epsilon(\phi_i)\iota(\psi_i^t) \tag{12}$$

is an isomorphism of vector spaces (not of algebras). Under this isomorphism, $\wedge(V)$ is characterized as the endomorphisms A such that $[A, \epsilon(v)] = 0$ for all $v \in V$, while $\wedge(V^*)$ is characterized as the endomorphisms such that $[A, \iota(\alpha)] = 0$ for all $\alpha \in V^*$.

Proof. By dimension count, it suffices to show that the map (12) is one-to-one. Let e_a be a basis of V, with dual basis e^a, and suppose

$$x = \sum x_J^I e_I \otimes e^J$$

with $x^I_J \in \mathbb{K}$ is in the kernel of (12). Note that $\iota((e^J)^t)e_K = 0$ if $J \not\subset K$, and is $\pm e_{K-J}$ if $J \subset K$. Suppose $x^I_J = 0$ for $|J| < k$. If $|K| = k$, we have

$$\Big(\sum x^I_J \epsilon(e_I) \otimes \iota(e^J)^t\Big)\, e_K = \sum_J x^K_J e^J$$

which shows $x^K_J = 0$. Hence, by induction we find that $x^I_J = 0$ for all I, J, thus $x = 0$.

Suppose now that the endomorphism A commutes with all exterior multiplications. Let $\phi = A(1) \in \wedge(V)$. For all $\phi' \in \wedge(V)$,

$$A(\phi') = A(1 \wedge (\phi')) = A(1) \wedge \phi' = \phi \wedge \phi' = \epsilon(\phi)(\phi').$$

Similarly, if A commutes with contractions, the dual map $A^* \in \mathrm{End}(\wedge(V^*))$ commutes with exterior multiplication. Hence $A^* = \epsilon(\psi^t)$ for some ψ, and hence $A = \epsilon(\psi^t)^* = \iota(\psi)$. □

REMARK 1.4. The isomorphism (13) should not to be confused with the standard isomorphism $E_2 \otimes E_1^* = \mathrm{Hom}(E_1, E_2)$ for $E_1 = E_2 = \wedge(V)$. The latter takes x to the linear map

$$\phi \mapsto \sum_{IJ} x^I_J e_I \big(\iota(e^J)^t \phi\big)_{[0]}.$$

REMARK 1.5. Thinking of elements of $\wedge(V)$ as functions on an odd 'superspace', the derivations correspond to vector fields on this superspace while the more general expressions $\sum_i \epsilon(\phi_i)\iota(\psi_i^t)$ correspond to differential operators.

There is an almost parallel discussion for the symmetric algebra $S(V)$. Elements of $S(V)$ can be viewed as polynomial functions on V^*. Any element of V^* defines a derivation of $S(V)$ (the directional derivative). The space of all derivations is isomorphic to $S(V) \otimes V^*$, and is identified with the vector fields with polynomial coefficients. Similarly, the space of finite order differential operators with polynomial coefficients is isomorphic to $S(V) \otimes S(V^*)$ as a vector space (not as an algebra). Note however that since $\dim S(V) = \infty$, the full space $\mathrm{End}(S(V))$ is much larger since it also contains differential operators of 'infinite order'.

By a similar argument, one proves:

LEMMA 1.6. *Suppose* $\dim V, \dim W < \infty$*, and let* $L\colon V \to W$ *be an arbitrary linear map. Denote by* $\wedge(L)$ *the extension of* L *to an algebra homomorphism* $\wedge(V) \to \wedge(W)$*. Then the linear map*

$$(13)\quad \wedge(W) \otimes \wedge(V^*) \to \mathrm{Hom}(\wedge(V), \wedge(W)), \quad \sum_i \phi_i \otimes \psi_j \mapsto \sum_i \epsilon(\phi_i) \circ \wedge L \circ \iota(\psi_i^t)$$

is an isomorphism of vector spaces. Under this isomorphism, $\wedge(W)$ *is characterized as the space of homomorphisms* A *such that* $A \circ \epsilon(v) = (-1)^{|A|}\epsilon(L(v)) \circ A$ *for all* $v \in V$*, while* $\wedge(V^*)$ *is characterized as the space of homomorphism such that* $\iota(\alpha) \circ A = (-1)^{|A|} A \circ \iota(L^*\,\alpha)$ *for all* $\alpha \in W^*$.

Consider again the 'usual' isomorphism $\mathrm{Hom}(E_1, E_2) = E_2 \otimes E_1^*$ for $E_1 = \wedge(W)$ and $E_2 = \wedge(V)$. Here $\phi \otimes \psi \in \wedge(V) \otimes \wedge(W^*)$ gives rise to the linear map $\chi \mapsto \phi\langle\psi, \chi\rangle$. Since the pairing is defined as

$$\langle\psi, \chi\rangle = (\iota(\psi^t)\chi)_{[0]},$$

and since the 'augmentation map' taking the degree 0 part is nothing but $\wedge(0)$, we see that the 'usual' homomorphism corresponds to $L = 0$.

Coproduct. The exterior algebra carries not only a product $m_\wedge \colon \wedge(V) \otimes \wedge(V) \to \wedge(V)$, but also a *coproduct*

$$\Delta\colon \wedge(V) \to \wedge(V) \otimes \wedge(V).$$

While $m_\wedge$ is induced by the addition map $V \oplus V \to V$, $v_1 \oplus v_2 \mapsto v_1 + v_2$, the coproduct is induced by the diagonal inclusion, $V \to V \oplus V$, $v \mapsto v \oplus v$. The dual map to addition map for V is the diagonal inclusion for V^*, and vice versa. Hence, if $\dim V < \infty$, the product (resp.coproduct) for $\wedge(V)$ is dual to the coproduct (resp. product) for $\wedge(V^*)$. This implies for instance that

$$\iota_\psi \circ m_\wedge = m_\wedge \circ \iota_{\Delta\psi}, \quad \psi \in \wedge(V^*)$$

as operators on $\wedge(V) \otimes \wedge(V)$.

2. $\wedge(V)$ as a Poisson algebra

A *graded Poisson algebra* is a graded algebra $\mathcal{A} = \bigoplus \mathcal{A}^k$, together with a bilinear map $P\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (called Poisson bracket) such that

(1) The space $\mathcal{A}[2]$ is a graded Lie algebra, with bracket P. [4]
(2) The Lie bracket and the algebra structure are compatible, in the sense that the map $\alpha \mapsto P(\alpha, \cdot)$ defines a graded Lie algebra homomorphism from $\mathcal{A}[2] \to \mathrm{Der}(\mathcal{A})$.

That is, for any $\alpha \in \mathcal{A}^k$, the map $P(\alpha, \cdot)$ is a degree $k-2$ derivation of the algebra structure.

Remark 2.1. Similarly, one defines consider graded Poisson algebras of degree n, by replacing $\mathcal{A}[2]$ with $\mathcal{A}[n]$ in the above definition. See Cattaneo-Fiorenza-Longini, 'graded Poisson algebras' (Preprint, 2005). For instance, any graded algebra is a graded Poisson algebra of degree 0, by taking the Lie bracket to be the commutator.

Note that any Poisson bracket on a graded algebra is uniquely determined by its values on generators for the algebra. We will usually write $\{\alpha_1, \alpha_2\} = P(\alpha \otimes \alpha_2)$. In this notation, the defining conditions are,

$$\begin{aligned}\{\alpha_1, \{\alpha_2, \alpha_3\}\} + (-1)^{|\alpha_1|(|\alpha_2|+|\alpha_3|)}\{\alpha_2, \{\alpha_3, \alpha_1\}\} \\ + (-1)^{|\alpha_3|(|\alpha_1|+|\alpha_2|)}\{\alpha_3, \{\alpha_1, \alpha_2\}\} = 0,\end{aligned}$$

$$\{\alpha_1, \alpha_2\} = -(-1)^{|\alpha_1,\alpha_2|}\{\alpha_2, \alpha_1\},$$

$$\{\alpha_1, \alpha_2\alpha_3\} = \{\alpha_1, \alpha_2\}\alpha_3 + (-1)^{|\alpha_1||\alpha_2|}\alpha_2\{\alpha_1, \alpha_3\}.$$

Since $\mathcal{A}[2]$ is a graded Lie algebra, it follows in particular that $\mathfrak{g} = \mathcal{A}[2]^0 = \mathcal{A}^2$ is an ordinary Lie algebra. Poisson bracket by elements of $\mathfrak{g}$ makes $\mathcal{A}$ into a $\mathfrak{g}$-module.

[4] For any graded vector space V, the graded vector space $V[l]$ is equal to V with the shifted grading $V[l]^k = V^{k+l}$.

EXAMPLE 2.2. Suppose $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is any Lie algebra, and let $\mathcal{A} = S(\mathfrak{g})$ be the symmetric algebra, with grading

$$S(\mathfrak{g})^{2k} = S^k(\mathfrak{g}), \quad S(\mathfrak{g})^{2k+1} = 0.$$

Then $S(\mathfrak{g})$ carries a graded Poisson bracket, given on generators by the Lie bracket $\{\xi_1, \xi_2\} = [\xi_1, \xi_2]_{\mathfrak{g}}$. In this case, the Lie bracket on $\mathcal{A}^2$ just recovers the given Lie algebra. Conversely, if V is any vector space, then the structure of a graded Poisson algebra on $S(V)$ is *equivalent* to a Lie algebra structure on V.

Any symmetric bilinear form B on a vector space induces on $\mathcal{A} = \wedge(V)$ the structure of a graded Poisson algebra. The Poisson bracket is given on generators $v, w \in V = \wedge^1(V)$ by

$$\{v, w\} = B(v, w).$$

In this way, one obtains a one-to-one correspondence between Poisson brackets (of degree -2) on $\wedge(V)$ and symmetric bilinear forms B.

LEMMA 2.3. *Let $B^\flat \colon V \to V^*$ be the linear map defined by B. Then*

$$\{v, \cdot\} = \iota_{B^\flat(v)}$$

for all $v \in V$.

PROOF. Since both sides are in $\mathrm{Der}^{-1}(\wedge(V))$, it suffices to check on generators $w \in V = \wedge^1(V)$. But on w, both sides give $B(v, w)$. □

As remarked above, $\wedge^2(V)$ is a Lie subalgebra under Poisson bracket.

LEMMA 2.4. *Suppose B is non-degenerate. Then the Lie algebra $(\wedge^2(V), \{\cdot,\cdot\})$ is canonically isomorphic to $\mathfrak{o}(V, B)$.*

PROOF. For any $\phi \in \wedge^2(V)$, let $A_\phi \colon V \to V$ be the linear map $A_\phi(v) = \{\phi, v\}$. Then $A_\phi \in \mathfrak{o}(V)$ since

$$\begin{aligned} B(A_\phi(v), w) + B(v, A_\phi(w) &= \{\{\phi, v\}, w\} + \{v, \{\phi, w\}\} \\ &= \{\phi, \{v, w\}\} \\ &= \{\phi, B(v, w)\} \\ &= B(v, w)\{\phi, 1\} = 0. \end{aligned}$$

The kernel of the map $\phi \mapsto A_\phi$ consists of all ϕ with $\{\phi, v\} = 0$ for all $v \in V$. We may write this condition as $\iota(B^\flat(v))\phi = 0$. Since B is non-degenerate, this just means $\phi = 0$. It follows that

$$\wedge^2(V) \to \mathfrak{o}(V, B), \ \phi \mapsto A_\phi$$

is an isomorphism, since both spaces have dimension $\binom{n}{2}$ with $n = \dim V$. □

The inverse map may be described as follows: Let $e_1, \ldots, e_n \in V$ be a basis, and let $e^1, \ldots, e^n \in V$ be the B-dual basis. Suppose $A \in \mathfrak{o}(V; B)$. Then $A = A_\phi$ for

$$\phi = \tfrac{1}{2} \sum_a A(e_a)\, e^a.$$

To check that this is the inverse map, we compute

$$\begin{aligned}\{\phi, v\} &= \tfrac{1}{2}\sum_a A(e_a)B(e^a, v) - \tfrac{1}{2}\sum_a B(A(e_a), v)e^a \\ &= \tfrac{1}{2}A(v) + \tfrac{1}{2}\sum_a B(e_a, A(v))e^a \\ &= A(v).\end{aligned}$$

REMARK 2.5. If B is non-degenerate, but $\dim V = \infty$, the image of the map $\phi \mapsto A_\phi$ consists of the subalgebra $\mathfrak{o}(V;B)_{\mathrm{fin}}$ of infinitesimal orthogonal transformations of finite rank.

3. Clifford algebras: definition and first properties

Let V be a vector space over a field $\mathbb{K}$ of characteristic zero, equipped with a symmetric bilinear form $B\colon V \times V \to \mathbb{K}$ (possibly degenerate).

DEFINITION 3.1. The *Clifford algebra* $\mathrm{Cl}(V;B)$ is the quotient

$$\mathrm{Cl}(V;B) = T(V)/\mathcal{I}(V;B) \tag{14}$$

where $\mathcal{I}(V;B) \subset \mathcal{T}(V)$ is the two-sided ideal generated by expressions of the form

$$v \otimes w + w \otimes v - B(v,w)1. \tag{15}$$

Note that

$$\mathrm{Cl}(V;0) = \wedge(V).$$

It is not immediately obvious that $\mathcal{I}(V;B)$ is a proper ideal, i.e. that $\mathrm{Cl}(V;B)$ is non-trivial. To this end, consider the linear map

$$\phi\colon V \to \mathrm{End}(\wedge(V)),\ v \mapsto \epsilon(v) + \tfrac{1}{2}\iota(B^\flat(v)).$$

and its extension to an algebra homomorphism $\phi_T\colon T(V) \to \mathrm{End}(\wedge(V))$. Since

$$\phi(v)\phi(w) + \phi(w)\phi(v) = B(v,w)1,$$

the map ϕ_T vanishes on the ideal $\mathcal{I}(V;B)$, and hence descends to the Clifford algebra. In particular, using $\phi_T(1) = 1$, we see that the inclusion $\mathbb{K} \hookrightarrow T(V)$ descends to an inclusion $\mathbb{K} \hookrightarrow \mathrm{Cl}(V;B)$. That is, $\mathrm{Cl}(V;B)$ is a *unital* algebra. Similarly, using $\phi_T(v).1 = v$ we also see that the inclusion $V \hookrightarrow T(V)$ descends to an inclusion $V \hookrightarrow \mathrm{Cl}(V;B)$. Thus $\mathrm{Cl}(V;B)$ is the unital associative algebra, generated by the elements of V subject to the relations

$$vw + wv = B(v,w)1, \quad v, w \in V. \tag{16}$$

The symbol map. The algebra homomorphism

$$\phi_{\mathrm{Cl}}\colon \mathrm{Cl}(V;B) \to \mathrm{End}(\wedge(V)),$$

introduced above is in fact an injection. To see this we consider the important *symbol map*,

$$\sigma\colon \mathrm{Cl}(V;B) \to \wedge(V),\ x \mapsto \phi_{\mathrm{Cl}}(x).1$$

where $1 \in \wedge^0(V) = \mathbb{K}$.

PROPOSITION 3.2. *The symbol map is an isomorphism of vector spaces. In low degrees,*

$$\begin{aligned}\sigma(1) &= 1\\ \sigma(v) &= v\\ \sigma(v_1v_2) &= v_1 \wedge v_2 + \tfrac{1}{2}B(v_1, v_2),\\ \sigma(v_1v_2v_3) &= v_1 \wedge v_2 \wedge v_3 + \tfrac{1}{2}(B(v_2, v_3)v_1 - B(v_1, v_3)v_2 + B(v_1, v_2)v_3).\end{aligned}$$

PROOF. Let $e_i \in V$ be an orthogonal basis. Since the operators $\phi(e_i)$ commute (in the grade sense), we find

$$\sigma(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

This directly shows that the symbol map is an isomorphism. The formulas in low degrees are obtained by straightforward calculation. □

The inverse map is called the *quantization map*

$$q\colon \wedge(V) \to \mathrm{Cl}(V),$$

and is given in terms of the basis by $q(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{i_1} \cdots e_{i_k}$. In low degrees,

$$\begin{aligned}q(1) &= 1,\\ q(v) &= v,\\ q(v_1 \wedge v_2) &= v_1v_2 - \tfrac{1}{2}B(v_1, v_2),\\ q(v_1 \wedge v_2 \wedge v_3) &= v_1v_2v_3 - \tfrac{1}{2}(B(v_2, v_3)v_1 - B(v_1, v_3)v_2 + B(v_1, v_2)v_3).\end{aligned}$$

The $\mathbb{Z}$-filtration. The increasing filtration

$$T_{(0)}(V) \subset T_{(1)}(V) \subset \cdots$$

with $T_{(k)}(V) = \bigoplus_{j \le k} T^j(V)$ descends to a filtration

$$\mathrm{Cl}_{(0)}(V; B) \subset \mathrm{Cl}_{(1)}(V; B) \subset \cdots$$

of the Clifford algebra, with $\mathrm{Cl}_{(k)}(V; B)$ the image of $T_{(k)}(V)$ under the quotient map. Equivalently, $\mathrm{Cl}_{(k)}(V; B)$ consists of linear combinations of products $v_1 \cdots v_l$ with $l \le k$ (including scalars, viewed as products of length 0). The filtration is compatible with product map, that is,

$$\mathrm{Cl}_{(k_1)}(V; B)\mathrm{Cl}_{(k_2)}(V; B) \subset \mathrm{Cl}_{(k_1+k_2)}(V; B).$$

Thus, $\mathrm{Cl}(V; B)$ is a *filtered algebra*. As a special case $B = 0$, we can also view $\wedge(V)$ as a filtered algebra.

PROPOSITION 3.3. *The symbol map induces an isomorphism of associated graded algebras*

$$\mathrm{gr}(\sigma)\colon \mathrm{gr}(\mathrm{Cl}(V)) \to \mathrm{gr}(\wedge(V)) = \wedge(V).$$

PROOF. The symbol map is filtration preserving, and hence descends to an isomorphism of associated graded vector spaces. In terms of generators, this map is given by

$$\Big(v_1 \cdots v_k \mod \mathrm{Cl}(V)_{(k-1)}\Big) \mapsto v_1 \wedge \cdots \wedge v_k$$

which clearly is an algebra homomorphism. □

It follows that if e_i is *any* basis of V (not necessarily orthogonal), then the products $e_I = e_{i_1} \cdots e_{i_k}$ with $i_1 < \cdots < i_k$ together with $e_\emptyset = 1$ form a basis of $\mathrm{Cl}(V)$.

$\mathbb{Z}_2$-grading. Let $T(V)$ carry the $\mathbb{Z}_2$-grading

$$T^{\bar{0}}(V) = \bigoplus_{k=0}^{\infty} T^{2k}(V),\ T^{\bar{1}}(V) = \bigoplus_{k=0}^{\infty} T^{2k+1}(V).$$

(Here $\bar{k}$ denotes $k \mod 2$.) Observe that since the elements $v \otimes w + w \otimes v - B(v,w)1$ are even, the ideal $\mathcal{I}(V;B)$ is $\mathbb{Z}_2$ graded, i.e. it is a direct sum of the subspaces $\mathcal{I}^{\bar{k}}(V;B) = \mathcal{I}(V;B) \cap T^{\bar{k}}(V)$ for $k = 0, 1$. Thus, $\mathrm{Cl}(V)$ inherits a $\mathbb{Z}_2$-grading,

$$\mathrm{Cl}(V;B) = \mathrm{Cl}^{\bar{0}}(V;B) \oplus \mathrm{Cl}^{\bar{1}}(V;B).$$

Note that the symbol map $\sigma\colon \mathrm{Cl}(V;B) \to \wedge(V)$ preserves the $\mathbb{Z}_2$-grading.

From now on, commutators $[\cdot,\cdot]$ in $\mathrm{Cl}(V;B)$ will denote $\mathbb{Z}_2$-graded commutators. (We will write $[\cdot,\cdot]_{\mathrm{Cl}}$ if there is risk of confusion.) In this notation, the defining relations for the Clifford algebra become

$$[v,w] = B(v,w), \quad v,w \in V.$$

The even (resp. odd) elements of $\mathrm{Cl}(V;B)$ are linear combinations of products $v_1 \cdots v_k$ with k even (resp. odd). The filtration and the $\mathbb{Z}_2$-grading are *compatible* in the sense that each $\mathrm{Cl}_{(k)}(V;B)$ is a $\mathbb{Z}_2$-graded subspace, with

$$\mathrm{Cl}^{\bar{0}}_{(2k)}(V;B) = \mathrm{Cl}^{\bar{0}}_{(2k+1)}(V;B),\ \ \mathrm{Cl}^{\bar{1}}_{(2k+1)}(V;B) = \mathrm{Cl}^{\bar{1}}_{(2k+2)}(V;B). \tag{17}$$

The Clifford algebra as a quantization of the exterior algebra. Using the quantization map, the Clifford algebra $\mathrm{Cl}(V;B)$ may be thought of as $\wedge(V)$ with a new associative product.

We can make more precise (following Kostant-Sternberg) in which sense the Clifford algebra is a quantization of the exterior algebra. Suppose $\mathcal{A}$ is an $\mathbb{Z}_2$-graded algebra, equipped with a filtration $\mathcal{A}_{(k)}$ that is compatible with the $\mathbb{Z}_2$-grading. Thus each $\mathcal{A}_{(k)}$ is a $\mathbb{Z}_2$-graded subspace, and

$$\mathcal{A}^{\bar{0}}_{(2k)}(V;B) = \mathcal{A}^{\bar{0}}_{(2k+1)}(V;B),\ \ \mathcal{A}^{\bar{1}}_{(2k+1)}(V;B) = \mathcal{A}^{\bar{1}}_{(2k+2)}(V;B). \tag{18}$$

Then the $\mathbb{Z}_2$-grading on the associated graded algebra $\mathrm{gr}(\mathcal{A})$ is just the mod 2 reduction of the $\mathbb{Z}$-grading. Suppose now that $\mathrm{gr}(\mathcal{A})$ is graded commutative. In other words, the multiplication in $\mathcal{A}$ is commutative 'up to lower order terms'. Then

$$[\mathcal{A}_{(k)}, \mathcal{A}_{(l)}] \subset \mathcal{A}_{(k+l-2)}.$$

The commutator lies in $\mathcal{A}_{(k+l-1)}$ since $\mathrm{gr}(\mathcal{A})$ is commutative, but in fact lies in $\mathcal{A}_{(k+l-2)}$ for parity reasons. Hence we can define a bracket on $\mathrm{gr}(A)$ as follows,

$$\{[x],\ [y]\} := [x,y]_{\mathcal{A}} \mod \mathcal{A}_{(k+l-3)}$$

for $x \in \mathcal{A}_{(k)}$ and $y \in \mathcal{A}_{(l)}$, where $[x],[y]$ denote the images in the associated graded algebra. It is easy to see that $\{\cdot,\cdot\}$ is a graded Poisson bracket (of degree -2).

If we apply these considerations to the Clifford algebra, we get a Poisson bracket on $\mathrm{gr}(\mathrm{Cl}(V)) = \wedge(V)$. On generators, $\{v,w\} = B(v,w)$, so this is just the Poisson bracket introduced earlier. The quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V)$ intertwines Poisson brackets with commutators, up to lower order terms. That is, for $\lambda_i \in \wedge^{k_i}(V)$,

$$q(\{\lambda_1, \lambda_2\}) - [q(\lambda_1), q(\lambda_2)] \in \mathrm{Cl}_{(k_1+k_2-4)}(V).$$

But this is clearly analogous to the notion of quantization in 'quantum mechanics'.

Note that we can also carry out a semi-classical limit: If we rescale $B \rightsquigarrow \hbar B$, $\hbar \in \mathbb{K}$, the Clifford product goes back to the wedge product.

4. Further properties of Clifford algebras

Universal property. Similar to the exterior algebra, the Clifford algebra is characterized by a universal property:

PROPOSITION 4.1. *If $\mathcal{A}$ is an associative algebra, and $\phi\colon V \to \mathcal{A}$ a linear map with $\phi(v_1)\phi(v_2) + \phi(v_2)\phi(v_1) = B(v_1, v_2) \cdot 1$, then ϕ extends uniquely to an algebra homomorphism* $\mathrm{Cl}(V; B) \to \mathcal{A}$.

PROOF. By the universal property of the tensor algebra, ϕ extends to an algebra homomorphism $\phi_{T(V)}\colon T(V) \to \mathcal{A}$. The property $\phi(v_1)\phi(v_2) + \phi(v_2)\phi(v_1) = B(v_1, v_2) \cdot 1$ shows that ϕ vanishes on the ideal $\mathcal{I}(V; B)$, and hence descends to the Clifford algebra. Uniqueness is clear, since the Clifford algebra is generated by elements of V. $\square$

Functoriality. Suppose B_1, B_2 are symmetric bilinear forms on V_1, V_2, and $\phi\colon V_1 \to V_2$ is a linear map such that

$$B_2(\phi(v), \phi(w)) = B_1(v, w), \quad v, w \in V_1.$$

By composing ϕ with the map $V_2 \to \mathrm{Cl}(V_2; B_2)$, and using the universal property, we see that ϕ extends to an algebra homomorphism

$$\mathrm{Cl}(\phi)\colon \mathrm{Cl}(V_1; B_1) \to \mathrm{Cl}(V_2; B_2).$$

For instance, if $F \subset V$ is an isotropic subspace of V, there is an algebra homomorphism $\wedge(F) = \mathrm{Cl}(F) \to \mathrm{Cl}(V; B)$. Clearly, $\mathrm{Cl}(\phi_1 \circ \phi_2) = \mathrm{Cl}(\phi_1) \circ \mathrm{Cl}(\phi_2)$.

Taking $V_1 = V_2 = V$, and restricting attention to invertible linear maps, one obtains a group homomorphism

$$\mathrm{O}(V; B) \to \mathrm{Aut}(\mathrm{Cl}(V; B)).$$

The extension of A to an algebra automorphism $\mathrm{Cl}(A)$ of the Clifford algebra will again be denoted A. As a simple example, the involution $\Pi(v) = -v$ lies in $O(V; B)$, hence it defines an involutive algebra automorphism Π of $\mathrm{Cl}(V; B)$. The ± 1 eigenspaces are the even and odd part of the Clifford algebra, respectively.

Direct sums. Let $(V_{,}B_1)$ and (V_2, B_2) be two vector spaces with symmetric bilinear forms, and consider the direct sum $(V_1 \oplus V_2, B_1 \oplus B_2)$. Then

$$\mathrm{Cl}(V_1 \oplus V_2, B_1 \oplus B_2) = \mathrm{Cl}(V_1, B_1) \otimes \mathrm{Cl}(V_2, B_2)$$

(tensor product of $\mathbb{Z}_2$-graded algebras). This isomorphism follows from the universal property of the Clifford algebra, applied to the linear map

$$V_1 \oplus V_2 \to \mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2), \;\; v_1 \oplus v_2 \mapsto v_1 \otimes 1 + 1 \otimes v_2.$$

The fact that this map is indeed an isomorphism will become clear later. In particular, if $\mathrm{Cl}(n,m)$ denotes the Clifford algebra for $\mathbb{R}^{n,m}$ we have

$$\mathrm{Cl}(n,m) = \mathrm{Cl}(1,0) \otimes \cdots \otimes \mathrm{Cl}(1,0) \otimes \mathrm{Cl}(0,1) \otimes \cdots \otimes \mathrm{Cl}(0,1),$$

with $\mathbb{Z}_2$-graded tensor products.

EXAMPLES 4.2. For n,m small it is easy to determine the algebras $\mathrm{Cl}(n,m)$ by hand. The Clifford algebra $\mathrm{Cl}(0,1)$ is generated by e_1 with a single relation $2e_1e_1 = -1$. Thus, letting $i = \frac{1}{\sqrt{2}}e_1$, we see that

$$\mathrm{Cl}(0,1) = \mathbb{C}$$

(viewed as an algebra over $\mathbb{R}$). For $\mathrm{Cl}(1,0)$, the relation is replaced by $2e_1e_1 = 1$. We hence find

$$\mathrm{Cl}(1,0) = \mathbb{R} \oplus \mathbb{R}$$

(direct sum of algebras), by the isomorphism $e_1 \mapsto \frac{1}{\sqrt{2}}(1 \oplus -1)$ and $1 \mapsto (1 \oplus 1)$. Next,

$$\mathrm{Cl}(0,2) \cong \mathbb{H},$$

by the isomorphism $e_1 \mapsto \frac{1}{\sqrt{2}}i, \quad e_2 \mapsto \frac{1}{\sqrt{2}}j, \quad e_3 \mapsto \frac{1}{2}k, \quad 1 \mapsto 1$. Here i,j,k are the unit quaternions i,j,k. For more results on the classification of the Clifford algebras $\mathrm{Cl}(n,m)$ (featuring the remarkable *periodicity theorem*), see e.g. the paper by Atiyah-Bott-Shapiro or the books by Lawson-Michelsohn or Garcia-Bondia et al.

The anti-automorphism. The canonical anti-automorphism of the tensor algebra, given by transposition, preserves the ideal $\mathcal{I}(V;B)$. It hence descends to an anti-automorphism of $\mathrm{Cl}(V;B)$, still called transposition, with

$$(v_1 \cdots v_k)^t = v_k \cdots v_1.$$

The symbol map and its inverse, the quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V;B)$ intertwines the transposition maps for $\wedge(V)$ and $\mathrm{Cl}(V;B)$. This information is sometimes useful for computations.

EXAMPLE 4.3. Recall that the transposition map is given by a sign $(-1)^{k(k-1)/2}$ on $\wedge^k(V)$. Thus, on $\wedge^k(V)$ it is equal to 1 if $k = 0,1 \bmod 4$ and equal to -1 if $k = 2,3 \bmod 4$. If $\phi \in \wedge^3(V)$, the element

$$2\sigma(q(\phi)^2) = \sigma([q(\phi), q(\phi)]).$$

lies in $\wedge_{(4)}(V) = \wedge^4(V) \oplus \wedge^2(V) \oplus \wedge^0(V)$, with leading term $\{\phi,\phi\}$. Using the transposition map, we can argue that the component in $\wedge^2(V)$ must be zero. Indeed, since $q(\phi)^t = q(\phi^t) = -q(\phi)$ we have

$$[q(\phi), q(\phi)]^t = [q(\phi)^t, q(\phi)^t] = [q(\phi), q(\phi)].$$

But $(\cdot)^t = -1$ on $\wedge^2(V)$. Hence

$$\sigma([q(\phi), q(\phi)]) - \{\phi,\phi\} \in \wedge^0(V) = \mathbb{K}.$$

Of particular interest are elements ϕ with $\{\phi,\phi\} = 0$, since then $\{\phi,\cdot\}$ resp. $[q(\phi),\cdot]$ define differentials on $\wedge(V)$ resp. on $\mathrm{Cl}(V;B)$ (i.e. they are operators squaring to 0.) Later we well see that the structure constants tensor of any semi-simple Lie algebra (and more generally of any quadratic Lie algebra) has this property.

Contraction operators. Let V be a vector space with symmetric bilinear form B. Simplifying our earlier notation, we will now write

$$\iota(v) := \iota(B^\flat(v)) \in \mathrm{Der}^{-1}(\wedge(V))$$

for the operator of contraction by $B^\flat(v) \in V$. We had seen that $\iota(v)$ can be written in terms of the Poisson bracket as

$$\iota(v) = \{v, \cdot\}.$$

Similarly, we define contraction operators $\iota(v) \in \mathrm{Der}^{\bar{1}}(\mathrm{Cl}(V;B))$ on the Clifford algebra,

$$\iota(v) := [v, \cdot].$$

The contraction operators extend to homomorphisms of $\mathbb{Z}_2$-graded algebras,

$$\iota\colon \wedge(V) \to \mathrm{End}(\wedge(V)), \quad \iota\colon \wedge(V) \to \mathrm{End}(\mathrm{Cl}(V;B)).$$

PROPOSITION 4.4. *The quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V;B)$ intertwines the contraction operators on the exterior and Clifford algebras. That is,*

$$q \circ \iota(\phi) = \iota(\phi) \circ q$$

for all $\phi \in \wedge(V)$.

PROOF. It is enough to check in an orthogonal basis e_i. For indices $j_1 < \ldots j_k$, the contraction operators in the Clifford algebra are given by

$$\iota(e_i)(e_{j_1} \cdots e_{j_k}) = \sum_{r=1}^{k} (-1)^{r+1} B(e_i, e_{j_r})\; e_{j_1} \cdots e_{j_{r-1}} \widehat{e_{j_r}} e_{j_{r+1}} \cdots e_{j_k}$$

The contraction operators in the exterior algebra have a similar description, with Clifford products replaced by wedge products. Since the quantization map takes any wedge product $e_{j_1} \wedge \cdots \wedge e_{j_k}$ with $i_1 < \cdots < i_k$ to $e_{j_1} \cdots e_{j_k}$, this proves $q \circ \iota(e_i) = \iota(e_i) \circ q$. ∎

The action of $\mathfrak{o}(V;B)$. Let V be a vector space with symmetric bilinear form B. Recall that we had constructed a Lie algebra homomorphism

$$\wedge^2(V) \to \mathfrak{o}(V;B), \;\; \phi \mapsto A_\phi$$

where the bracket on $\wedge^2(V)$ is the Poisson bracket, and $A_\phi(v) = \{\phi, v\} = -\iota(v)\phi$.

PROPOSITION 4.5. *The quantization map restricts to a Lie algebra homomorphism*

$$q\colon \wedge^2(V) \to \mathrm{Cl}^{\bar{0}}(V;B).$$

For any $\phi \in \wedge^2(V)$, the operator $A_\phi \in \mathfrak{o}(V;B)$ may be described as a Clifford commutator,

$$A_\phi(v) = [q(\phi), v].$$

PROOF. We have

$$\sigma([q(\phi), q(\psi)]) - \{\phi, \psi\} \in \wedge^0(V) = \mathbb{K}.$$

The first part of the Proposition says that the scalar term is in fact zero. We argue using the transposition map: We have $q(\phi)^t = -q(\phi)$, $q(\psi)^t = -q(\psi)$ and hence

$$\begin{aligned}
[q(\phi), q(\psi)]^t &= \big(q(\phi)q(\psi) - q(\psi)q(\phi)\big)^t \\
&= q(\psi)q(\phi) - q(\phi)q(\psi) \\
&= -[q(\phi), q(\psi)].
\end{aligned}$$

Since $(\cdot)^t = 1$ on $\wedge^0(V)$, the scalar term must be zero. The second claim follows from

$$A_{q(\phi)}(v) = [q(\phi), v] = -\iota(v)q(\phi) = -q(\iota(v)\phi) = q(A_\phi(v)) = A_\phi(v).$$

$\square$

We saw that if B is non-degenerate, the map $\wedge^2(V) \to \mathrm{O}(V; B)$ is an isomorphism. Denote the inverse map by

$$\lambda\colon \mathfrak{o}(V;B) \to \wedge^2(V),\ A \mapsto \lambda(A) = \tfrac{1}{2}\sum_a A(e_a) \wedge e^a \tag{19}$$

(here $e_a \in V$ is an orthogonal basis, and e^a is the B-dual basis). Ith thus follows that $q(\wedge^2(V)) \to \mathrm{o}(V; B)$ is an isomorphism as well, with inverse $\gamma = q \circ \lambda$. In the orthogonal basis e_a we have,

PROPOSITION 4.6. *The inverse map is given by*

$$\gamma\colon \mathfrak{o}(V;B) \to q(\wedge^2(V;B)) \subset \mathrm{Cl}(V;B),\ A \mapsto \tfrac{1}{2}\sum_a A(e_a)\, e^a.$$

PROOF. We have

$$q(\sum_a A(e_a) \wedge e^a) = \sum_a A(e_a)e^a + \tfrac{1}{2}\sum_a B(A(e_a), e^a).$$

Since A is skew-symmetric,

$$\sum_a B(A(e_a), e^a) = -\sum_a B(e_a, A(e^a)) = -\sum_a B(A(e^a), e_a)$$

Together with

$$\sum_a B(A(e_a), e^a) = \sum_a B(A(e^a), e_a)$$

(since the expression does not depend on the choice of orthogonal basis), this shows $\sum_a B(A(e_a), e^a) = 0$. $\square$

The chirality element Let $\dim V = n$. Then any generator $\Gamma_\wedge \in \wedge^n(V)$ quantizes to given an element $\Gamma = q(\Gamma_\wedge)$. This element (or suitable normalizations of this element) is called the *chirality element* of the Clifford algebra.

Square Γ^2 of the chirality element is always a scalar. Indeed, this is immediate by choosing an orthogonal basis e_i, and letting $\Gamma = e_1 \cdots e_n$. If $\mathbb{K} = \mathbb{R}$, the choice of $\Gamma_\wedge$ is unique up to sign, and hence the sign of Γ^2 is canonical. In fact one finds, for an orthogonal basis e_i,

$$\Gamma^2 = (-1)^{n(n-1)/2} \prod B(e_i, e_i)$$

which shows that the sign is $(-1)^{s+n(n-1)/2}$ if B has signature (p, s). In particular, we can always normalize Γ to satisfy

$$\Gamma^2 = (-1)^{s+n(n-1)/2}.$$

In the complex case, we can even normalize Γ to satisfy $\Gamma^2 = 1$. In both cases, the normalization determines Γ up to a sign.

Returning to the general case, we note that the element $\Gamma = q(\Gamma_\wedge)$ satisfies, for any $v \in V$

$$\Gamma v = \begin{cases} v\Gamma & \text{if } n \text{ is odd} \\ -v\Gamma & \text{if } n \text{ is even} \end{cases}$$

Thus, if n is odd then Γ lies in the center of $\mathrm{Cl}(V)$, viewed as an ordinary algebra. (The center of $\mathrm{Cl}(V)$ as a super-algebra is trivial.) In the case n even, we obtain

$$\Pi(x) = \Gamma x \Gamma^{-1},$$

for all $x \in \mathrm{Cl}(V)$, i.e. the chirality element implements the parity operator.

The trace. Suppose $\dim V = n$, so that $\mathrm{Cl}_{(n)}(V;B) = \mathrm{Cl}(V;B)$. Let $\det(V) = \wedge^n(V)$ be the top exterior power of V. There is a canonical map,

$$\mathrm{tr}_s \colon \mathrm{Cl}(V;B) \to \det(V)$$

taking an element of the Clifford algebra to its image in $\mathrm{Cl}_{(n)}(V;B)/\mathrm{Cl}_{(n-1)}(V;B)$. This map has the property

$$\mathrm{tr}_s([x,y]) = 0$$

for all $x, y \in \mathrm{Cl}(V;B)$, where the bracket is the $\mathbb{Z}_2$-graded commutator. Thus, once we choose a generator of $\det(V)$ the map tr_s is a super-trace on the super-algebra $\mathrm{Cl}(V;B)$. As remarked above, the exterior algebra, and in particular the line $\det(V)$ carries a symmetric bilinear form induced from B. If $\mathbb{K} = \mathbb{R}$ and B is positive definite, it is natural to take the generator to be of length 1; the choice of sign of the generator amounts to the choice of orientation on V.

There is also a canonical trace of $\mathrm{Cl}(V;B)$ regarded as an 'ordinary' algebra (i.e. forgetting about the $\mathbb{Z}_2$-grading). Let $\Gamma_\wedge$ be a generator of $\wedge^n(V)$, and Γ be the corresponding chirality element in the Clifford algebra. Define $\mathrm{tr} \colon \mathrm{Cl}(V;B) \to \mathbb{K}$ by the formula,

$$\mathrm{tr}_s(\Gamma x) = \mathrm{tr}(x)\,\Gamma_\wedge.$$

PROPOSITION 4.7. tr *is a trace on the Clifford algebra, viewed as an ungraded algebra. That is,*

$$\mathrm{tr}(xy) = \mathrm{tr}(yx)$$

for all $x, y \in \mathrm{Cl}(V;B)$.

PROOF. Pick an orthogonal basis e_i, and let $\Gamma_\wedge = e_1 \wedge \cdots \wedge e_l$. It suffices to check for basis vectors $x = e_I, y = e_J$. The trace property is obvious for $I = J$, so let us suppose $I \neq J$. Note that $xy = c e_K$ where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$, and consequently $\Gamma x y = c' e_{K^c}$ where K^c is the complement of K. Thus, $\mathrm{tr}_s(\Gamma e_I e_J)$ and similarly $\mathrm{tr}_s(\Gamma e_J e_I)$ vanish if $I \neq J$. $\square$

Observe that $\mathrm{tr}(1) = 1$. Along with any trace comes a symmetric bilinear form, $(x,y) \mapsto \mathrm{tr}(xy)$. In particular, for $v, w \in V$,

$$\mathrm{tr}(vw) = \tfrac{1}{2}\,\mathrm{tr}(vw + wv) = \tfrac{1}{2}B(v,w)\,\mathrm{tr}(1) = \tfrac{1}{2}B(v,w).$$

5. A formula for the Clifford product

It is sometimes useful to express the Clifford multiplication

$$m_{\mathrm{Cl}} \colon \mathrm{Cl}(V \oplus V) = \mathrm{Cl}(V) \otimes \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

in terms of the exterior algebra multiplication,

$$m_\wedge \colon \wedge(V \oplus V) = \wedge(V) \otimes \wedge(V) \to \wedge(V).$$

Recall that by definition of the isomorphism $\wedge(V\oplus V)=\wedge(V)\otimes\wedge(V)$, if $\phi,\psi\in\wedge(V^*)$, the element $\phi\otimes\psi\in\wedge(V^*)\otimes\wedge(V^*)$ is identified with the element $(\phi\oplus 0)\wedge(0\oplus\psi)\in\wedge(V^*\oplus V^*)$. Similarly fro the Clifford algebra.

PROPOSITION 5.1. *Let $e_i\in V$ be an orthogonal basis, $e^i\in V^*$ the dual basis. Then*

$$m_{\mathrm{Cl}}\circ q=q\circ m_\wedge\circ\iota(\Psi)$$

where

$$\Psi=\sum_I\frac{1}{(-2)^{|I|}}\;e^I\otimes B^\flat((e_I)^t)$$

(sum over all subsets $I\subset\{1,\ldots,n\}$).

PROOF. Let V_i be the 1-dimensional subspace spanned by e_i. Then $\wedge(V)$ is the graded tensor product over all $\wedge(V_i)$, and similarly $\mathrm{Cl}(V)$ is the graded tensor product over all $\mathrm{Cl}(V_i)$. The formula for Ψ factorizes as

$$\Psi=\prod_{i=1}^n\big(1-\tfrac{1}{2}e^i\otimes B^\flat(e_i)\big).$$

For instance, if $n=2$,

$$\Psi=1-\frac{1}{2}\sum_i e^i\otimes B^\flat(e_i)+\frac{1}{4}(e^1\wedge e^2)\otimes B^\flat(e_2\wedge e_1).$$

is a product $(1-\frac{1}{2}e^1\otimes B^\flat(e_1))(1-\frac{1}{2}e^2\otimes B^\flat(e_2))$

It hence suffices to prove the formula for the case $V=V_1$. We have,

$$\begin{aligned}q\circ m_\wedge\circ\iota\big(1-\tfrac{1}{2}e^1\otimes B^\flat(e_1)\big)(e_1\otimes e_1)&=q\circ m_\wedge\big(e_1\otimes e_1+\tfrac{1}{2}B(e_1,e_1)\big)\\&=q\big(\tfrac{1}{2}B(e_1,e_1)\big)\\&=e_1e_1.\end{aligned}$$

□

If $\mathrm{char}(\mathbb{K})=0$, we may also write the element Ψ as an exponential:

$$\Psi=\exp\big(-\tfrac{1}{2}\sum_i e^i\otimes B^\flat(e_i)\big).$$

This follows directly by rewriting the exponential of a sum as a product of exponentials,

$$\prod_i\exp\big(-\tfrac{1}{2}e^i\otimes B^\flat(e_i)\big)=\prod_i\big(1-\tfrac{1}{2}e^i\otimes B^\flat(e_i)\big).$$

REMARK 5.2. Consider the addition map

$$\mathrm{Add}\colon V\oplus V\to V,\;\;v\oplus w\mapsto v+w.$$

This map is linear, and hence to an algebra homomorphism

$$\wedge(\mathrm{Add})\colon\;\wedge(V\oplus V)\to\wedge(V).$$

In terms of the identification $\wedge(V\oplus V)=\wedge(V)\otimes\wedge(V)$, this is exactly the map $m_\wedge$. The dual map $\mathrm{Add}^*\colon V^*\to V^*\oplus V^*$ is the diagonal inclusion. The composition

$$\tilde{m}_{\mathrm{Cl}}=\sigma\circ m_{\mathrm{Cl}}\circ q\colon\;\wedge(V\oplus V)\to\wedge(V)$$

has the property,

$$\tilde{m}_{\mathrm{Cl}}\circ\iota(\mathrm{Add}^*(\alpha))=\iota(\alpha)\circ\tilde{m}_{\mathrm{Cl}}$$

for all $\alpha \in V^*$. Hence, by Lemma 1.6, there exists a unique element $\Psi \in \wedge(V^* \oplus V^*)$ such that $\tilde{m}_{\mathrm{Cl}} = m_\wedge \circ \iota(\Psi)$, and this is the element determined in the Proposition.

6. The Clifford group and the spin group

Let V be a vector space over a field $\mathbb{K}$ of characteristic $\neq 2$, equipped with a non-degenerate symmetric bilinear form B. Recall that $\Pi\colon \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ denotes the parity automorphism of the Clifford algebra. Let $\mathrm{Cl}(V)^\times$ be the group of invertible elements in $\mathrm{Cl}(V)$.

DEFINITION 6.1. The *Clifford group* $\Gamma(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$, consisting of all $x \in \mathrm{Cl}(V)^\times$ such that $A_x(v) := \Pi(x)vx^{-1} \in V$ for all $v \in V \subset \mathrm{Cl}(V)$.

Hence, by definition the Clifford group comes with a natural representation, $\Gamma(V) \to \mathrm{GL}(V)$, $x \mapsto A_x$. Let $\Gamma^{\bar{0}}(V) = \Gamma(V) \cap \mathrm{Cl}^{\bar{0}}(V)^\times$ denote the *even Clifford group.*

THEOREM 6.2. *The canonical representation of the Clifford group takes values in* $\mathrm{O}(V)$*, and defines an exact sequence,*

$$1 \longrightarrow \mathbb{K}^\times \longrightarrow \Gamma(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1.$$

It restricts to a similar exact sequence for the even Clifford group $\Gamma^{\bar{0}}(V)$*, with image* $\mathrm{SO}(V)$*. The elements of* $\Gamma(V)$ *are all products* $x = v_1 \cdots v_k$ *where* $v_1, \ldots, v_k \in V$ *are non-isotropic.* $\Gamma_0(V)$ *consists of similar products, with* k *even. The corresponding element* A_x *is a product of reflections:*

$$A_{v_1 \cdots v_k} = R_{v_1} \cdots R_{v_k}.$$

PROOF. The transformation A_x is trivial if and only if $\Pi(x)v = vx$ for all $v \in V$, i.e. if and only if $[v, x] = 0$ for all $v \in V$. That is, it is the intersection of the center $\mathbb{K} \subset \mathrm{Cl}(V)$ with $\Gamma(V)$. This shows that the kernel of the homomorphism $\Gamma(V) \to \mathrm{GL}(V)$ is the group $\mathbb{K}^\times$ of invertible scalars.

Applying $-\Pi$ to the definition of A_x, we obtain $A_x(v) = xv\Pi(x)^{-1} = A_{\Pi(x)}(v)$. This shows $A_{\Pi(x)} = A_x$ for $x \in \Gamma(V)$. For $x \in \Gamma(V)$ and $v, w \in V$ we have,

$$\begin{aligned}
B(A_x(v), A_x(w)) &= \big(A_x(v)A_x(w) + A_x(w)A_x(v)\big) \\
&= \big(A_x(v)A_{\Pi(x)}(w) + A_x(w)A_{\Pi(x)}(v)\big) \\
&= \big(\Pi(x)(vw + wv)\Pi(x^{-1})\big) \\
&= B(v, w)\Pi(x)\Pi(x^{-1}) \\
&= B(v, w).
\end{aligned}$$

This proves that $A_x \in \mathrm{O}(V)$ for all $x \in \Gamma(V)$.

Observe next that any non-isotropic $v \in V$ lies in the Clifford group, with $A_v = R_v$ the reflection defined by v. To check $A_v(w) = R_v(w)$, it suffices to consider the two cases $w = v$ and $B(v, w) = 0$. In the first case,

$$A_v(v) = \Pi(v)vv^{-1} = \Pi(v) = -v = R_v(v).$$

In the second case,

$$A_v(w) = \Pi(v)wv^{-1} = -\Pi(v)v^{-1}w = vv^{-1}w = w = R_v(w).$$

More generally, this shows $A_{v_1 \cdots v_k} = R_{v_1} \cdots R_{v_k}$. By the E. Cartan-Dieudonné theorem, any $A \in \mathrm{O}(V)$ is of this form. This shows the map $x \mapsto A_x$ is onto $\mathrm{O}(V)$, and that $\Gamma(V)$ is generated by the non-isotropic vectors in V. ☐

Since all $x \in \Gamma(V)$ can be written in the form $x = v_1 \cdots v_k$ with non-isotropic vectors v_i, it follows that the element $x^t x$ lies in $\mathbb{K}^\times$. This defines the *norm homomorphism*

$$N \colon \Gamma(V) \to \mathbb{K}^\times, \; x \mapsto x^t x.$$

It has the obvious property

$$N(\lambda x) = \lambda^2 N(x)$$

for $\lambda \in \mathbb{K}^\times$. If $\mathbb{K} = \mathbb{R}$, any x can be rescaled to satisfy $N(x) = \pm 1$. One defines, [5]

DEFINITION 6.3. Suppose $\mathbb{K} = \mathbb{R}$. The *Pin group* $\mathrm{Pin}(V)$ is the pre-image of $\{1, -1\}$ under the norm homomorphism $N \colon \Gamma(V) \to \mathbb{K}^\times$. Its intersection with $\Gamma^{\bar{0}}(V)$ is called the *Spin group*, and is denoted $\mathrm{Spin}(V)$.

The exact sequence for the Clifford group restricts to an exact sequence,

$$1 \longrightarrow \{\lambda |\; \lambda^2 = \pm 1\} \longrightarrow \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1,$$

so that $\mathrm{Pin}(V)$ is a double cover of $\mathrm{O}(V)$. Similarly,

$$1 \longrightarrow \{\lambda |\; \lambda^2 = \pm 1\} \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1,$$

defines a double cover of $\mathrm{SO}(V)$. Elements in $\mathrm{Pin}(V)$ are products $x = v_1 \cdots v_k$ with $B(v_i, v_i) = \pm 2$. The group $\mathrm{Spin}(V)$ consists of similar products, with k even.

For $V = \mathbb{R}^{n,m}$, with the scalar product of signature n, m, let $\mathrm{Spin}(V) = \mathrm{Spin}(n, m)$ and $\mathrm{Pin}(V) = \mathrm{Pin}(n, m)$. Also, let $\mathrm{Spin}_0(n, m)$ denote the preimage of the identity component, $\mathrm{SO}_0(n, m)$. As usual, we will write $\mathrm{Pin}(n) = \mathrm{Pin}(n, 0)$ and $\mathrm{Spin}(n) = \mathrm{Spin}(n, 0)$.

THEOREM 6.4. *Let $\mathbb{K} = \mathbb{R}$, and $V \cong \mathbb{R}^{n,m}$. If $n \geq 2$ or $m \geq 2$, the group $\mathrm{Spin}_0(V)$ is connected.*

PROOF. The pre-image of the group unit $e \in \mathrm{SO}_0(V)$ in $\mathrm{Spin}(V)$ are the elements $+1, -1 \in \mathrm{Cl}(V)$. To show that $\mathrm{Spin}_0(V)$ is connected, it suffices to show that ± 1 are in the same connected component. Let

$$v(\theta) \in V, \quad 0 \leq \theta \leq \pi$$

be a continuous family of non-isotropic vectors with the property

$$v(\pi) = -v(0).$$

Such a family exists, since V contains a 2-dimensional subspace isomorphic to $\mathbb{R}^{2,0}$ or $\mathbb{R}^{0,2}$. We may normalize the vectors $v(\theta)$ to satisfy

$$B(v(\theta), v(\theta)) = \pm 2.$$

Then $v(\theta)v(0) \in \mathrm{Spin}(V) \subset \mathrm{Cl}^{\bar{0}}(V)$ equals ± 1 for $\theta = 0$, and ∓ 1 for $\theta = \pi$. This shows that 1 and -1 are in the same component of $\mathrm{Spin}_0(V)$, as desired. ☐

Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$, the connected double cover $\mathrm{Spin}(n)$ is the universal cover in that case. In low dimensions, we had determined these universal covers to be

$$\mathrm{Spin}(3) = \mathrm{SU}(2), \quad \mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2).$$

It can also be shown that $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ (the group of norm-preserving automorphisms of the quaternionic vector space $\mathbb{H}^2$) and $\mathrm{Spin}(6) = \mathrm{SU}(4)$. For $n \geq 7$, the groups $\mathrm{Spin}(n)$ are all simple and non-isomorphic to the other classical groups.

[5] The definition also makes sense for arbitrary fields. However, the natural representation need not be onto. Cf. Grove p. 78.

The groups $\mathrm{Spin}_0(n,m)$ are usually not simply connected. Indeed since since $\mathrm{SO}_0(n,m)$ has maximal compact subgroup $\mathrm{SO}(n)\times\mathrm{SO}(m)$, the fundamental group is

$$\pi_1(\mathrm{SO}_0(n,m)) = \pi_1(\mathrm{SO}(n))\times\pi_1(\mathrm{SO}(m))$$

Hence, only in the cases $n\geq 3,\quad m=0,1$ or $n=0,1,\quad m\geq 3$ we obtain $\pi_1(\mathrm{SO}_0(n,m))=\mathbb{Z}_2$, and only in those cases $\mathrm{Spin}_0(n,m)$ is a universal cover.

Let us now turn to the case $\mathbb{K}=\mathbb{C}$, so that $V\cong\mathbb{C}^n$ with the standard bilinear form. In that case, we can rescale any $x\in\Gamma(V)=\Gamma(n,\mathbb{C})$ to satisfy $N(x)=+1$. Hence define[6]

$$\mathrm{Pin}(n,\mathbb{C}) = \{x\in\Gamma(n,\mathbb{C}))|\, N(x)=+1\}$$

and $\mathrm{Spin}(n,\mathbb{C})=\mathrm{Pin}(n,\mathbb{C})\cap\Gamma^{\bar{0}}(n,\mathbb{C})$.

PROPOSITION 6.5. $\mathrm{Pin}(n,\mathbb{C})$ *and* $\mathrm{Spin}(n,\mathbb{C})$ *are double covers of* $\mathrm{O}(n,\mathbb{C})$ *and* $\mathrm{SO}(n,\mathbb{C})$. *Furthermore,* $\mathrm{Spin}(n,\mathbb{C})$ *is connected and simply connected, i.e. it is the universal cover of* $\mathrm{SO}(n,\mathbb{C})$.

PROOF. The first part is clear, since the condition $N(x)=1$ determines the scalar multiple of x up to a sign. The second part follows by the same argument as in the real case, or alternatively by observing that ±1 are in the same component of $\mathrm{Spin}(n,\mathbb{R})\subset\mathrm{Spin}(n,\mathbb{C})$. □

Assume $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Recall the isomorphism $\lambda\colon\mathfrak{o}(V)\to\wedge^2(V)$, and let

$$\gamma = q\circ\lambda\colon \mathfrak{o}(V)\to\mathrm{Cl}(V).$$

Then $A(v)=[\gamma(A),v]$ for $v\in V$, and accordingly

$$\exp(A)(v) = e^{[\gamma(A),\cdot]}v = e^{\gamma(A)}ve^{-\gamma(A)}.$$

Here

$$e^{[\gamma(A),\cdot]}v = \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{[\gamma(A),[\gamma(A),[\cdots[\gamma(A)}_{n\text{ times}},v]\cdots]]]$$

and

$$e^{\gamma(A)} = \sum_{n=0}^{\infty}\frac{1}{n!}\gamma(A)^n.$$

By definition of the Clifford group, this shows that $e^{\gamma(A)}\in\Gamma^{\bar{0}}(V)$. The element $\gamma(A)$ satisfies $\gamma(A)^t=-\gamma(A)$. Hence,

$$(e^{\gamma(A)})^t = e^{\gamma(A)^t} = e^{-\gamma(A)},$$

and therefore $N(e^{\gamma(A)})=1$. That is,

$$e^{\gamma(A)}\in\mathrm{Spin}(V)$$

Since $\theta\mapsto e^{\theta\gamma(A)}$ defines a curve in $\mathrm{Spin}(V)$, connecting 1 with $e^{\gamma(A)}$, it follows that $e^{\gamma(A)}$ is in the identity component $\mathrm{Spin}_0(V)$.

In other words, the group $\mathrm{Spin}(V)\subset\mathrm{Cl}(V)^\times$ constructed above has Lie algebra $\gamma(\mathfrak{o}(V))\subset\mathrm{Cl}^{\bar{0}}(V)$. Indeed, if $\mathbb{K}=\mathbb{R}$ and the bilinear form B is positive definite, we can directly define $\mathrm{Spin}(V)$ as the set of elements $e^{\gamma(\mathfrak{o}(V))}$. This follows because $\mathrm{Spin}(V)$, as a double cover of the compact group $\mathrm{SO}(V)$, is compact, and for compact Lie groups the exponential map is onto.

[6]There seem to be no standard conventions for the definitions for the complex case.

EXAMPLE 6.6. Let $V = \mathbb{R}^2$ with the standard bilinear form, and consider the element $A \in \mathfrak{o}(V)$ defined by $\lambda(A) = e_1 \wedge e_2$. Then $\gamma(A) = e_1 e_2$. The 1-parameter group of elements

$$x(\theta) = \exp(\theta e_1 e_2) \in \mathrm{Spin}(V).$$

is given by the formula,

$$\begin{aligned} x(\theta) &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!}(e_1 e_2)^n \\ &= \sum_{m=0}^{\infty} \frac{\theta^{2m}}{(2m)!}\frac{(-1)^m}{2^{2m}} + \sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!}\frac{(-1)^k}{2^{2k}} e_1 e_2 \\ &= \cos(\theta/2) + \sin(\theta/2)(2e_1e_2) \end{aligned}$$

which shows in particular that $x(\theta + 2\pi) = -x(\theta)$. To find its action on V, we compute

$$\begin{aligned} x(\theta)e_1 x(-\theta) &= (\cos(\theta/2) + \sin(\theta/2)(2e_1e_2))e_1(\cos(\theta/2) - \sin(\theta/2)(2e_1e_2)) \\ &= (\cos(\theta/2)e_1 - \sin(\theta/2)e_2)(\cos(\theta/2) - \sin(\theta/2)(2e_1e_2)) \\ &= (\cos^2(\theta/2) - \sin^2(\theta/2))e_1 - 2\sin(\theta/2)\cos(\theta/2)e_2 \\ &= \cos(\theta)e_1 - \sin(\theta)e_2 \end{aligned}$$

This verifies that $x(\theta) = \exp(\theta e_1 e_2)$ acts as rotations by θ, but $x(\theta + 2\pi) = -x(\theta)$.

7. The groups $\mathrm{Pin}_c(V)$ and $\mathrm{Spin}_c(V)$

Let V be a vector space over $\mathbb{K} = \mathbb{R}$, with a positive definite symmetric bilinear form B. Denote by $V^{\mathbb{C}}$ the complexification of V. The complex conjugation mapping $v \mapsto \overline{v}$ extends to an anti-linear algebra automorphism $x \mapsto \overline{x}$ of the complexified Clifford algebra $\mathbb{C}l(V) = \mathrm{Cl}(V^{\mathbb{C}}) = \mathrm{Cl}(V)^{\mathbb{C}}$. Let $x^* = \overline{x}^t$.

DEFINITION 7.1. The group $\mathrm{Pin}_c(V)$ is the subgroup of $\Gamma(V^{\mathbb{C}})$ given as

$$\mathrm{Pin}_c(V) = \{x \in \Gamma(V^{\mathbb{C}}) |\ x^* x = \pm 1\}.$$

Its intersection with the even part of the Clifford algebra is denoted $\mathrm{Spin}_c(V)$.

The point of this definition is that for $x \in \mathrm{Pin}_c(V)$, the automorphism A_x of $V^{\mathbb{C}}$ preserves the real subspace V. That is, the representation $\Gamma(V^{\mathbb{C}}) \to \mathrm{O}(V^{\mathbb{C}})$ restricts to group homomorphisms

$$\mathrm{Pin}_c(V) \to \mathrm{O}(V), \quad \mathrm{Spin}_c(V) \to \mathrm{SO}(V).$$

PROPOSITION 7.2. *Each of these two group homomorphism is onto, and has kernel* $\mathrm{U}(1) \subset \mathbb{C}^\times$.

PROOF. Since $\mathrm{Pin}_c(V) \supset \mathrm{Pin}(V)$, and similarly $\mathrm{Spin}_c(V) \supset \mathrm{Spin}(V)$, it is clear that the two homomorphisms are onto. The kernel is obtained as the intersection with the kernel of $\Gamma(V^{\mathbb{C}}) \to \mathrm{O}(V^{\mathbb{C}})$. But the latter consists of $\mathbb{C}^\times$, and

$$\mathbb{C}^\times \cap \mathrm{Pin}_c(V) = \mathbb{C}^\times \cap \mathrm{Spin}_c(V) = \mathrm{U}(1).$$

□

We have thus constructed central extensions,

$$1 \to \mathrm{U}(1) \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \to 1,$$

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \to 1.$$

Of course, one could directly define these groups as the subgroup generated by $\mathrm{Pin}(V)$ resp. $\mathrm{Spin}(V)$ together with $\mathrm{U}(1)$. More precisely, $\mathrm{Spin}_c(V)$ is the quotient of $\mathrm{Spin}(V) \times \mathrm{U}(1)$ by the relation

$$(x, e^{i\psi}) \sim (-x, -e^{i\psi})$$

and similarly for $\mathrm{Pin}_c(V)$. The norm homomorphism for $\Gamma(V^{\mathbb{C}})$ restricts to a group homomorphism,

$$\mathrm{Pin}_c(V) \to \mathrm{U}(1),\ x \mapsto x^t x.$$

Together with the map to $\mathrm{O}(V)$ this defines exact sequences,

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \times \mathrm{U}(1) \to 1,$$

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \times \mathrm{U}(1) \to 1$$

One of the motivations for the group $\mathrm{Spin}_c(V)$ is the following.

Suppose B is positive definite. J is an orthogonal complex structure on V, that is, $J \in \mathrm{O}(V)$ and $J^2 = -I$. Such a J exists if and only if $n = \dim V$ is even, and turns V into a vector space over $\mathbb{C}$, with scalar multiplication

$$(a + \sqrt{-1}b)x = ax + bJx.$$

Let $U_J(V) \subset \mathrm{SO}(V)$ be the corresponding unitary group (i.e. the elements of $\mathrm{SO}(V)$ preserving J).

THEOREM 7.3. *The inclusion $U_J(V) \hookrightarrow \mathrm{SO}(V)$ admits a unique lift to a group homomorphism $U_J(V) \hookrightarrow \mathrm{Spin}_c(V)$, in such a way that the composition with the map to $\mathrm{U}(1)$ is the map $\mathrm{U}_J(V) \to \mathrm{U}(1),\ A \mapsto \det_J(A)$ (complex determinant).*

PROOF. We may choose an orthogonal basis $e_1, \ldots, e_{2n}$ of V, with the property $J(e_i) = e_{n+i}$ for $i = 1, \ldots, n$. This identifies $V \cong \mathbb{C}^k$, and $U_J(V)$ with $U(k)$.

We are trying to construct a lift of the map

$$U(k) \to \mathrm{SO}(2k) \times \mathrm{U}(1),\ A \mapsto (A,\ \det_{\mathbb{C}}(A))$$

to the double cover. Since $\mathrm{U}(k)$ is connected, if such a lift exists then it is unique. To prove existence, it suffices to check that any loop representing a generator of $\pi_1(\mathrm{U}(k)) \cong \mathbb{Z}$ lifts to a loop in $\mathrm{Spin}_c(V)$. Since the inclusion $\mathrm{U}(1) \to \mathrm{U}(k)$ induces an isomorphism of fundamental groups, it is enough to check this for $k = 1$, i.e. $n = 2$. Hence, our task is to lift the map

$$\mathrm{U}(1) \to \mathrm{SO}(2) \times \mathrm{U}(1),\ e^{i\theta} \mapsto (R(\theta), e^{i\theta})$$

to the double cover, $\mathrm{Spin}(V) \times \mathrm{U}(1)/\mathbb{Z}_2$. We had found in example 6.6 that the curve $R(\theta)$ lifts to

$$x(\theta) = \exp(\theta e_1 e_2) = \cos(\theta/2) + \sin(\theta/2)\, 2e_1 e_2$$

which has the property $x(\theta + 2\pi) = -x(\theta)$. The desired lift is explicitly given as,

$$e^{i\theta} \mapsto \big[\big(x(\theta), e^{i\theta/2}\big)\big].$$

where the brackets indicate the equivalence relation $(x, e^{i\psi}) \cong (-x, -e^{i\psi})$. $\square$

REMARK 7.4. The two possible square roots of $\det_{\mathbb{C}}(A)$ for $A \in \mathrm{U}(k)$ define a double cover of $\mathrm{U}(k)$,

$$\tilde{U}(k) = \{(A, z) \in \mathrm{U}(k) \times \mathbb{C}^{\times} \mid z^2 = \det_{\mathbb{C}}(A)\}.$$

While the inclusion $\mathrm{U}(k) \hookrightarrow \mathrm{SO}(2k)$ does not live to the Spin group, the above proof shows that there exists a lift for this double cover (i.e. the double cover is identified with the pre-image of $\mathrm{U}(k)$).

CHAPTER 3

Clifford modules and Spinors

1. Clifford modules

Let V be a vector space with symmetric bilinear form B, and $\mathrm{Cl}(V)$ the corresponding Clifford algebra. A $\mathbb{Z}_2$-graded left module over $\mathrm{Cl}(V)$ for B is called a *Clifford module.*[1] That is, a Clifford module is a $\mathbb{Z}_2$-graded vector space E together with a homomorphism of $\mathbb{Z}_2$-graded algebras,

$$\rho_E\colon \mathrm{Cl}(V) \to \mathrm{End}(E).$$

Equivalently, a Clifford module is given by a linear map $\rho_E\colon V \to \mathrm{End}^{\bar{1}}(E)$ such that $\rho_E(v)\rho_E(w) + \rho_E(w)\rho_E(v) = B(v,w)1$ for all $v, w \in V$. The first example of a Clifford module is the Clifford algebra $\mathrm{Cl}(V)$ itself, with module structure given by multiplication from the left. The identification $\sigma\colon \mathrm{Cl}(V) \cong \wedge(V)$ makes the exterior algebra into a Clifford module. Similarly, if $X \subset \mathrm{Cl}(V)$ is any $\mathbb{Z}_2$-graded subspace, the left-ideal $\mathrm{Cl}(V) \cdot S$ becomes a Clifford module.

Submodules, quotient modules. A submodule of a Clifford module E is a $\mathbb{Z}_2$-graded subspace E_1 which is stable under the module action. In this case, the quotient E/E_1 becomes a Clifford module in an obvious way. A Clifford module E is called *irreducible* if there are no submodules other than E and $\{0\}$.

Direct sum. The direct sum of two Clifford modules E_1, E_2 is again a Clifford module, with $\rho_{E_1 \oplus E_2} = \rho_{E_1} \oplus \rho_{E_2}$. A Clifford module is *completely reducible* if it is a direct sum of irreducible Clifford modules.

Tensor products. Suppose V_1, V_2 are vector spaces with bilinear forms B_1, B_2. If E_1 is a $\mathrm{Cl}(V_1)$-module and E_2 is a $\mathrm{Cl}(V_2)$-module, the tensor product $E_1 \otimes E_2$ is a module over $\mathrm{Cl}(V_1) \otimes \mathrm{Cl}(V_2) = \mathrm{Cl}(V_1 \oplus V_2)$, with

$$\rho_{E_1 \otimes E_2}(x_1 \otimes x_2) = \rho_{E_1}(x_1) \otimes \rho_{E_2}(x_2).$$

In particular, $\mathrm{Cl}(V)$-modules E can be tensored with $\mathbb{Z}_2$-graded vector spaces, viewed as modules over the Clifford algebra for the trivial vector space $\{0\}$.

Dual modules. If E is any Clifford module, the dual space E^* becomes a Clifford module, with module structure defined in terms of the canonical anti-automorphisms of $\mathrm{Cl}(V)$ by

$$\rho_{E^*}(x) = \rho_E(x^t)^*, \ x \in \mathrm{Cl}(V).$$

[1]One can also consider ungraded Clifford modules, i.e. (ordinary) vector spaces together with an algebra homomorphism $\mathrm{Cl}(V) \to \mathrm{End}(E)$ of (ordinary) algebras.

That is, $\langle \rho_{E^*}(x)\psi, \beta\rangle = (-1)^{|\psi||x|}\langle \psi, \rho_E(x^t)\beta\rangle$ for $\psi \in E^*$ and $\beta \in E$.

Right modules. Occasionally, it is useful to consider also *right*-modules over the Clifford algebra $\mathrm{Cl}(V)$. That is, $\mathbb{Z}_2$-graded vector spaces E with linear maps $\rho'\colon \mathrm{Cl}(V) \to \mathrm{End}(E)$ such that $\rho'(x_1)\rho'(x_2) = (-1)^{|x_1||x_2|}\rho'(x_2x_1)$. An example is $E = \mathrm{Cl}(V)$, with action given by right multiplication. Any right Clifford module can be turned into an ordinary Clifford module, using the canonical anti-homomorphism of $\mathrm{Cl}(V)$ to define $\rho(x) = \rho'(x^t)$.

Induction. Suppose $\mathcal{A}$ is any $\mathbb{Z}_2$-graded subalgebra of $\mathrm{Cl}(V)$. Then any $\mathcal{A}$-module E_1 gives rise to a Clifford module,

$$E = \mathrm{ind}_{\mathcal{A}}^{\mathrm{Cl}(V)}(E_1) = \mathrm{Cl}(V) \otimes_{\mathcal{A}} E_1.$$

Here the tensor product over $\mathcal{A}$ is the quotient of the usual tensor product $\mathrm{Cl}(V) \otimes E_1$ by the subspace spanned by all $x \otimes a.y - xa \otimes y$, and the $\mathrm{Cl}(V)$-action is inherited from the action by left multiplication.

2. The Spinor module

A particularly important case of the induction procedure arises if the bilinear form B is split, and $F \subset V$ is a Lagrangian subspace. Then $\wedge(F)$ is a subalgebra of $\mathrm{Cl}(V)$, and we obtain a Clifford module by induction from the trivial $\wedge(F)$-module,

$$\mathcal{S} = \mathrm{ind}_{\wedge F}^{\mathrm{Cl}(V)}(\mathbb{C}) = \mathrm{Cl}(V) \otimes_{\wedge F} \mathbb{C}.$$

This is called the *Spinor module* of the Clifford algebra $\mathrm{Cl}(V)$. The Spinor module can be described more explicitly, by choosing a Lagrangian complement F' to F.

LEMMA 2.1. *The map $\wedge(F') \otimes \wedge(F) \to \mathrm{Cl}(V)$, $y' \otimes y \mapsto y'y$ is an isomorphism of vector spaces.*

PROOF. The map preserves filtrations, and the associated graded map is the usual isomorphism

$$\wedge(F') \otimes \wedge(F) \to \wedge(V),\ y' \otimes y \mapsto y' \wedge y.$$

But a morphism of finite-dimensional filtered vector spaces is an isomorphism if and only if the associated graded map is an isomorphism. □

It follows that $\mathrm{Cl}(V) \otimes_{\wedge(F)} \mathbb{C} = \wedge(F')$. Using the bilinear form to identify $F' \cong F^*$, we obtain

$$\mathcal{S} = \mathrm{ind}_{\wedge(F)}^{\mathrm{Cl}(V)} \cong \wedge(F^*)$$

Here the elements of $F^* \cong F' \subset V$ act by exterior multiplication, and the elements of F act by contraction.

These identifications also show that the spinor module $\mathcal{S}$ is independent of the choice of F, up to isomorphism. Indeed, $\mathcal{S}$ can be explicitly described in an adapted basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ with $B(e_i, f^j) = \delta_i^j$ and $B(e_i, e_j) = 0 = B(f^i, f^j)$: Letting $F = \mathrm{span}\{e_1, \ldots, e_k\}$ and $F' = \mathrm{span}\{f^1, \ldots, f^k\}$, we have

$$\mathcal{S} = \mathrm{span}\{f^I\}$$

with Clifford generators acting by

$$f^i \cdot f^I = f^i \wedge f^I, \qquad e_i \cdot f^I = \iota(e_i)(f^I) = \begin{cases} 0 & \text{if } i \notin I \\ \pm f^{I-\{i\}} & \text{if } i \in I \end{cases}$$

THEOREM 2.2. *The spinor module is irreducible, and the module map*

$$\rho\colon\ \mathrm{Cl}(V) \to \mathrm{End}(\mathcal{S})$$

is an isomorphism of $\mathbb{Z}_2$*-graded algebras. Similarly,* $\mathcal{S}^{\bar{0}}$ *and* $\mathcal{S}^{\bar{1}}$ *are irreducible modules over* $\mathrm{Cl}^{\bar{0}}(V)$.

PROOF. The first part is just rephrasing Theorem 1.3 for $\wedge(F^*) = \mathcal{S}$; the second part follows since $\mathrm{End}^{\bar{0}}(\wedge(F^*)) = \mathrm{End}(\wedge^{\bar{0}}F^*) \oplus \mathrm{End}(\wedge^{\bar{1}}F^*)$. □

The spinor module $\mathcal{S} = \mathrm{ind}_{\wedge F}^{\mathrm{Cl}(V)}(\mathbb{K})$ can also be viewed in other ways. Indeed, by definition of the tensor product over $\wedge F$, $\mathcal{S}$ is simply the quotient of $\mathrm{Cl}(V)\otimes\mathbb{K} = \mathrm{Cl}(V)$ by the subspace spanned by elements of the form xf with $f \in \oplus_{j\geq 1} \wedge^j F$. But this subspace is just the left-ideal generated by F. Thus

$$\mathcal{S} = \mathrm{Cl}(V)/\mathrm{Cl}(V)\cdot F.$$

One can also directly view the Spinor module as a submodule of $\mathrm{Cl}(V)$, namely the left-module generated by the determinant line $\det(F) \subset \wedge(F) \subset \mathrm{Cl}(V)$:

$$\mathcal{S} \cong \mathrm{Cl}(V)\cdot \det(F).$$

The isomorphism depends on the choice of a generator of $\det(F)$: More precisely, the map $\mathrm{Cl}(V)\otimes\det(F) \to \mathrm{Cl}(V)\cdot\det(F)$ given by Clifford multiplication descends to give a canonical isomorphism

$$\mathcal{S}\otimes\det(F) = \mathrm{Cl}(V)\otimes_{\wedge F}\det(F) = \mathrm{Cl}(V)\cdot\det(F).$$

The identification $\mathrm{Cl}(V) = \wedge(F^*)\otimes\wedge(F)$ gives $\mathrm{Cl}(V)/\mathrm{Cl}(V)\cdot F = \wedge(F^*)$, and also $\mathrm{Cl}(V)\cdot\det(F) = \wedge(F^*)\otimes\det(F) \cong \wedge(F^*)$.

Let us discuss the dependence of the spinor module on the choice of F. Let us recall that the orthogonal group $\mathrm{O}(V)$ acts transitively on the set $\mathrm{Lag}(V)$ of Lagrangian subspaces of V. We say that $F, F_1 \in \mathrm{Lag}(V)$ have equal parity if they are related by a transformation $g \in \mathrm{SO}(V)$ and *odd* otherwise.

EXAMPLE 2.3. Let $V = \mathbb{C}^2$ with the standard bilinear form, viewed as a vector space over $\mathbb{K} = \mathbb{C}$. Then $F = \mathrm{span}(e_1 + \sqrt{-1}e_2)$ is a Lagrangian subspace, and so is $F' = \mathrm{span}(e_1 - \sqrt{-1}e_2)$. The orthogonal transformation $e_1 \mapsto e_1, e_2 \mapsto -e_2$ takes F to F'. However, it is impossible to take F to F' by a *special* orthogonal transformation.

The relative parity of Lagrangian subspaces may also be viewed as follows. Recall that for a given splitting $V = F'\oplus F$ into transversal Lagrangian subspaces, the subgroup of $\mathrm{O}(V)$ fixing each element of F *pointwise* is identified with the space of skew-adjoint linear maps $d:\ F^* \to F$. That is , it consists of matrices of block form

$$\begin{pmatrix} I & 0 \\ d & I \end{pmatrix}$$

Together with the group $\mathrm{GL}(F)$, sitting inside $\mathrm{O}(V)$ as transformations of the form

$$\begin{pmatrix} (a^{-1})^t & 0 \\ 0 & a \end{pmatrix}$$

this generates the group of orthogonal transformation taking F to itself. Since all these transformations have determinant one, we conclude that the stabilizer subgroup of F in $\mathrm{Lag}(V)$ is contained in $\mathrm{SO}(V)$. Hence, the determinant function $\det\colon\ \mathrm{O}(V) \to \mathbb{Z}_2$ descends to the set of Lagrangian subspaces, $\mathrm{Lag}(V) \to \mathbb{Z}_2$. This gives the distinction between even and odd Lagrangians. This becomes even more explicit in the case $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^{n,n}$, where we had identified $\mathrm{Lag}(V) \cong \mathrm{O}(n)$.

PROPOSITION 2.4. *If F, F_1 are Lagrangian subspace of V of equal parity, the corresponding spinor modules $\mathcal{S} = \mathrm{Cl}(V)/\mathrm{Cl}(V).F$ and $\mathcal{S}_1 = \mathrm{Cl}(V)/\mathrm{Cl}(V).F_1$ are isomorphic as $\mathbb{Z}_2$-graded $\mathrm{Cl}(V)$-modules. If F, F_1 have opposite parity, the spinor modules are related by a parity reversing isomorphism.*

PROOF. Choose $g \in \mathrm{O}(V)$ with $g.F = F_1$, and let $x \in \Gamma(V)$ be an element of the Clifford group with $\pi(x) = g$, where $\pi \colon \Gamma(V) \to \mathrm{O}(V)$ is the quotient map. Then $\Pi(x)Fx^{-1} = F_1$ (under Clifford multiplication). [2] Then

$$\mathrm{Cl}(V)Fx^{-1} = \mathrm{Cl}(V)x^{-1}F_1 = \mathrm{Cl}(V)F_1.$$

Thus, right multiplication by x^{-1} on $\mathrm{Cl}(V)$ descends to an isomorphism of Clifford modules $\mathcal{S} \to \mathcal{S}_1$. Note that it preserves parity if and only if x is even, i.e. if and only if $\pi(x) \in \mathrm{SO}(V)$. □

REMARK 2.5. As mentioned above, we can also consider $\mathrm{Cl}(V)$ as an ungraded algebra, and $\mathcal{S}$ as an ordinary module. If $\mathbb{K} = \mathbb{C}$, we can turn any ungraded module over the Clifford algebra into a graded module, by using the chirality element $\Gamma = i^{n(n-1)/2} e_1 \cdots e_n$ (relative to an orthonormal basis). The $\mathbb{Z}_2$-grading is given simply as the ± 1 eigenspaces of Γ on $\mathcal{S}$. This is compatible with the original grading, since $\Gamma v \Gamma^{-1} = -v$.

Any finite-dimensional module for a matrix algebra $M_n(\mathbb{K}) = \mathrm{End}(\mathbb{K}^n)$ is of the form $\mathcal{E} = E \otimes \mathbb{K}^n$ for some vector space E. (Exercise.) It is irreducible if and only if $\dim E = 1$. Since we have seen that $\mathrm{Cl}(V)$ is the algebra of the spinor module $\mathcal{S}$ defined by a Lagrangian subspace F, it follows that the most general module is isomorphic to a module of the form $\mathcal{E} = E \otimes \mathcal{S}$. For this reason, it is common practice to call any irreducible $\mathrm{Cl}(V)$-module a spinor module.

3. The Spin representation

The Spinor representation of the algebra $\mathrm{Cl}(V)$ restricts to a group representation of the Clifford group $\Gamma(V)$, called the *spin representation* of $\Gamma(V)$. The action of $\Gamma^{\bar{0}}(V)$ preserves the splitting $\mathcal{S} = \mathcal{S}^{\bar{0}} \oplus \mathcal{S}^{\bar{1}}$, one calls $\mathcal{S}^{\bar{0}}$ and $\mathcal{S}^{\bar{1}}$ the *half-spin representations* of $\Gamma^{\bar{0}}(V)$.

THEOREM 3.1. *The representation of $\Gamma(V)$ on $\mathcal{S}$ is irreducible. Similarly, each of the half-spin representations $\mathcal{S}^{\bar{0}}$ and $\mathcal{S}^{\bar{1}}$ is an irreducible representation of $\Gamma^{\bar{0}}(V)$. (If $\mathbb{K} = \mathbb{R}$ we can replace $\Gamma(V)$ with $\mathrm{Pin}(V)$ and $\Gamma^{\bar{0}}(V)$ with $\mathrm{Spin}(V)$.)*

PROOF. If a subspace is invariant under the action of $\Gamma(V)$, the it is also invariant under the subalgebra generated by $\Gamma(V)$. Since the spinor representation of the Clifford algebra is irreducible, it suffices to show that this subalgebra is all of $\mathrm{Cl}(V)$. The subalgebra consist of linear combinations of products

$$x = v_1 \cdots v_k \tag{20}$$

with $v_i \in V$ non-isotropic. But any vector $v \in V$ can be written as a sum of non-isotropic vectors: If v is isotropic, pick any non-isotropic vector $y \in \mathrm{span}(v)^{\perp}$;

[2] We may replace $\Pi(x) = \pm x$ with x, since any element of the Clifford group is either even or odd.

then $v + y$ is non-isotropic, and $v = (v + y) - y$. Hence, the subalgebra consist of linear combinations (20) with no restrictions on the v_i, and therefore equals $\mathrm{Cl}(V)$. Similarly, the subalgebra generated by $\Gamma^{\bar{0}}(V)$ equals $\mathrm{Cl}^{\bar{0}}(V)$. □

REMARK 3.2. Suppose (V, B) is an even-dimensional vector space over $\mathbb{K} = \mathbb{R}$, with positive definite bilinear form B. Let $V^{\mathbb{C}}$ be its complexification. Then $V^{\mathbb{C}}$ admits (complex) Lagrangian subspaces. In fact, there is a 1-1 correspondence between complex structures J on V and Lagrangian subspaces of $V^{\mathbb{C}}$, taking any J to the $+i$-eigenspace of J on $V^{\mathbb{C}}$. The complex conjugate $\bar{F}$ is a Lagrangian complement, equal to the $-i$ eigenspace. The spinor module for $\mathrm{Cl}(V^{\mathbb{C}})$ defines a representation of $\mathrm{Pin}(V)$ and half-spin representations of $\mathrm{Spin}(V) \subset \mathrm{Spin}(V^{\mathbb{C}})$. As before, we see that these representations are irreducible.

4. Pure spinors

Let $\rho\colon \mathrm{Cl}(V) \to \mathrm{End}(\mathcal{S})$ be a spinor module defined by F (or any other irreducible $\mathbb{Z}_2$-graded module). If $w \in \mathcal{S}$ is a spinor, we can consider the space

$$F(w) = \{v \in V |\ \rho(v)w = 0\}.$$

LEMMA 4.1. *For all non-zero spinors $w \in \mathcal{S}$, the space $F(w)$ is an isotropic subspace of V.*

PROOF. If $v_1, v_2 \in F(w)$ we have

$$0 = (\rho(v_1)\rho(v_2) + \rho(v_2)\rho(v_1))w = B(v_1, v_2)w,$$

hence $B(v_1, v_2) = 0$. □

DEFINITION 4.2. A spinor $w \in \mathcal{S}$ is called *pure* if the subspace $F(w)$ is Lagrangian.

In the model $\mathcal{S} = \mathrm{Cl}(V)/\mathrm{Cl}(V).F$ of the spinor module, the non-zero multiples of $w_0 = 1 \mod \mathrm{Cl}(V).F$ and (using the identification $\mathcal{S} \cong \wedge(F^*)$) the non-zero elements of $\det(F^*)$, are examples of pure spinors.

THEOREM 4.3. *If w is a pure Spinor and $x \in \Gamma(V)$, then $\rho(x)w$ is again a pure spinor. The map*

$$\left\{ \begin{array}{c} \textit{pure} \\ \textit{spinors} \end{array} \right\} \to \mathrm{Lag}(V),\ \ w \mapsto F(w)$$

is $\Gamma(V)$-equivariant, with fibers $\mathbb{K}^\times$: i.e. if $F(w) = F(w')$, then w, w' coincide up to a non-zero scalar. All pure spinors w are either even or odd, and their parity coincides with that of the corresponding Lagrangian subspace $F(w)$.

PROOF. For any $x \in \Gamma(V)$,

$$F(x.w) = xF(w)x^{-1} = \pi(x).F(w).$$

It follows that for any pure spinor w, the element $x.w$ is again a pure spinor. Moreover, since $\mathrm{O}(V)$ acts transitively on Lagrangian subspaces, one can always arrange that $F(x.w) = F$, the given Lagrangian subspace F. For the second part, it suffices to show that if $F(w) = F$, then $w \in \mathcal{S}$ is a scalar multiple of w_0. Choose a complementary Lagrangian subspace F' to identify $V = F^* \oplus F$ and $\mathcal{S} = \wedge F^*$, as above. Under this identification, $w_0 = 1$. Since $w \in \wedge F^*$ is annihilated by all $\rho(v) = \iota(v)$ with $v \in F(w) = F$, it lies in $\wedge^0 F^* = \mathbb{K}$ and is hence a multiple of w_0.

The last statement is clear, since any element of the Clifford group is either even or odd, thus $\rho(x)w_0$ is even or odd depending on the parity of x. $\square$

REMARK 4.4. It turns out that the action of $\Gamma(V)$ on the set of pure spinors does not descend to an action of $\mathrm{O}(V)$. In fact, we will see below that the bundle just constructed is a *square root* of the bundle over $\mathrm{Lag}(V)$ with fibers $\det(F_1)^\times$ over F_1.

Suppose $\mathcal{S}$ is an arbitrary (not yet graded) irreducible $\mathrm{Cl}(V)$-module, and $w_0 \in \mathcal{S}$ is a pure spinor. Define $F_0 := F(w_0)$. Then the map $\mathrm{Cl}(V) \to \mathcal{S},\ x \mapsto \rho(x)w_0$ vanishes on the left-ideal $\mathrm{Cl}(V)F_0$, and hence descends to a homomorphism of $\mathrm{Cl}(V)$-modules,

$$\mathrm{Cl}(V)/\mathrm{Cl}(V)F_0 \to \mathcal{S}.$$

This map is onto (otherwise its image would be a $\mathrm{Cl}(V)$-invariant subspace), and 1-1 (otherwise its kernel would be a $\mathrm{Cl}(V)$-invariant subspace). Hence, the choice of a pure spinor gives an identification of $\mathcal{S}$ with the standard model of the spinor module defined by the Lagrangian subspace F_0.

Pure spinors have some rather interesting applications, see for example Chapter 4.9 of Lawson-Michelsohn. For instance, if V is a real vector space with a positive definite symmetric bilinear form, a pure spinor σ for $V^{\mathbb{C}}$ defines a Lagrangian subspace F_σ, hence an orthogonal complex structure J_σ. It follows if an element $x \in \mathrm{Spin}(V)$ preserves σ, then in particular it preserves J_σ. This defines an injective homomorphism

$$\mathrm{Spin}(V)_\sigma \to \mathrm{U}(V; J).$$

But the spinor σ contains more information than just J_σ – it turns out that it also defines a trivialization of the bundle $\det_J(V)$ (top exterior power of V viewed as a complex vector space). In fact, the image of the above map is just $\mathrm{SU}(V; J)$. Thus

$$G_\sigma \cong \mathrm{SU}(m)$$

for any pure spinor on $V^{\mathbb{C}} = \mathbb{C}^{2m}$.

5. The action of the Lie algebra $\mathfrak{o}(F^* \oplus F)$

Let V be a vector space with split bilinear form, and $F \subset V$ a Lagrangian subspace. The spinor module $\mathcal{S}$ over the Clifford algebra $\mathrm{Cl}(V)$ restricts not only to the Spin group but also to the Lie algebra $\mathfrak{o}(V)$, viewed as a Lie subalgebra of $\mathrm{Cl}(V)$. To study the representation in more detail, we pick a transverse Lagrangian subspace to identify

$$V = F^* \oplus F$$

with the bilinear form given by the pairing, $B(\mu_1+v_1, \mu_2+v_2) = \langle \mu_1, v_2\rangle + \langle \mu_2, v_1\rangle$. Equivalently, writing elements of $F^* \oplus F$ as column vectors,

$$B(\mu_1 \oplus v_1, \mu_2 \oplus v_2) = \left\langle \begin{pmatrix} \mu_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id}_F \\ \mathrm{Id}_{F^*} & 0 \end{pmatrix} \begin{pmatrix} \mu_2 \\ v_2 \end{pmatrix} \right\rangle$$

Any endomorphism of $F^* \oplus F$ may be written in block form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{21}$$

where $\alpha\colon F^* \to F^*$, $\beta\colon F \to F^*$, $\gamma\colon F^* \to F$, $\delta\colon F \to F$. Such a matrix lies in $\mathfrak{o}(F^* \oplus F)$ if and only if the linear map $F \oplus F^* \to F^* \oplus F$ given by

$$\begin{pmatrix} 0 & \mathrm{Id}_F \\ \mathrm{Id}_{F^*} & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}$$

is skew-adjoint. This gives the conditions, $\alpha^* = -\delta$, $\beta^* = -\beta$, $\gamma^* = -\gamma$. We conclude that the elements of $\mathfrak{o}(F^* \oplus F)$ are matrices in block form,

$$\begin{pmatrix} -A^* & E_2 \\ E_1 & A \end{pmatrix}$$

where both $E_1\colon F^* \to F$ and $E_2\colon F \to F^*$ are skew-adjoint, and $A \in \mathfrak{gl}(F)$ is arbitrary. Identify the space of skew-adjoint linear maps $F^* \to F$ with $\wedge^2 F$, by the map taking E_1 to $\lambda(E_1)$ with $\iota_\mu \lambda(E_1) = -E_1(\mu)$. Similarly, let $\lambda(E_2) \in \wedge^2 F^*$ denote the element corresponding to the skew-adjoint linear map E_2. We have shown:

Proposition 5.1. *The Lie algebra $\mathfrak{o}(F^* \oplus F)$ is a direct sum*

$$\mathfrak{o}(F^* \oplus F) \cong \wedge^2(F^*) \oplus \mathfrak{gl}(F) \oplus \wedge^2(F).$$

Here each summand is a Lie subalgebra, acting on $F^* \oplus F$ as follows:

1. $\phi \in \wedge^2(F^*)$ *acts by*

$$\mu \oplus v \mapsto -\iota_v \phi \oplus 0.$$

2. $\psi \in \wedge^2(F)$ *acts by*

$$\mu \oplus v \mapsto 0 \oplus -\iota_\mu \psi.$$

3. $A \in \mathfrak{gl}(F)$ *acts by*

$$\mu \oplus v \mapsto (-A^* \mu) \oplus Av.$$

Under the map $\gamma\colon \mathfrak{o}(F^* \oplus F) \to \mathrm{Cl}(F^* \oplus F)$, the summands $\wedge^2 F$ and $\wedge^2 F^*$ just go to the corresponding subspaces of $\wedge(F), \wedge(F^*) \subset \mathrm{Cl}(F^* \oplus F)$. In particular, their action in the spinor representation is contraction and exterior multiplication, respectively. Let us describe the action of $A \in \mathfrak{gl}(F) \subset \mathfrak{o}(F \oplus F^*)$. The corresponding element $\lambda(A) \in \wedge^2(F^* \oplus F)$ is given by

$$\lambda(A) = -\sum_i f^i \wedge A(e_i).$$

This quantizes to

$$\begin{aligned} \gamma(A) = q(\lambda(A)) &= -\tfrac{1}{2} \sum_i (f^i A(e_i) - A(e_i) f^i) \\ &= -\sum_i f^i A(e_i) + \tfrac{1}{2}\,\mathrm{tr}(A). \end{aligned}$$

Letting D_{A^*} denote the derivation of $\wedge(F^*)$, given on F^* by A^*, it follows that the action of A on the spinor module is

$$\rho(A) = -D_{A^*} + \tfrac{1}{2}\,\mathrm{tr}(A).$$

5.1. The group $\mathrm{O}(F^* \oplus F)$. Let us now turn to the orthogonal group. An endomorphism of $F^* \oplus F$, written in block form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{22}$$

defines an orthogonal transformation of $F^* \oplus F$ if and only if

$$\begin{aligned}\begin{pmatrix} 0 & \mathrm{Id}_F \\ \mathrm{Id}_{F^*} & 0 \end{pmatrix} &= \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} 0 & \mathrm{Id}_F \\ \mathrm{Id}_{F^*} & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^*c + c^*a & a^*d + c^*b \\ d^*a + b^*c & b^*d + d^*b \end{pmatrix}\end{aligned}$$

This gives the conditions,

$$a^*c + c^*a = 0, \quad b^*d + d^*b = 0, \quad a^*d + c^*b = I.$$

From this may identify the following three subgroups, which are of course just the exponentials of the three Lie subalgebras of $\mathfrak{o}(F^* \oplus F)$ described above:

1. $\wedge^2(F^*)$, where $\phi \in \wedge^2(F^*)$ acts by $\mu \oplus v \mapsto (\mu + \iota_v\phi) \oplus v$. This subgroup is embedded as matrices in block form
$$\begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix}$$
where $E_2\colon F \to F^*$ is a skew-adjoint linear map.
2. $\wedge^2(F)$, where $\psi \in \wedge^2(F)$ acts by $\mu \oplus v \mapsto \mu \oplus (v + \iota_\mu\psi)$ This subgroup is embedded as matrices in block form
$$\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix}$$
where $E_1\colon F^* \to F$ is a skew-adjoint linear map.
3. $\mathrm{GL}(F)$, where $R \in \mathrm{GL}(F)$ acts by $\mu \oplus v \mapsto (R^{-1})^*\mu \oplus Rv$. This corresponds to block diagonal matrices
$$\begin{pmatrix} (R^{-1})^* & 0 \\ 0 & R \end{pmatrix}$$

PROPOSITION 5.2 (Factorization formulas).

(1) *The map*
$$\wedge^2(F) \times \mathrm{GL}(F) \times \wedge^2(F^*) \to \mathrm{O}(F^* \oplus F), \ (g_1, g_2, g_3) \mapsto g_1g_2g_3$$
is 1-1, with image the set of all orthogonal transformations for which the block $a\colon F^* \to F^*$ is invertible.

(2) *The map*
$$\wedge^2(F^*) \times \mathrm{GL}(F) \times \wedge^2(F) \to \mathrm{O}(F^* \oplus F), \ (g_1, g_2, g_3) \mapsto g_1g_2g_3$$
is 1-1, with image the set of all orthogonal transformations for which the block $d\colon F \to F$ is invertible.

PROOF. For (1) we calculate

$$\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix}\begin{pmatrix} (R^{-1})^* & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} (R^{-1})^* & (R^{-1})^*E_2 \\ E_1(R^{-1})^* & R + E_1(R^{-1})^*E_2 \end{pmatrix}.$$

If a is invertible, we can solve for E_1, E_2, R in terms of the blocks a, b, c:

$$R = (a^{-1})^*, \; E_1 = ca^{-1}, \; E_2 = a^{-1}b.$$

(Note that $d = (a^{-1})^*(I - c^*b)$ if a is invertible.) The proof of (2) is similar. □

Note that each of the three factors is contained in $\mathrm{SO}(F^* \oplus F)$. They gives rise to three subgroups of the double cover $\mathrm{Spin}(F^* \oplus F)$. We are interested in the action of these factors in the Spin representation on $\wedge(F^*)$.

The inclusion of $\wedge^2(F^*)$ into $\mathrm{SO}(F^* \oplus F)$ lifts to an inclusion into $\mathrm{Spin}(F^* \oplus F) \subset \mathrm{Cl}(F^* \oplus F)$ as elements of the form $e^\psi \in \wedge(F^*) \subset \mathrm{Cl}(F^* \oplus F)$ with $\psi \in \wedge^2(F^*)$. Similarly, $\wedge^2(F) \hookrightarrow \mathrm{SO}(F^* \oplus F)$ lifts to the Spin group as exponentials. since they are just vector spaces.

Let $\tilde{E}_1 = e^{\lambda(E_1)}, \tilde{E}_2 = e^{\lambda(E_2)} \in \mathrm{Spin}(F^* \oplus F)$ be these natural lifts of the elements $\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix}$ and $\begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix}$. Also, let $\tilde{GL}(F)$ denote the pre-image of $\mathrm{GL}(F)$ in $\mathrm{Spin}(F^* \oplus F)$. Let $\tilde{R} \in \tilde{GL}(F) \subset \mathrm{Spin}(F^* \oplus F)$ denote lifts of $R \in \mathrm{GL}(F)$.

THEOREM 5.3. *Let $\mathbb{K} = \mathbb{C}$. Under the spin representation on $\wedge(F^*)$,*

$$\rho(\tilde{E}_1)\phi = \iota(e^{\lambda(E_1)})\phi, \qquad \rho(\tilde{E}_2)\phi = e^{\lambda(E_2)} \wedge \phi$$

and

$$\rho(\tilde{R})\phi = \sqrt{\det R}\; (R^{-1})^*(\phi).$$

Here the sign of the square root $\sqrt{\det R}$ depends on the choice of lift, and we use the same notation for $(R^{-1})^* \in \mathrm{End}(F^*)$ and its extension to an automorphism of the exterior algebra.

PROOF. The action of the $\wedge^2(F)$ and $\wedge^2(F^*)$ factors is clear, since the subalgebras $\wedge(F^*), \wedge(F)$ act by exterior multiplication and contractions, respectively. The formula for $\rho(\tilde{R})$ follows by exponentiating the corresponding formula for the Lie algebra, using that

$$\exp(\tfrac{1}{2}\,\mathrm{tr}(A)) = \sqrt{\det \exp(A)}.$$

(Note that $\mathrm{GL}(F)$ is connected since $\mathbb{K} = \mathbb{C}$, and that $\tilde{GL}(F)$ is connected as well.) □

REMARK 5.4. We get the following simple characterization for the group $\tilde{GL}(F)$,

$$\tilde{GL}(F) = \{(R, z) \in \mathrm{GL}(F) \times \mathbb{C}^\times \mid \; \det(A) = z^2\}.$$

To gain some confidence in this result, let us first work out an example.

EXAMPLE 5.5. Let $V = \mathbb{C}^2$ with isotropic basis e, f such that $B(e, f) = 1$. We take $F = \mathrm{span}\{e\}$, with complementary subspace $F' = \mathrm{span}\{f\}$ (identified with F^*). Thus, the spinor module $\mathcal{S} = \wedge F^*$ has basis $\{1, f\}$. Let $R \in \mathrm{GL}(F) = \mathbb{C}^\times$ be given by $u \in \mathbb{C}^\times$. A lift of the corresponding matrix

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$$

to the Spin group is given by,

$$x = u^{1/2} - (u^{1/2} - u^{-1/2})fe$$

where $u^{1/2} \in \mathbb{C}^\times$ is one of the two possible choices of square root. To verify this, note that $efef = ef$, which gives $x^t x = 1$ and $x^{-1} = x^t = u^{-1/2} + (u^{1/2} - u^{-1/2})fe$. Furthermore,

$$\begin{aligned} xex^{-1} &= u^{1/2} e(u^{-1/2} + (u^{1/2} - u^{-1/2})fe) = ue, \\ xfx^{-1} &= (u^{1/2} - (u^{1/2} - u^{-1/2})fe)fu^{-1/2} = u^{-1} f. \end{aligned}$$

The action of x in the spin representation is given by

$$\rho(x)1 = u^{1/2}, \quad \rho(x)f = u^{-1/2} f$$

which is consistent with the formula given above.

For the case $\mathbb{K} = \mathbb{R}$, we can use the result for $\mathbb{K} = \mathbb{C}$, viewing $\mathrm{GL}(F)$ as a subgroup of $\mathrm{GL}(F^{\mathbb{C}})$. Over the subgroup $\mathrm{GL}^+(F)$, of automorphisms of determinant > 0, the resulting formula is the same as for the complex case. However for $\det(R) < 0$, the formula does not make sense since in that case $\det(R)$ has no square root in $\mathbb{R}$.

Remark 5.6. Recall on the other hand that our definition of the Spin group was slightly different in the real case: we used the weaker condition $x^t x = \pm 1$ rather than $x^t x = 1$. If we once again define $\tilde{\mathrm{GL}}(F)$ as the preimage of $\mathrm{GL}(F)$ in the spin group, we obtain a similar formula as before, replacing $\det^{1/2}(R)$ with $|\det|^{1/2}(R)$.

We may also switch the roles of F and F^*, and consider the Spin representation of $\mathrm{Spin}(F \oplus F^*)$ on $\wedge(F)$. Here the subgroups $\wedge^2 F^*$ and $\wedge^2 F$ act by exterior multiplication and contraction of the exponentials of these elements, and the formula for the action of $R \in \mathrm{GL}(F)$ becomes, for $\mathbb{K} = \mathbb{C}$,

$$\rho(\tilde{E}_1)\psi = e^{\lambda(E_1)} \wedge \psi, \quad \rho(\tilde{E}_2)\psi = \iota(e^{\lambda(E_2)})\,\psi, \quad \rho(\tilde{R})\psi = \frac{R.\psi}{\sqrt{\det R}}$$

and similarly for $\mathbb{K} = \mathbb{R}$ and $R \in \mathrm{GL}^+(F)$.

6. The quantization map revisited

Until now, we discussed the spinor representation only for vector spaces with split bilinear form B. However, the spinor representation may be used to study the Clifford algebras $\mathrm{Cl}(V)$ for *arbitrary* quadratic vector spaces (V, B). This is based on the following simple observation.

Suppose V is a vector space with a symmetric bilinear form, B (possibly degenerate). Then the map

$$j : V \mapsto V \oplus V^*, \ v \mapsto v \oplus B^\flat(v/2)$$

(where $V \oplus V^*$ carries the bilinear form coming from the pairing) preserves B:

$$\begin{aligned} B_{V\oplus V^*}(j(v_1), j(v_2)) &= \tfrac{1}{2}\langle B^\flat(v_1), v_2\rangle + \tfrac{1}{2}\langle B^\flat(v_2), v_1\rangle \\ &= B(v_1, v_2). \end{aligned}$$

Hence, it extends to a 1-1 homomorphism of Clifford algebras,

$$j \colon \mathrm{Cl}(V) \to \mathrm{Cl}(V \oplus V^*).$$

PROPOSITION 6.1. *The composition*

$$\mathrm{Cl}(V) \xrightarrow{j} \mathrm{Cl}(V \oplus V^*) \cong \mathrm{End}(\wedge(V))$$

is equal to the standard representation of $\mathrm{Cl}(V)$ *on* $\wedge(V)$. *In particular, the symbol map can be written in terms of the spinor representation as,*

$$\sigma(x) = j(x).1$$

PROOF. The elements $j(v) = v \oplus B^\flat(v/2)$ act as $\epsilon(v) + \frac{1}{2}\iota(B^\flat(v))$, as required. $\square$

We will use this fact, together with our results on the spin representation of $\mathrm{Spin}(F^* \oplus F)$ (here $F = V^*$) to prove explicit formulas for the elements $\exp(\gamma(A)) \in \mathrm{Spin}(V)$ of the spin group, generalizing our formula for the special case $V = \mathbb{R}^2$,

$$\exp(\theta e_1 e_2) = \cos(\theta/2) + \sin(\theta/2) 2 e_1 e_2.$$

We will assume that the bilinear form B on V is non-degenerate, and use B to identify V^* and V. Write $W = V \oplus V^*$, with bilinear form B_W coming from the pairing. Let V^- denote the vector space V with bilinear form $-B$. Then the map

$$\kappa \colon V \oplus V^- \to W,\ v_1 \oplus v_2 \mapsto (v_1 + v_2) \oplus \tfrac{1}{2}(v_1 - v_2)$$

is an isomorphism of quadratic vector spaces, with inverse $\kappa^{-1}(y_1 \oplus y_2) = (\frac{1}{2}y_1 + y_2) \oplus (\frac{1}{2}y_1 - y_2)$. Indeed, if $w = (v_1 + v_2) \oplus \frac{1}{2}(v_1 - v_2)$ then

$$\begin{aligned} B_W(w, w) &= 2B(v_1 + v_2, \tfrac{1}{2}(v_1 - v_2)) \\ &= B(v_1, v_1) - B(v_2, v_2). \end{aligned}$$

Written in matrix form,

$$\kappa = \begin{pmatrix} I & I \\ I/2 & -I/2 \end{pmatrix}, \qquad \kappa^{-1} = \begin{pmatrix} I/2 & I \\ I/2 & -I \end{pmatrix}$$

PROPOSITION 6.2. *For any* $C \in \mathrm{SO}(V)$ *such that* $C + I$ *is invertible,*

$$\kappa \circ \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \circ \kappa^{-1} = \begin{pmatrix} I & 2E \\ 0 & I \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & (R^{-1})^t \end{pmatrix} \begin{pmatrix} I & 0 \\ E/2 & I \end{pmatrix}$$

where $R = \frac{2}{I + C^{-1}}$, $E = \frac{C - I}{C + I}$.

PROOF. A direct calculation shows that both sides are equal to

$$\begin{pmatrix} (C+I)/2 & C - I \\ (C-I)/4 & (C+I)/2 \end{pmatrix} \tag{23}$$

$\square$

From the known action of the factors in the spinor representation, we may therefore deduce:

COROLLARY 6.3. *Let* $\tilde{C} \in \mathrm{Spin}(V)$ *be a lift of* $C \in \mathrm{SO}(V)$. *Then the action of* $\tilde{C}$ *on* $\psi \in \wedge V$ *is given by the formula,*

$$\tilde{C}.\psi = \frac{1}{\sqrt{\det R}}\ e^{2\lambda(E)}\ R.\iota(e^{\lambda(E/2)})\psi$$

where $R = \frac{2}{I+C^{-1}}$, $E = \frac{C-I}{C+I}$, *and where the sign in* $\sqrt{\det R}$ *depends on the choice of lift. In particular, taking* $\psi = 1$ *we find*

$$\sigma(\tilde{C}) = 2^{-\dim V/2}\sqrt{\det(I+C)}\ \exp\big(2\frac{C-I}{C+I}\big).$$

COROLLARY 6.4. *The lift of each of the two functions*

$$\mathrm{SO}(V) \to \mathbb{C},\ C \mapsto \det(C \pm I)$$

to $\mathrm{Spin}(V)$ *has a well-defined holomorphic square root if lifted to* $\mathrm{Spin}(V)$. *Here the choice of square root of* $\det(C-I)$ *depends on the choice of an orientation of* V.

PROOF. We use the symbol of $\tilde{C}$ to define these square roots: the form degree 0 part of $\sigma(\tilde{C})$ is $2^{-\dim V/2}\sqrt{\det(C+I)}$, while[3] the form degree $\dim V$ part is $2^{-\dim V/2}\sqrt{\det(C-I)}$. $\square$

Let us specialize to the case $C = \exp(A)$, so that

$$E = \tanh(A/2), \quad R = 2(I - e^{-A})^{-1}.$$

There is a distinguished lift $\tilde{C} = \exp(\gamma(A)) \in \mathrm{Spin}(V)$. The formula for the above choice $D = 2\tanh(A/2)$ now reads:

PROPOSITION 6.5. *The function* $\mathfrak{o}(V) \to \mathbb{C}$, $A \mapsto \det(\cosh(A/2))$ *admits a global square root, equal to* 1 *at* $A = 0$ *and holomorphic everywhere on* $\mathfrak{o}(V)$. *With this choice of square root, the differential form* $\det^{1/2}\big(\cosh(A/2)\big)e^{2\lambda(\tanh(A/2))}$ *is holomorphic everywhere, and*

$$\sigma(e^{\gamma(A)}) = \sqrt{\det\cosh(A/2)}e^{2\lambda(\tanh(A/2))}.$$

for a suitable choice of square root (defined by this equation).

EXAMPLE 6.6. Let us check this formula for $V = \mathbb{R}^2$, $\lambda(A) = \theta e_1 \wedge e_2$. Here $A = \theta J$ where J is the standard complex structure on $\mathbb{R}^2$. It follows that $\exp(A) = \cos(\theta) + \sin(\theta)J$, and therefore $\cosh(A/2) = \cos(\theta/2)$, $\sinh(A/2) = \sin(\theta/2)J$ and $\tanh(A/2) = \tan(\theta/2)J$. This yields $\lambda(\tanh(A/2)) = \tan(\theta/2)e_1 \wedge e_2$ and

$$\det{}^{1/2}(\cosh(A/2)) = \cos(\theta/2), \quad e^{2\lambda(\tanh(A/2))} = 1 + 2\tan(\theta/2)e_1 \wedge e_2.$$

Hence, finally

$$\det{}^{1/2}(\cosh(A/2))e^{2\lambda(\tanh(A/2))} = \cos(\theta/2) + 2\sin(\theta/2)e_1 \wedge e_2.$$

Other types of factorizations of the matrix (23) defined by C lead to different formulas for symbols. We will use the following factorization, in which block-diagonal matrices are all the way to the right:

PROPOSITION 6.7. *Suppose that* $C \in \mathrm{O}(V)$ *has no eigenvalue equal to* 1. *For all* $D \in \mathfrak{o}(V)$ *such that* D *is invertible and commutes with* C, *the following factorization formula holds true:*

$$\kappa \circ \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \circ \kappa^{-1}$$
$$= \begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix}\begin{pmatrix} I & D \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ E_2 & I \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & (R^{-1})^t \end{pmatrix}.$$

[3]We recall that for $A \in \mathfrak{o}(V)$, the top degree part of $\exp(\lambda(A)) \in \wedge(V)$ is $\sqrt{\det A}\,\Gamma_\wedge$, where $\Gamma_\wedge$ is a generator of $\det(V)$ with $g(\Gamma_\wedge, \Gamma_\wedge) = 1$. This is the well-known *Pfaffian*.

Here

$$E_1 = \frac{1}{2}\frac{C+I}{C-I} - \frac{1}{D}, \quad E_2 = \frac{1}{D^2}\left(\frac{C-C^{-1}}{2} - D\right), \quad R = \frac{D}{I-C^{-1}}. \tag{24}$$

PROOF. The matrix product on the left hand side of the desired equality is given by (23), while the right hand side is, by direct computation,

$$\text{r.h.s.} \;=\begin{pmatrix} (I+DE_2)R & D(R^{-1})^t \\ (E_1+E_1DE_2+E_2)R & (I+DE_1)(R^{-1})^t \end{pmatrix}$$

The two expressions coincide if and only if E_1, E_2, R are given by (24). For instance, a comparison of the upper right corners gives

$$R = ((D^{-1}(C-I))^{-1})^t = \frac{D^t}{(C-I)^t} = \frac{D}{I-C^{-1}}.$$

Similarly, one finds E_1, E_2 by comparing the upper left and lower right corners. Finally, a direct computation shows that with these choice of E_1, E_2, R, the lower left corners match as well. □

From the known action of the factors in the spinor representation, we may therefore deduce:

COROLLARY 6.8. *Let $\tilde{C} \in \mathrm{Spin}(V)$ be a lift of $C \in \mathrm{SO}(V)$. For any choice of D as above, the action of $\tilde{C}$ on $\psi \in \wedge V$ is given by the formula,*

$$\tilde{C}.\psi = \frac{1}{\sqrt{\det R}}\ \iota\big(e^{\lambda(E_1)}\big)\ e^{\lambda(D)}\ \iota\big(e^{\lambda(E_2)}\big)R.\psi$$

where the sign in $\sqrt{\det R}$ depends on the choice of lift. In particular, taking $\psi = 1$ we find

$$\sigma(\tilde{C}) = \frac{1}{\sqrt{\det R}}\ \iota\big(e^{\lambda(E_1)}\big)\ e^{\lambda(D)}.$$

Different choice of D give different formulas. One very natural choice is $D = 2\frac{C-I}{C+I}$ since then $E_1 = 0$, but this will just recover our first formula for $\sigma(\tilde{C})$. Instead, specialize to $C = \exp A$, and take $D = A$. Then

$$E_1 = f(A), \quad E_2 = g(A), \quad R = j^L(A)$$

where we have introduced the following functions of $z \in \mathbb{C}$,

$$f(z) = \frac{1}{2}\coth(\frac{z}{2}) - \frac{1}{z}, \quad g(z) = \frac{\sinh(z) - z}{z^2},$$

$$j^L(z) = \frac{1-e^{-z}}{z}, \quad j(z) = \frac{\sinh(z/2)}{z/2}, \quad j^R(z) = \frac{e^z - 1}{z}.$$

Note that g, j^L, j^R, j are holomorphic, while f is meromorphic with poles at $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. Observe that $f(A), g(A)$ are again in $\mathfrak{o}(V)$, while $j(A)^t = j(A)$ and $j^L(A)^t = j^R(A)$. Furthermore, $j^L(A), j^R(A), j(A)$ are invertible if and only if A has no eigenvalues of the form $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. The resulting formula for the symbol gives:

THEOREM 6.9. *For all $A \in \mathfrak{o}(V)$ with the property that A has no eigenvalue $2\pi\sqrt{-1}k$ with $k \neq 0$, the symbol of $\exp(\gamma(A)) \in \mathrm{Cl}(V)$ is given by the formula,*

$$\sigma(\exp(\gamma(A)) = \iota(\mathcal{S}(A)) \exp(\lambda(A)).$$

where $\mathcal{S}\colon \mathfrak{o}(V) \to \wedge(V)$ *is the map*

$$\mathcal{S}(A) = \sqrt{\det(j(A))}\ \exp\big(\lambda(f(A))\big).$$

Once again, while Proposition 6.7 requires that A is invertible (which never happens if $\dim V$ is odd), the resulting formula holds without this assumption. In case A is invertible we can directly write

$$\mathcal{S}(A) = \det{}^{1/2}\big(\tfrac{\sinh(A/2)}{A/2}\big)\ e^{\lambda\left(\frac{1}{2}\coth(\frac{A}{2})-\frac{1}{A}\right)}.$$

The significance of the formula

$$e^{q(\lambda(A))} = q\Big(\iota(\mathcal{S}(A))e^{\lambda(A)}\Big)$$

is that it compares the exponentials of elements of $\wedge^2(V)$ under Clifford and exterior algebra multiplication. The formula has a useful generalization, allowing linear terms. Let P be a vector space of "parameters". Let e_i be a basis of V, and consider expressions $e_i \otimes \tau^i \in V \otimes P$ with $\tau^i \in P$. We can then compare the exponentials of elements

$$\lambda(A) - \sum_i e_i\tau^i \in \wedge(V)\otimes\wedge(P)$$

and

$$\gamma(A) - \sum_i e_i\tau^i \in \mathrm{Cl}(V)\otimes\wedge(P)$$

Note that we can view that parameters τ^i as the components $\tau(e^i)$ of a linear map $V \to P$.

THEOREM 6.10. *With* $\mathcal{S}(A)$ *as above,*

$$e^{\gamma(A)-\sum_i e_i\tau^i} = q\Big(\iota(\mathcal{S}(A))e^{\lambda(A)-\sum_i e_i\tau^i}\Big).$$

It is somewhat remarkable that we do not have to introduce a τ-dependence into $\mathcal{S}$.

PROOF. Identify $\wedge(V)\otimes\wedge(P) = \wedge(V\oplus P)$, and think of $\mathrm{Cl}(V)\otimes\wedge(P) = \mathrm{Cl}(V\oplus P)$ as the Clifford algebra for the degenerate bilinear form $B\oplus 0$. Pick an arbitrary non-degenerate symmetric bilinear form B_P on P, and consider the bilinear form $B\oplus\epsilon B_P$ on $V\oplus P$. Then $\lambda(A)-\sum_i e_i\tau^i = \lambda(\tilde{A}_\epsilon)$ with

$$\tilde{A}_\epsilon = \begin{pmatrix} A & 0 \\ \tau & 0 \end{pmatrix} + O(\epsilon).$$

where $\tau^t\colon P \to V$ denotes the transpose map relative to the bilinear forms B, B_P. All powers of $\tilde{A}_s$ have a similar scaling behavior with respect to ϵ: That is,

$$\tilde{A}_\epsilon^m = \begin{pmatrix} A^m & 0 \\ \tau A^{m-1} & 0 \end{pmatrix} + O(\epsilon).$$

Hence

$$f(\tilde{A}_\epsilon) = \begin{pmatrix} f(A) & 0 \\ Q & 0 \end{pmatrix} + O(\epsilon),$$

with $Q = \tau f(A)A^{-1}$ independent of ϵ. Now $f(\tilde{A}_\epsilon)$ is an endomorphism of $V\oplus P$. In order to take the limit in the resulting formula, we have to compose with $(B_V \oplus \epsilon B_P)^\flat$, to produce a skew-adjoint linear map $V\oplus P \to V^*\oplus P^*$, or equivalently an element $\lambda(\tilde{A}_\epsilon) \in \wedge^2(V^* \oplus P^*$. . But this introduces another factor of ϵ in front of Q, i.e.

$$\lambda(f(\tilde{A}_\epsilon)) = \lambda(f(A)) + O(\epsilon)$$

as an element of $\wedge^2(V^* \oplus P^*)$. Similarly,

$$\det(j(\tilde{A}_\epsilon)) = \det(j(\mathcal{S}(A))),$$

since only the block diagonal term contributes. The Theorem now follows by letting $\epsilon \to 0$ in our general formula,

$$\exp(\gamma(\tilde{A}_\epsilon)) = \iota(\mathcal{S}(\tilde{A}_\epsilon)) \exp(\lambda(\tilde{A}_\epsilon)).$$

□

LEMMA 6.11. *Let $\chi, \psi \in \wedge(E)$, and suppose that the top degree part $\chi_{[\dim E]} \in \det(E)$ is non-zero. Then there is a unique solution $\phi \in \wedge(E^*)$ of the equation*

$$\psi = \iota(\phi)\chi.$$

If χ, ψ depend continuously (smoothly, holomorphically) on parameters, then so does the solution ϕ.

PROOF. Fix a generator $\Gamma \in \det(E^*)$. Then the desired equation $\psi = \iota(\phi)\chi$ is equivalent to

$$\iota(\psi)\Gamma = \phi \wedge \iota(\chi)\Gamma.$$

Since $\chi_{[\dim E]} \neq 0$, we have $(\iota(\chi)\Gamma)_{[0]} \neq 0$, i.e. $\iota(\chi)\Gamma$ is invertible. Thus

$$\phi = (\iota(\psi)\Gamma) \wedge (\iota(\chi)\Gamma)^{-1}$$

This shows existence and uniqueness, and also implies the statements regarding dependence on parameters. □

THEOREM 6.12. *The function $A \mapsto \mathcal{S}(A)$ extends to a global holomorphic function $\mathfrak{o}(V) \to \wedge(V)$. In particular, its degree zero part*

$$A \mapsto \det{}^{1/2}(j(A)) = \det{}^{1/2}\left(\tfrac{\sinh(A/2)}{A/2}\right)$$

is a well-defined holomorphic function $\mathfrak{o}(V) \to \mathbb{C}$.

PROOF. Let $P = V^*$ in Theorem 6.10, and $\tau^i = f^i$ a dual basis to e_i. Then $\exp(\lambda(A) - e_i f^i)$ has a non-vanishing part of top degree $2 \dim V$. By the Lemma, there exists a holomorphic function $\mathcal{S}'(A)$ satisfying

$$\iota(\mathcal{S}'(A))\Gamma(A) = q^{-1}(\exp(\gamma(A) - \sum_i e_i f^i)).$$

By uniqueness, this function coincides with the function $\tilde{S}(A)$ defined above. □

CHAPTER 4

Lie groups

1. Preliminaries

We review some basic material on Lie groups, mainly to refresh our memory, and to fix our notational conventions.

A (real) *Lie group* is a group G, equipped with a (real) manifold structure such that the group operations of multiplication and inversion are smooth. For example, $\mathrm{GL}(N,\mathbb{R})$, with manifold structure as an open subset of $\mathrm{Mat}_N(\mathbb{R})$, is a obviously Lie group. According to theorem of E. Cartan, any (topologically) closed subgroup H of a Lie group G is a Lie subgroup: I.e. the smoothness is automatic. Hence, it is immediate that e.g. that $\mathrm{SO}(n)$, $\mathrm{GL}(N,\mathbb{C})$, $\mathrm{U}(n)$ etc. are again Lie groups. A related result is that if G_1, G_2 are Lie groups, then any continuous group homomorphism $G_1 \to G_2$ is smooth. Consequently, a given topological group cannot carry more than one smooth structure making it into a Lie group.

An *action of a Lie group on a manifold* M is a group homomorphism $\mathcal{A}\colon G \to \mathrm{Diff}(M)$ into the group of diffeomorphisms of M, with the property that the action map $G\times M \to M,\ (g,x)\mapsto \mathcal{A}(g)(x)$ is smooth. It induces actions on the tangent bundle and cotangent bundle, and hence there are notions of invariant vector fields $\mathfrak{X}(M)^G$, invariant differential forms $\Omega(M)^G$ and so on.

There are three important actions of a Lie group on itself: The actions by left-and right-multiplication,

$$\mathcal{A}^L(g)(a) = ga,\ \mathcal{A}^R(g)(a) = ag^{-1}$$

and the adjoint action,

$$\mathrm{Ad}(g)(a) = gag^{-1}.$$

Let $\mathfrak{X}^L(G) \subset \mathfrak{X}(G)$ denote the Lie algebra of left-invariant vector fields. Any element of $\mathfrak{X}^L(G)$ is determined by its value at the group unit $e \in G$. This gives a vector space isomorphism $T_eG \to \mathfrak{X}^L(G),\ \xi \mapsto \xi^L$. One calls

$$\mathfrak{g} = T_eG \cong \mathfrak{X}^L(G),$$

with Lie bracket induced from that on $\mathfrak{X}^L(G)$, the Lie algebra of G. Lie's third theorem asserts that any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ arises in this way from a Lie group G, and in fact there is a unique connected, simply connected Lie group having $\mathfrak{g}$ as its Lie algebra.

If $G = \mathrm{GL}(N,\mathbb{R})$, the tangent space $\mathfrak{g} = T_eG$ is canonically identified with the space $\mathrm{Mat}_N(\mathbb{R})$ of $N\times N$-matrices, and one may verify that the Lie bracket is simply the commutator of matrices. (This is the main reason for working with $\mathfrak{X}^L(G)$ rather than $\mathfrak{X}^R(G)$, since the latter choice would have produced *minus* the commutator.)

The adjoint action of G on itself defines a homomorphism,

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(G) \subset \mathrm{Diff}(G).$$

where $\mathrm{Aut}(G)$ are the Lie group automorphisms of G. The adjoint action fixes e, and hence induces a linear action on $T_eG = \mathfrak{g}$ by Lie algebra automorphisms, which again is referred to as the adjoint action,

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}).$$

One also defines an infinitesimal adjoint action,

$$\mathrm{ad}_\mu\colon \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}_\mu(\xi) = [\mu, \xi]_{\mathfrak{g}}.$$

ad defines a Lie algebra homomorphism

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}),$$

where a linear map $A \in \mathrm{End}(\mathfrak{g})$ is a derivation of the Lie bracket if and only if $A[\xi_1, \xi_2] = [A\xi_1, \xi_2] + [\xi_1, A\xi_2]$. This referred to as the adjoint representation of $\mathfrak{g}$. The adjoint representation ad of $\mathfrak{g}$ is the infinitesimal version of the adjoint representation Ad of G on $\mathfrak{g}$, in the following sense: Let $g(t) \in G$ be a smooth curve with $g(0) = e$, and let $\mu = \frac{\partial}{\partial t}\big|_{t=0} g(t) \in \mathfrak{g}$ be the tangent vector represented by that curve. Then

$$\frac{\partial}{\partial t}|_{t=0} \mathrm{Ad}_{g(t)}\, \xi = \mathrm{ad}_\mu(\xi).$$

For matrix Lie algebras, the adjoint actions Ad are just conjugation, while ad is just a commutator.

2. The exponential map

Any $\xi \in \mathfrak{g} = T_eG$ determines a unique 1-parameter subgroup $\phi_\xi\colon \mathbb{R} \to G$ such that

$$\phi_\xi(t_1 + t_2) = \phi_\xi(t_1)\phi_\xi(t_2), \;\; \phi_\xi(0) = e, \;\; \frac{\partial \phi_\xi}{\partial t}|_{t=0} = \xi.$$

One defines the *exponential map*

$$\exp\colon \mathfrak{g} \to G, \;\; \xi \mapsto \phi_\xi(1).$$

For matrices, the abstract exponential map coincides with the usual exponential of matrices as a Taylor series. The 1-parameter subgroup may be written in terms of the exponential map as $\phi_\xi(t) = \exp(t\xi)$.

LEMMA 2.1. *One has the equality of operators on* $\mathfrak{g}$*,*

$$\exp(\mathrm{ad}_\mu) = \mathrm{Ad}(\exp(\mu)),$$

where on the left hand side exp *is the exponential of an element of the algebra* $\mathrm{End}(\mathfrak{g})$*, while on the right hand side* exp *is the exponential map for Lie algebras.*

PROOF.

$$\frac{\partial}{\partial t} \mathrm{Ad}(\exp(t\mu))\zeta = \mathrm{Ad}(\exp(t\mu))\, \mathrm{ad}_\mu \zeta,$$

but also

$$\frac{\partial}{\partial t} \exp(t\, \mathrm{ad}_\mu)\zeta = \exp(t\, \mathrm{ad}_\mu)\, \mathrm{ad}_\mu \zeta.$$

By uniqueness of solutions of ODE's, this implies $\mathrm{Ad}(\exp(t\mu)) = \exp(t\, \mathrm{ad}_\mu)$. Now set $t = 1$. ☐

It is easy to see that the differential of the exponential map at the origin is $\mathrm{d}_0 \exp = \mathrm{id}$. Hence, by the implicit function theorem the exponential map gives a diffeomorphism from an open neighborhood of 0 in $\mathfrak{g}$ to an open neighborhood of e in G. We are interested in the differential of $\exp\colon \mathfrak{g} \to G$ at any given point $\mu \in \mathfrak{g}$. It is a linear operator $\mathrm{d}_\mu \exp\colon \mathfrak{g} = T_\mu\mathfrak{g} \to T_gG$. Since $\mathfrak{g}$ is a vector space, $T_\mu\mathfrak{g} \cong \mathfrak{g}$ canonically. On the other hand, we may use the left-action to obtain an isomorphism, $\mathrm{d}_e\mathcal{A}^L(g)\colon T_gG \cong \mathfrak{g}$.

THEOREM 2.2. *The differential of the exponential map* $\exp\colon \mathfrak{g} \to G$ *at* $\mu \in \mathfrak{g}$ *is the linear operator* $d_\mu \exp\colon \mathfrak{g} \to \mathfrak{g}$ *given by the formula,*

$$d_\mu \exp = d_e\mathcal{A}^L(\exp(\mu)) \circ j^L(\mathrm{ad}_\mu)$$

Here $j^L(z) = \frac{1-e^{-z}}{z}$ is the holomorphic function introduced in the last chapter. Thus, in terms of left trivialization of the tangent bundle, the differential is given by $j^L(\mathrm{ad}_\mu)$.

PROOF. The differential $\mathrm{d}_\mu \exp(\zeta) = \frac{\partial}{\partial t}\big|_{t=0} \exp(\mu + \mathrm{t}\zeta)$ may be written,

$$\mathrm{d}_\mu \exp(\zeta) = d_e\mathcal{A}^L(\exp(\mu)) \left.\frac{\partial}{\partial t}\right|_{t=0} \Big(\exp(\mu)^{-1}\exp(\mu + t\zeta)\Big)$$

Let $\exp_s(\nu) := \exp(s\nu)$ and (for any given μ, ζ)

$$\phi(s,t) = \exp_s(\mu)^{-1}\exp_s(\mu + t\zeta).$$

Write $\psi(s) = \frac{\partial \phi}{\partial t}|_{t=0} \in \mathfrak{g}$. Thus $\psi(1) = \mathrm{d}_\mu \exp(\zeta)$, while $\psi(0) = 0$ (since $\psi(t, 0) = e$ for all t). Taking the t-derivative of the equation

$$\mu + t\zeta = \frac{\partial}{\partial s}\Big(\exp_s(\mu + t\zeta)\Big)\ \exp_s(\mu + t\zeta)^{-1} = \frac{\partial}{\partial s}\Big(\exp_s(\mu)\phi\Big)\ \phi^{-1}\exp_s(\mu)^{-1}$$

at $t = 0$, we obtain

$$\begin{aligned}
\zeta &= \frac{\partial}{\partial s}\Big(\exp_s(\mu)\psi\Big)\ \exp_s(\mu)^{-1} - \frac{\partial}{\partial s}\big(\exp_s(\mu)\big)\ \psi\exp_s(\mu)^{-1}\\
&= \exp_s(\mu)\frac{\partial\psi}{\partial s}\exp_s(\mu)^{-1}\\
&= \mathrm{Ad}(\exp_s(\mu))\frac{\partial\psi}{\partial s}\\
&= \exp(s\,\mathrm{ad}_\mu)\frac{\partial\psi}{\partial s}.
\end{aligned}$$

That is, $\frac{\partial\psi}{\partial s} = \exp(-s\,\mathrm{ad}_\mu)\zeta$. Integrating,

$$\psi(1) = \Big(\int_0^1 \exp(-s\,\mathrm{ad}_\mu)\mathrm{d}s\Big)\,\zeta = \frac{1 - \exp(-\mathrm{ad}_\mu)}{\mathrm{ad}_\mu}\zeta = j^L(\mathrm{ad}_\mu)\zeta.$$

□

REMARKS 2.3. (1) Instead of the left-action, we may also identify $T_gG \cong \mathfrak{g}$ using the right action. This choice yields,

$$\mathrm{d}_\mu \exp = d_e\mathcal{A}^R(\exp(\mu)) \circ j^R(\mathrm{ad}_\mu)$$

This follows from the formula for the left trivialization, because the adjoint action of $\exp\mu$ on $\mathfrak{g}$ is

$$\mathrm{Ad}(\exp\mu) = \mathrm{d}_e\mathcal{A}^R(\exp\mu)^{-1} \circ \mathrm{d}_a\mathcal{A}^L(\exp\mu),$$

and since

$$\mathrm{Ad}(\exp\mu) j^L(\mathrm{ad}_\mu) = e^{\mathrm{ad}_\mu} j^L(\mathrm{ad}_\mu) = j^R(\mathrm{ad}_\mu).$$

(2) In particular, the Jacobian of the exponential map relative to the left-invariant volume form is the function, $\mu \mapsto \det(j^L(\mathrm{ad}_\mu))$. while for the right-invariant volume form one obtains $\det(j^R(\mathrm{ad}_\mu))$. In general, the two Jacobians are not the same: Their quotient is the function

$$\det(e^{\mathrm{ad}_\mu}) = e^{\mathrm{tr}(\mathrm{ad}_\mu)}.$$

The function $G \to \mathbb{R}^\times,\ g \mapsto \det(\mathrm{Ad}(g))$ is a group homomorphism called the unimodular character, while $\mathfrak{g} \to \mathbb{R},\ \mu \mapsto \mathrm{tr}(\mathrm{ad}_\mu)$ is a Lie algebra homomorphism called the (infinitesimal) unimodular character. A Lie group is called unimodular if the unimodular character is trivial. For instance, any compact Lie group, and any semi-simple Lie group, is unimodular. The simply connected Lie group corresponding to the non-trivial 2-dimensional Lie algebra is not unimodular.

If G is connected and $\mathfrak{g}$ is quadratic (i.e. it admits an Ad-invariant quadratic form), then G is unimodular. This follows because in that case, ad_μ is skew-adjoint, so its trace vanishes. In the quadratic case, the determinants of $j^L(\mathrm{ad}_\mu)$ and $j^R(\mathrm{ad}_\mu)$ coincide, and are equal to

$$J(\mu) := \det j(\mathrm{ad}_\mu) = \det\Big(\frac{\sinh \mathrm{ad}_\mu/2}{\mathrm{ad}_\mu/2}\Big).$$

By our results from the last section, this function admits a global analytic square root.

The unimodular character arises from the fact that for a general Lie group G, the left- and right-invariant volume forms Γ^L and Γ^R may be different. The quotient of the two volume forms at $g \in G$ is given $\det(\mathrm{Ad}_g)$.

3. The vector field $\frac{1}{2}(\xi^L + \xi^R)$

Given a Lie group action

$$\mathcal{A}\colon G \to \mathrm{Diff}(M),$$

its differential defines a Lie algebra homomorphism (which we denote by the same letter)

$$\mathcal{A}\colon \mathfrak{g} \to \mathfrak{X}(M).$$

In terms of the actions of vector fields on functions,

$$\mathcal{A}(\xi) f = \frac{\partial}{\partial t}|_{t=0} \exp(-t\xi)^* f.$$

One calls $\mathcal{A}(\xi)$ the generating vector field for the G-action, also denote ξ_M. (Some authors use opposite sign conventions, so that ξ_M is an anti-homomorphism.)

The generating vector fields $\mathrm{Ad}(\xi) \in \mathfrak{X}(\mathfrak{g})$ for the adjoint action of G on $\mathfrak{g}$ are

$$\mathrm{Ad}(\xi)|_\mu = \mathrm{ad}_\mu(\xi)$$

(using the identifications $T_\mu \mathfrak{g} = \mathfrak{g}$). The generating vector fields for the three natural actions of G on itself are

$$\mathcal{A}^L(\xi) = -\xi^R,\ \ \mathcal{A}^R(\xi) = \xi^L,\ \ \mathrm{Ad}(\xi) = \xi^L - \xi^R.$$

(Note that the vector field $\mathcal{A}^R(\xi)$ must be left-invariant, since the action $\mathcal{A}^R(g)$ commutes with the left-action.) We have $[\xi^L, \zeta^R] = 0$, since the left and right actions commute.

Any $\xi \in \mathfrak{g}$ may be viewed as a constant vector field on $\mathfrak{g}$. The half-sum $\xi^\sharp = \frac{1}{2}(\xi^L + \xi^R) \in \mathfrak{X}(G)$ is the closest counterpart of the constant vector field $\xi \in \mathfrak{X}(\mathfrak{g})$. For example, the vector fields $\xi^\sharp$ 'almost' commute in the sense that

$$[\xi^\sharp, \zeta^\sharp] = \frac{1}{4}[\xi, \zeta]^L - \frac{1}{4}[\xi, \zeta]^R = \frac{1}{4}\operatorname{Ad}([\xi, \zeta])$$

vanishes at $e \in G$. Note also that

$$[\operatorname{Ad}(\xi), \zeta^\sharp] = [\xi, \zeta]^\sharp,$$

parallel to a property of the constant vector field on $\mathfrak{g}$.

Let $\mathfrak{g}' \subset \mathfrak{g}$ denote the subset where the exponential map has maximal rank. By the formula for $\mathrm{d}_\mu \exp$, this is the subset where $\mathrm{ad}_\mu \colon \mathfrak{g} \to \mathfrak{g}$ has no eigenvalue of the form $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. Given a vector field $X \in \mathfrak{X}(G)$, one has a well-defined vector field $\exp^*(X) \in \mathfrak{X}(\mathfrak{g}')$ such that $\exp^*(X)_\mu = (\mathrm{d}_\mu \exp)^{-1}(X_{\exp \mu})$ for all $\mu \in \mathfrak{g}^*$. In particular, for $\xi \in \mathfrak{g}$ we can consider

$$\exp^* \xi^L,\ \exp^* \xi^R,\ \exp^* \xi^\sharp.$$

Since $T_\mu \mathfrak{g} \cong \mathfrak{g}$, both of these vector fields define elements of $C^\infty(\mathfrak{g}') \otimes \mathfrak{g}$, depending linearly on ξ. The map taking ξ to this vector field is therefore an element of $C^\infty(\mathfrak{g}^*) \otimes \operatorname{End}(\mathfrak{g})$.

Using left-trivialization of the tangent bundle, we have

$$(\exp^* \xi^L)_\mu = (j^L(\mathrm{ad}_\mu))^{-1}(\xi) = \frac{\mathrm{ad}_\mu}{1 - e^{-\mathrm{ad}_\mu}}\xi$$

Similarly,

$$(\exp^* \xi^R)_\mu = (j^R(\mathrm{ad}_\mu))^{-1}(\xi) = \frac{\mathrm{ad}_\mu}{e^{\mathrm{ad}_\mu} - 1}\xi.$$

The difference with the constant vector field ξ is,

$$\begin{aligned}(\exp^* \xi^L)_\mu - \xi &= \mathrm{ad}_\mu f^L(\mathrm{ad}_\mu)(\xi) = \operatorname{Ad}(f^L(\mathrm{ad}_\mu)\xi),\\ (\exp^* \xi^R)_\mu - \xi &= \mathrm{ad}_\mu f^R(\mathrm{ad}_\mu)(\xi) = \operatorname{Ad}(f^R(\mathrm{ad}_\mu)\xi).\end{aligned}$$

where

$$f^L(z) = \frac{1}{1 - e^{-z}} - \frac{1}{z}, \quad f^R(z) = \frac{1}{e^z - 1} - \frac{1}{z}.$$

Note that $f^L(\mathrm{ad}_\mu), f^R(\mathrm{ad}_\mu) \in \operatorname{End}(\mathfrak{g})$ are well-defined for all $\mu \in \mathfrak{g}'$. The formula shows that the difference between the vector fields $\exp^* \xi^L, \exp^* \xi^R$ and the constant vector field ξ is a vector field in the direction of the orbits of the adjoint action. Put differently, the radial part of these vector fields equals ξ. Finally,

$$(\frac{1}{2}\exp^*(\xi^L + \xi^R))_\mu - \xi = f(\mathrm{ad}_\mu)(\mathrm{ad}_\mu \xi) = \operatorname{Ad}\big(f(\mathrm{ad}_\mu)\xi\big)\big|_\mu,$$

where $f = \frac{1}{2}(f^L + f^R)$. That is,

$$f(z) = \frac{1}{2}\Big(\frac{1}{e^z - 1} + \frac{1}{1 - e^{-z}}\Big) - \frac{1}{z} = \frac{1}{2}\coth\big(\frac{z}{2}\big) - \frac{1}{z}.$$

REMARKS 3.1. The function $j^R(z)^{-1} = \frac{z}{e^z-1}$ is the well-known generating functions for the *Bernoulli numbers* B_n:

$$j^R(z)^{-1} = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

The Bernoulli numbers for odd $n \geq 3$ are all zero, while $B_0 = 1,\ B_1 = -\frac{1}{2}$, and

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66}, \ldots$$

The expansion of the function $f(z)$ reads,

$$f(z) = \frac{1}{2}\coth(\frac{z}{2}) - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n-1}.$$

4. Maurer-Cartan forms

The left-invariant Maurer-Cartan form $\theta^L \in \Omega^1(G) \otimes \mathfrak{g}$ is defined in terms of its contractions with left-invariant vector fields by

$$\iota(\xi^L)\theta^L = \xi$$

Similarly, one defines the right-invariant Maurer-Cartan form $\theta^R \in \Omega^1(G)^R \otimes \mathfrak{g}$ by

$$\iota(\xi^R)\theta^R = \xi.$$

For matrix Lie groups, one has the formulas

$$\theta^L = g^{-1}\mathrm{d}g,\ \theta^R = \mathrm{d}g\ g^{-1}.$$

(More precisely, $\mathrm{d}g$ is a matrix-valued 1-form on G, to be interpreted as the pull-back of the coordinate differentials on $\mathrm{Mat}_N(\mathbb{R}) \cong \mathbb{R}^{N^2}$ under the inclusion map $G \to \mathrm{Mat}_N(\mathbb{R})$.)

PROPOSITION 4.1 (Properties of Maurer-Cartan forms).

(1) *The Maurer-Cartan forms are related by*

$$\theta^R_g = \mathrm{Ad}_g(\theta^L_g),$$

(2) *The differential of* θ^L, θ^R *is given by the* Maurer-Cartan equations

$$d\theta^L + \tfrac{1}{2}[\theta^L, \theta^L] = 0, \quad d\theta^R - \tfrac{1}{2}[\theta^R, \theta^R] = 0.$$

(3) *The pull-back of* θ^L, θ^R *under group multiplication* $\mathrm{Mult}\colon G \times G \to G, (g_1, g_2) \mapsto g_1 g_2$ *is given by the formula,*

$$\mathrm{Mult}^*\theta^L = \mathrm{Ad}_{g_2^{-1}} \mathrm{pr}_1^*\theta^L + \mathrm{pr}_2^*\theta^L$$
$$\mathrm{Mult}^*\theta^R = \mathrm{Ad}_{g_1} \mathrm{pr}_2^*\theta^R + \mathrm{pr}_1^*\theta^L.$$

where $\mathrm{pr}_1, \mathrm{pr}_2\colon G \times G \to G$ *are the two projections.*

For matrix Lie groups, all of these results are easily proved from $\theta^L = g^{-1}\mathrm{d}g$ and $\theta^R = \mathrm{d}gg^{-1}$ (although the general case it not much harder). For instance, $\mathrm{Mult}^*\theta^L$ is computed as follows:

$$(g_1g_2)^{-1}\mathrm{d}(g_1g_2) = g_2^{-1}g_1^{-1}\mathrm{d}g_1g_1^{-1} + g_2^{-1}\mathrm{d}g_2.$$

Consider now the pull-back of the Maurer-Cartan forms under the exponential map, $\exp^*\theta^L, \exp^*\theta^R \in \Omega^1(\mathfrak{g}) \otimes \mathfrak{g}$. At any given point $\mu \in \mathfrak{g}$, these are elements of $T^*_\mu\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}^* \to \mathfrak{g}$. Thus, we can view $\exp^*\theta^L, \exp^*\theta^R$ as maps $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

THEOREM 4.2. *The maps* $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *determined by* $\exp^* \theta^L, \exp^* \theta^R$ *are given by*

$$\mu \mapsto j^L(\mathrm{ad}_\mu), \quad \mu \mapsto j^R(\mathrm{ad}\,\mu),$$

respectively.

PROOF. Let $\mu \in \mathfrak{g}$ and $\xi \in T_\mu \mathfrak{g} = \mathfrak{g}$. Then, using left-trivialization of the tangent bundle,

$$\begin{aligned}\iota(\xi)(\exp^* \theta^L)_\mu &= \iota(\mathrm{d}_\mu \exp(\xi))\theta^L_{\exp \mu}\\ &= \mathrm{d}_\mu \exp(\xi)\\ &= j^L(\mathrm{ad}_\mu)(\xi).\end{aligned}$$

For θ^R, just use $\theta^R = \mathrm{Ad}_g \theta^L$. □

In a basis $e_i \in \mathfrak{g}$, the Maurer-Cartan forms can be written $\theta^L = \sum_i \theta^{L,i} \otimes e_i$. Letting μ^i be the coordinate functions on $\mathfrak{g}$ and $\mathrm{d}\mu^i$ their differentials, the Theorem says that

$$\exp^* \theta^{L,i} = \sum_j j^L(\mathrm{ad}_\mu)^i_j \, \mathrm{d}\mu^j$$

where $j^L(\mathrm{ad}_\mu)^i_j$ are the components of the matrix describing $j^L(\mathrm{ad}_\mu)$. Dropping indices, we may write this as $\exp^* \theta^L = j^L(\mathrm{ad}_\mu)(\mathrm{d}\mu)$, where $\mathrm{d}\mu \in \Omega^1(\mathfrak{g}) \otimes \mathfrak{g}$ is the tautological 1-form.

The half-sum $\frac{1}{2}(\theta^L + \theta^R)$ is the most natural counterpart of $\mathrm{d}\mu \in \Omega^1(\mathfrak{g}; \mathfrak{g})$. The theorem tells us that

$$\tfrac{1}{2} \exp^*(\theta^L + \theta^R) - \mathrm{d}\mu = \frac{\sinh(\mathrm{ad}_\mu) - \mathrm{ad}_\mu}{\mathrm{ad}_\mu}(\mathrm{d}\mu) = g(\mathrm{ad}_\mu)\, \mathrm{ad}_\mu(\mathrm{d}\mu)$$

where

$$g(z) = \frac{\sinh z - z}{z^2}$$

is another of the functions introduced in the last chapter.

5. Quadratic Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A bilinear form B on $\mathfrak{g}$ is called invariant if it is invariant under the adjoint action:

$$B(\mathrm{Ad}_g(\xi), \mathrm{Ad}_g(\zeta)) = B(\xi, \zeta)$$

for all ξ, ζ. An important example of an invariant bilinear form is the *Killing form*

$$B(\xi, \zeta) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_\xi \, \mathrm{ad}_\zeta).$$

It is a well-known fact that the Killing form on $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g}$ is semi-simple, i.e. a direct sum of simple ideals. For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$, one can also directly take the trace form $B(\xi, \zeta) = \mathrm{tr}(\xi\zeta)$.

But there are more general examples of symmetric bilinear forms: For instance, if $\mathfrak{h}$ is any given Lie algebra, let $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{h}^*$ be the semi-direct product. That is, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ as a vector space, with bracket relations

$$[\xi_1 \oplus \mu_1, \xi_2 \oplus \mu_2] = [\xi_1, \xi_2] \oplus (-\mathrm{ad}^*_{\xi_1} \mu_2 + \mathrm{ad}^*_{\xi_2} \mu_1).$$

Then the bilinear form given by the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ is invariant, but $\mathfrak{g}$ is not semi-simple since $\mathfrak{h}^* \subset \mathfrak{g}$ is an ideal.

DEFINITION 5.1. A Lie algebra $\mathfrak{g}$ with an invariant, non-degenerate symmetric bilinear form B is called a quadratic Lie algebra. A Lie group is called quadratic if its Lie algebra carries such a quadratic form.

Given an invariant symmetric bilinear form B, on $\mathfrak{g}$, one can construct an important 3-form on the group,

$$\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G)$$

PROPOSITION 5.2. *The 3-form η is closed: $d\eta = 0$. Hence it defines a de Rham cohomology class $[\eta] \in H^3(G, \mathbb{R})$.*

PROOF. Using the Maurer-Cartan-equation $\mathrm{d}\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0$, we have

$$\mathrm{d}\eta = -\frac{1}{24} B([\theta^L, \theta^L], [\theta^L, \theta^L]) = -\frac{1}{24} B(\theta^L, [\theta^L, [\theta^L, \theta^L]]).$$

But $[\theta^L, [\theta^L, \theta^L]] = 0$ by the Jacobi identity for $\mathfrak{g}$. $\square$

The pull-back of $\exp^* \eta$ of the closed 3-form η to $\mathfrak{g}$ is exact. In fact, the Poincaré lemma gives an explicit primitive $\varpi \in \Omega^2(\mathfrak{g})$ with $\mathrm{d}\varpi = \Phi^* \eta$. Now, $\Omega^2(\mathfrak{g}) = C^\infty(\mathfrak{g}) \otimes \wedge^2 \mathfrak{g}^*$ is identified, using B, with $C^\infty(\mathfrak{g}) \otimes \mathfrak{o}(\mathfrak{g})$. What is this function?

PROPOSITION 5.3. *The function $\mathfrak{g} \to \mathfrak{o}(\mathfrak{g})$ corresponding to the 2-form ϖ is $\mu \mapsto g(\mathrm{ad}_\mu)$, where $g(z) = z^{-2}(\sinh(z) - z)$.*

PROOF. $\square$

CHAPTER 5

Enveloping algebras

1. The universal enveloping algebra

For any Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, one defines the *universal enveloping algebra* $U\mathfrak{g} = T(\mathfrak{g})/\mathcal{I}$ as a quotient of the tensor algebra by the two-sided ideal $\mathcal{I}$ generated by elements of the form

$$\xi \otimes \zeta - \zeta \otimes \xi - [\xi, \zeta]_{\mathfrak{g}}.$$

Equivalently, the universal enveloping algebra is generated by elements $\xi \in \mathfrak{g}$ subject to relations $\xi\zeta - \zeta\xi = [\xi, \zeta]_{\mathfrak{g}}$.

Note that $\mathcal{I} \cap \mathbb{K} = 0$, which shows immediately that $U\mathfrak{g}$ is a non-trivial unital algebra. The projection $T(\mathfrak{g}) \to T^0(\mathfrak{g}) \to \mathbb{K}$ is an algebra homomorphism vanishing on $\mathcal{I}$, and therefore descends to an algebra homomorphism

$$U\mathfrak{g} \to \mathbb{K}$$

which is right inverse to the inclusion $\mathbb{K} \to U\mathfrak{g}$. This is the *augmentation map* for the enveloping algebra. (Note that for the Clifford algebra, there is no natural augmentation map which is also an algebra homomorphism.)

Suppose from now on that $\mathbb{K} = \mathbb{R}$. Most of the results proved below hold for more general fields, but have simpler proofs in the real case, by working with a Lie group G integrating $\mathfrak{g}$.

For any manifold M, let $\mathfrak{D}(M)$ denote the algebra of differential operators on M. By definition, this is the algebra of operators on $C^\infty(M)$, generated by $C^\infty(M)$ itself together with $\mathfrak{X}(M)$. Given a G-action on M, one can consider the subalgebra $\mathfrak{D}(M)^G$ of differential operators which commute with the G-action.

Let $\mathfrak{D}^L(G)$ denote the differential operators on G which commute with left translation. The Lie algebra homomorphism

$$\mathfrak{g} \mapsto \mathfrak{X}^L(G),\ \xi \mapsto \xi^L$$

extends to an algebra homomorphism $T(\mathfrak{g}) \to \mathfrak{D}^L(G)$, which vanishes on the ideal $\mathcal{I}$. Hence we get an induced algebra homomorphism

$$U(\mathfrak{g}) \to \mathfrak{D}^L(G),$$

taking the image of $\xi \in \mathfrak{g} = T^1(\mathfrak{g})$ to ξ^L. It follows in particular that the injection $\mathfrak{g} \to T(\mathfrak{g})$ descends to an injection $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$. From the definition of $U(\mathfrak{g})$, it is clear that $\mathfrak{g}$ is a Lie subalgebra of $U(\mathfrak{g})$ (where the bracket on $U(\mathfrak{g})$ is the commutator).

THEOREM 1.1 (Universal property). *If $\mathcal{A}$ is an associative algebra, and $f\colon \mathfrak{g} \to \mathcal{A}$ is a homomorphism of Lie algebras, then f extends uniquely to an algebra homomorphism $U(\mathfrak{g}) \to \mathcal{A}$.*

PROOF. The map f extends to an algebra homomorphism $T(\mathfrak{g}) \to \mathcal{A}$. This algebra homomorphism vanishes on the ideal $\mathcal{I}$, and hence descends to an algebra homomorphism $U(\mathfrak{g}) \to \mathcal{A}$. This extension is unique, since $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra. $\square$

The map taking $\mathfrak{g} \to U(\mathfrak{g})$ is a functor from the category of Lie algebras to the category of associative algebras: Lie algebra homomorphisms go to algebra homomorphisms. (For instance, the augmentation map is induced by the map $\mathfrak{g} \to \{0\}$ to the zero Lie algebra.) Any module over the Lie algebra $\mathfrak{g}$ becomes a module over the algebra $U(\mathfrak{g})$.

The canonical anti-automorphism. The map $\xi \mapsto -\xi$ is an anti-automorphism of the Lie algebra $\mathfrak{g}$, i.e. it preserves the bracket up to a sign. Define an anti-automorphism of $T(\mathfrak{g})$ by $v_1 \otimes \cdots \otimes v_r \mapsto (-1)^r v_r \otimes \cdots \otimes v_1$ This preserves the ideal $\mathcal{I}$, and therefore descends to an anti-automorphism of $U\mathfrak{g}$, denoted $x \mapsto x^T$.

The left, right, and adjoint representations. The enveloping algebra carries a natural representation of $\mathfrak{g}$ by multiplication from the left: $\rho^L(\xi)x = \xi x$, and there is another representation by multiplication from the right, $\rho^R(\xi)x = -\xi x$. The two actions commute, and the diagonal action is the adjoint action, $\mathrm{ad}(\xi)x = \xi x - x\xi = [\xi, x]$. An element x lies in the center of $U(\mathfrak{g})$ if and only if it commutes with all generators ξ. That is, it consists exactly of the invariants for the adjoint action:

$$\mathrm{Cent}(U\mathfrak{g}) = (U\mathfrak{g})^{\mathfrak{g}}.$$

The $\mathbb{Z}$-filtration. Consider the $\mathbb{Z}$-filtration of the enveloping algebra, where $U^{(k)}(\mathfrak{g})$ is the image of $\bigoplus_{j \le k} T^j(\mathfrak{g})$ under the quotient map. Equivalently, $U^{(k)}\mathfrak{g}$ consists of linear combinations of products of $\le k$ elements in $\mathfrak{g}$.

LEMMA 1.2. *For any permutation σ of $\{1, \ldots, k\}$ and any $v_j \in \mathfrak{g}$,*

$$v_1 \cdots v_k - v_{\sigma(1)} \cdots v_{\sigma(k)} \in U^{(k-1)}(\mathfrak{g}).$$

PROOF. For transpositions of two adjacent elements, this is clear from the definition of the enveloping algebra. The general case follows since such transpositions generate the symmetric group. $\square$

From this Lemma, it follows that the commutator or two elements of filtration degree k, l has filtration degree $k+l-1$. Hence, the associated graded algebra $\mathrm{gr}(U\mathfrak{g})$ is commutative[1] , and the inclusion of $\mathfrak{g}$ extends to an algebra homomorphism

$$S\mathfrak{g} \to \mathrm{gr}(U\mathfrak{g}).$$

Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$. Given any finite sequence I of indices $i_1, \ldots, i_k \in \{0, \ldots, n\}$ (possibly with repetitions), let $e_I \in U_{(k)}\mathfrak{g}$ be the monomial,

$$e_I := e_{i_1} \cdots e_{i_k}$$

We set $e_\emptyset = 1$. Then these elements span the enveloping algebra (since $U\mathfrak{g}$ is generated by $\mathfrak{g}$). The Lemma shows that $U\mathfrak{g}$ is already spanned by elements e_I where I is weakly increasing, i.e. $i_1 \le i_2 \le \cdots$. Since the corresponding elements in $S\mathfrak{g}$ are clearly a basis, we therefore obtain a surjective linear map

$$S\mathfrak{g} \to U\mathfrak{g},\ e_I \to e_I. \tag{25}$$

[1]Here we mean commutativity in the plain sense, rather than the $\mathbb{Z}$-graded sense. To make everything for with our conventions for graded algebras, it is sometimes convenient to double the gradings – see below.

lifting the map $S\mathfrak{g} \to \mathrm{gr}(U\mathfrak{g})$.

2. The Poincaré-Birkhoff-Witt theorem

The Poincaré-Birkhoff-Witt theorem is of fundamental importance in Lie theory. We assume that $\mathbb{K}$ has characteristic zero.

THEOREM 2.1 (Poincaré-Birkhoff-Witt, version I). *Let e_i be a basis of $\mathfrak{g}$. Then the elements*

$$\{e_I \mid I \text{ is weakly increasing}\}$$

form a basis of $U\mathfrak{g}$.

Equivalently, the map (25) is an isomorphism. As explained above, it is easily seen that this map is onto, so the main point of the theorem is that it is 1-1 (i.e. that the ξ^I are linearly independent). Since a morphism of filtered vector spaces is an isomorphism if and only if the associated graded map is an isomorphism, one has the following equivalent basis-independent version:

THEOREM 2.2 (Poincaré-Birkhoff-Witt, version II). *The homomorphism $S\mathfrak{g} \to \mathrm{gr}(U\mathfrak{g})$ is an algebra isomorphism.*

An alternative lift of the map $S\mathfrak{g} \to \mathrm{gr}(U\mathfrak{g})$ is given by symmetrization,

$$\mathrm{sym}\colon S\mathfrak{g} \to U\mathfrak{g}, \xi_1 \cdots \xi_k \mapsto \frac{1}{k!} \sum_{s \in S_k} \xi_{s(1)} \cdots \xi_{s(k)}$$

Alternatively, sym may be characterized as the unique linear map such that

$$\mathrm{sym}(\xi^k) = \xi^k$$

for all $\xi \in \mathfrak{g}$ and all k, where on the left hand side the kth power $\xi^k = \xi \cdots \xi$ is a product in the symmetric algebra, while on the right hand side it is taken in the enveloping algebra. (Note that the elements ξ^k with $\xi \in \mathfrak{g}$ span $S^k(\mathfrak{g})$, by polarization.) The symmetrization map is the direct analogue of the quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V)$ for Clifford algebras, which was given by symmetrization in the graded sense.

THEOREM 2.3 (Poincaré-Birkhoff-Witt, version III). *The symmetrization map $\mathrm{sym}\colon S\mathfrak{g} \to U\mathfrak{g}$, is an isomorphism of filtered vector spaces.*

If $\mathbb{K} = \mathbb{R}$, so that $\mathfrak{g}$ is the Lie algebra of a simply connected Lie group G, the PBW-theorem also has a differential-geometric interpretation.

THEOREM 2.4 (Poincaré-Birkhoff-Witt, version IV). *The canonical map $U(\mathfrak{g}) \to \mathfrak{D}^L(G)$ is an isomorphism of algebras.*

We will sketch a proof of these theorems for $\mathbb{K} = \mathbb{R}$, assuming the existence of a Lie group G integrating $\mathfrak{g}$ (which exists by Lie's third theorem).

PROOF OF PBW FOR $\mathfrak{g} = \mathrm{Lie}(G)$. To begin, let us review some facts about differential operators on manifolds. For any manifold M, the algebra $\mathfrak{D}(M)$ of differential operators is a filtered algebra, where $\mathfrak{D}^{(k)}(M)$ consists of differential operators of degree $\leq k$. In local coordinates, any differential operator has the form

$$D = \sum_{|I| \leq k} a_I(x) (\frac{\partial}{\partial x})^I.$$

The function $\sum_{|I|\le k} a_I(x)p^I$ (called the *full symbol* of D) has rather complicated transformations properties under coordinate change, but its leading term

$$\sigma^k(D)(x,p) = \sum_{|I|=k} a_I(x)p^I$$

transforms very nicely: It defines a function on the cotangent bundle T^*M. This can be seen as follows: Let $\mu \in T^*_x M$ correspond to the point (x,p). For any function f such that $\mathrm{d}_x f = \mu$, one may verify that

$$\sigma^k(D)(\mu) = \lim_{t\to\infty}(t^{-k}e^{-tf}De^{tf}\Big|_x),$$

which gives a coordinate-free description of the principal symbol. The principal symbol gives a canonical linear map

$$\sigma^k : \mathfrak{D}^{(k)}(M) \to \mathrm{Pol}^k(T^*M)$$

where $\mathrm{Pol}^k(T^*M) \subset C^\infty(T^*M)$ are the functions on T^*M whose restriction to every cotangent fiber is a polynomial of degree k. Equivalently, this is the space $C^\infty(M; S^k(TM))$ of smooth sections of the kth symmetric power of the tangent bundle, $S^k(TM)$. We obtain an exact sequence,

$$0 \to \mathfrak{D}^{(k-1)}(M) \to \mathfrak{D}^{(k)}(M) \xrightarrow{\sigma^k} C^\infty(M; S^k(TM)) \to 0.$$

If D_1, D_2 are differential operators of degrees k_1, k_2, then $D_1 \circ D_2$ is a differential operator of degree $k_1 + k_2$ and

$$\sigma^{k_1+k_2}(D_1 \circ D_2) = \sigma^{k_1}(D_1)\sigma^{k_2}(D_2).$$

That is, the symbol map descends to an isomorphism of graded algebras,

$$\sigma^\bullet\colon\ \mathrm{gr}^\bullet \mathfrak{D}(M) \to C^\infty(M, S^\bullet(TM)).$$

Given a G-action on a manifold M, this map is equivariant and restricts to an isomorphism,

$$(\mathrm{gr}^\bullet \mathfrak{D}(M))^G \to C^\infty(M; S^\bullet(TM))^G.$$

We also have an injection $\mathrm{gr}^\bullet(\mathfrak{D}(M)^G) \to (\mathrm{gr}^\bullet \mathfrak{D}(M))^G$, but for non-compact Lie groups and ill-behaved actions this need not be an isomorphism.

In the special case $M = G$, with G acting by left translation, we have

$$C^\infty(G, S^\bullet(TG))^L = S(T_eG) = S(\mathfrak{g}).$$

Consider now the composition of maps

$$S^k\mathfrak{g} \to U^{(k)}\mathfrak{g} \to \mathfrak{D}^{(k),L}(G) \xrightarrow{\sigma} S^k\mathfrak{g},$$

where the first map is given either by symmetrization, or in terms of the basis by $e_I \mapsto e_I$. Note that the associated graded map $S\mathfrak{g} \to \mathrm{gr}(U\mathfrak{g})$ is independent of this choice, and we obtain a sequence of homomorphisms of graded algebras,

$$S\mathfrak{g} \to \mathrm{gr}(U\mathfrak{g}) \to \mathrm{gr}(\mathfrak{D}^L(G)) \to S\mathfrak{g} \tag{26}$$

We know that the last map is 1-1 (since it comes from the inclusion $\mathrm{gr}(\mathfrak{D}^L(G)) \hookrightarrow \mathrm{gr}(\mathfrak{D}(G))^L \cong S\mathfrak{g}$), and that the first map is onto (since the e_I span $U\mathfrak{g}$). On the other hand, the composition of all these maps is the identity. (For any e_i the symbol of the left-invariant vector field ξ^L is the left-invariant function on T^*G, given on $T^*_eG = \mathfrak{g}^*$ by ξ itself. Now use that all maps are algebra homomorphisms, up to lower order terms.) This implies that each map must be an isomorphism. This proves each of the four versions of the PBW theorem. $\square$

The proof of the PBW theorem given above is unsatisfactory since it is based on the highly nontrivial 'Lie's third theorem'. For this reason, it is better to have a purely algebraic proof of this result. This proof can be found in most textbooks on Lie theory. In a recent work, Emanuela Petracci (*Universal representations by coderivations of Lie algebras. Bulletin des Sciences Mathématiques 127 (2003), no. 5, 439-465* gives a new and very beautiful proof of the PBW theorem, exploiting the Hopf algebra structures on the symmetric and enveloping algebras. We will explain some of the ideas from her proof in Section 4 below.

Let us use the symmetrization map $\mathrm{sym}\colon S\mathfrak{g} \to U\mathfrak{g}$ to transfer the non-commutative product on $U\mathfrak{g}$ to a product $*$ on $S\mathfrak{g}$. By definition of symmetrization, and of the enveloping algebra, we have

$$v_1 v_2 = \tfrac{1}{2}(v_1 * v_2 + v_2 * v_1), \quad [v_1, v_2] = v_1 * v_2 - v_2 * v_1.$$

Hence

$$v_1 * v_2 = v_1 v_2 + \tfrac{1}{2}[v_1, v_2].$$

The triple product is already much more complicated. One finds, after cumbersome computation,

$$\begin{aligned} v_1 * v_2 * v_3 = v_1 v_2 v_3 &+ \frac{v_3[v_1, v_2] + v_1[v_2, v_3] + v_2[v_1, v_3]}{2} \\ &+ \frac{[v_1, [v_2, v_3]] - [v_3, [v_1, v_2]]}{6}. \end{aligned}$$

(As a consistency check, note that sym intertwines the anti-automorphism of $S\mathfrak{g}$ and $U\mathfrak{g}$. The anti-automorphism takes $v_1 * v_2 * v_3$ to $-v_3 * v_2 * v_1$. It follows that the odd degree terms on the right hand side must be preserved under a permutation of v_1, v_3, while the even degree term should change sign.)

Similar to the discussion for Clifford algebras, the isomorphism $\mathrm{gr}(U(\mathfrak{g})) \cong S(\mathfrak{g})$ induces a Poisson structure on $S\mathfrak{g}$. (To comply with the conventions from Section 2, it is necessary to double the grading and filtration: That is, let

$$(S\mathfrak{g})^{2k} = S^k\mathfrak{g}, \ (S\mathfrak{g})^{2k+1} = 0, \ (U\mathfrak{g})^{(2k)} = U^{(k)}\mathfrak{g}, \ (U\mathfrak{g})^{(2k+1)} = U^{(2k)}\mathfrak{g}.$$

The Poisson algebra structure on $S\mathfrak{g}$ is determined by its value on generators $v_1, v_2 \in \mathfrak{g}$, where it is given by $\{v_1, v_2\} = [v_1, v_2]$. That is, the resulting Poisson structure is just the Kirillov Poisson structure.

3. The Hopf algebra structure on $U\mathfrak{g}$

In this section, $\mathbb{K}$ is any field of characteristic 0. Recall that a co-algebra is defined similar to an algebra, but with 'arrows reversed'. An algebra may be viewed as a triple $(\mathcal{A}, m, i)$ consisting of a vector space $\mathcal{A}$, together with linear maps $m\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ (the *multiplication*) and $i\colon \mathbb{K} \to \mathcal{A}$ (the *unit*), such that

$$m \circ (m \otimes 1) = m \circ (1 \otimes m) \quad \text{(Associativity)}$$
$$m \circ (i \otimes 1) = m \circ (1 \otimes i) = 1 \quad \text{(Unit property)}.$$

It is called *commutative* if $m \circ \mathcal{T} = m$, where $\mathcal{T}\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}, \ x \otimes x' \mapsto x' \otimes x$ exchanges the two factors. Dualizing all these definitions one obtains,

Definition 3.1. A *co-algebra* is a vector space $\mathcal{C}$, together with linear maps

$$\Delta\colon \mathcal{C} \to \mathcal{C}\otimes\mathcal{C}, \quad \epsilon\colon \mathcal{C} \to \mathbb{K}$$

called *co-multiplication* and *co-unit*, such that

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta \quad \text{(Co-associativity)}$$
$$(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta = 1 \quad \text{(Co-unit property)}$$

It is called *co-commutative* if $\mathcal{T} \circ \Delta = \Delta$.

It is fairly obvious from the definition that the dual of any coalgebra is an algebra. The converse is not true: The dual of an algebra $\mathcal{A}$ is not a co-algebra, since the dual map $m^*\colon \mathcal{A}^* \to (\mathcal{A} \otimes \mathcal{A})^*$ need not take values in $\mathcal{A}^* \otimes \mathcal{A}^*$, in general. (Of course, this problem does not arise if $\dim \mathcal{A} < \infty$.) There is an obvious notion of homomorphism of co-algebras; for example the co-unit provides such a homomorphism.

A Hopf algebra is a vector space with compatible algebra and coalgebra structures, as follows:

DEFINITION 3.2. A *Hopf algebra* is a vector space $\mathcal{A}$, together with maps $m\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ (multiplication), $i\colon \mathbb{K} \to \mathcal{A}$ (unit), $\Delta\colon \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ (co-multiplication), $\epsilon\colon \mathcal{A} \to \mathbb{K}$ (co-unit), $s\colon \mathcal{A} \to A$ (antipode), such that

(1) $(\mathcal{A}, m, i)$ is an algebra,
(2) $(\mathcal{A}, \Delta, \epsilon)$ is a co-algebra,
(3) Δ and ϵ are algebra homomorphisms,
(4) s is a linear isomorphism and has the property,

$$m \circ (1 \otimes s) \circ \Delta = m \circ (s \otimes 1) \circ \Delta = i \circ \epsilon.$$

From the definition, one can show that m, i are co-algebra homomorphisms, that s is an algebra and co-algebra anti-homomorphism, and that

$$s \circ i = i, \quad \epsilon \circ s = \epsilon.$$

The dual of a finite-dimensional Hopf algebra $\mathcal{A}$ is again a Hopf-algebra. (In the infinite-dimensional case, this is usually false.)

Hopf algebras may be viewed as algebraic counterparts of groups, as the main example shows.

EXAMPLE 3.3 (Finite groups). Let $\mathcal{A} = C(\Gamma, \mathbb{K})$ be the algebra of functions on a finite group Γ, with m the pointwise multiplication and i given by the constant function. Define a co-multiplication

$$\Delta\colon C(\Gamma, \mathbb{K}) \to C(\Gamma, \mathbb{K}) \otimes C(\Gamma, \mathbb{K}) = C(\Gamma \times \Gamma, \mathbb{K}),$$

a co-unit ϵ, and an antipode s by

$$\Delta(f)(g_1, g_2) = f(g_1 g_2), \quad \epsilon(f) = f(e), \quad s(f)(g) = f(g^{-1}).$$

Then $(\mathcal{A}, m, i, \Delta, \epsilon, s)$ is a finite-dimensional Hopf algebra. Conversely, given a Hopf algebra $\mathcal{A}$, one obtains a group of algebra homomorphisms

$$\Gamma_{\mathcal{A}} = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{A}, \mathbb{K})$$

with product

$$\phi_1 \phi_2 = (\phi_1 \otimes \phi_2) \circ \Delta.$$

and group unit $e = \epsilon$. If $\mathcal{A}$ arises as the function algebra of a finite group Γ, then the natural map

$$\Gamma \to \Gamma_{\mathcal{A}}, \; g \mapsto \big[\mathrm{ev}_g\colon f \mapsto f(g)\big]$$

is an isomorphism. (Tannaka-Krein duality.)

REMARK 3.4. This example generalizes to topological groups, provided tensor products are completed in the appropriate way.

EXAMPLE 3.5. The dual Hopf algebra of $C(\Gamma,\mathbb{K})$ is the group algebra $\mathbb{K}[\Gamma]$, i.e. the space of linear combinations $\sum_{g\in\Gamma} a_g\, g$, with Hopf algebra structure defined in the basis $g\in\Gamma\subset\mathbb{K}[\Gamma]$ by

$$m(g\otimes g')=gg',\ \Delta(g)=g\otimes g, s(g)=g^{-1}$$

while $i\colon\mathbb{K}\to\mathbb{K}[\Gamma]$ is the inclusion as multiples of $e\in\Gamma$, while $\epsilon\colon\mathbb{K}[\Gamma]\to\mathbb{K}$ takes $\sum_{g\in\Gamma} a_g\, g$ to the coefficient a_e.

An element x of a Hopf algebra is called *group-like* if $\Delta(x)=x\otimes x$. Since Δ is an algebra homomorphism, the group-like elements form a semi-group under product, and the invertible group-like elements form a group. If $\mathcal{A}=\mathbb{K}[\Gamma]$ is the group algebra of a finite group Γ, the non-zero group like elements are exactly the elements of $\Gamma\subset\mathbb{K}[\Gamma]$. Indeed, if $x=\sum_g a_g\, g$ then $\Delta(x)=\sum_g a_g\, g\otimes g$, which coincides with $x\otimes x=\sum_{g,g'} a_g a_{g'}\, g\otimes g'$ if and only if $a_g^2=a_g$ for all g and $a_g a_{g'}=0$ for $g\neq g'$.

EXAMPLE 3.6. Let $(S(E),m,i)$ be the symmetric algebra over a vector space E. Then $S(E)$ becomes a Hopf algebra if we define

$$\Delta\colon S(E)\to S(E)\otimes S(E)=S(E\oplus E)$$

to be the map defined by the diagonal embedding $E\to E\oplus E$, $\epsilon\colon S(E)\to\mathbb{K}$ to be the augmentation map (induced by $E\to\{0\}$), and $s\colon S(E)\to S(E)$ the canonical anti-automorphism (equal to $v\mapsto -v$ on $E\subset S(E)$). More explicitly, the coproduct is given by

$$\Delta(v^k)=\sum_{j=0}^{k}\binom{k}{j}v^{k-j}\otimes v^j.$$

The formulas have a very nice description in terms of the 'generating function' $e^{tv}=\sum_{j=0}^{\infty}\frac{t^j}{j!}v^j$,

$$\Delta(e^{tv})=e^{tv}\otimes e^{tv},\quad \epsilon(e^{tv})=1,\quad s(e^{tv})=e^{-tv}$$

(to be interpreted as equalities of formal power series in t).

This example generalizes to enveloping algebra $U(\mathfrak{g})$. Again, we take

$$\Delta\colon U\mathfrak{g}\to U(\mathfrak{g})\otimes U(\mathfrak{g})=U(\mathfrak{g}\oplus\mathfrak{g})$$

to be induced by the Lie algebra homomorphism $\mathfrak{g}\to\mathfrak{g}\oplus\mathfrak{g}$, $\xi\mapsto\xi\oplus\xi$ and $\epsilon\colon U(\mathfrak{g})\to\mathbb{K}$ to be induced by the map $\mathfrak{g}\to\{0\}$, and we let the antipode $U(\mathfrak{g})\to U(\mathfrak{g})$ be the canonical anti-homomorphism of $U(\mathfrak{g})$.

THEOREM 3.7. *$U\mathfrak{g}$ with these definitions of Δ,ϵ,δ is a co-commutative Hopf algebra.*

PROOF. Clearly, Δ and ϵ are algebra homomorphisms. The co-associativity of Δ follows because both $(\Delta\otimes 1)\circ\Delta$ and $(1\otimes\Delta)\circ\Delta$ are the maps $U(\mathfrak{g})\to U(\mathfrak{g})\otimes U(\mathfrak{g})\otimes U(\mathfrak{g})=U(\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g})$ induced by the triagonal inclusion. The co-unital properties of ϵ are equally clear. The other two properties are easily checked on the generating function:

$$m\circ(1\otimes s)\circ\Delta(e^{t\xi})=m\circ(1\otimes s)(e^{t\xi}\otimes e^{t\xi})=m(e^{t\xi}\otimes e^{-t\xi})=1=\iota(\epsilon(e^{t\xi})).$$

Similarly, co-commutativity follows because $\Delta(e^{t\xi}) = e^{t\xi} \otimes e^{t\xi}$ is invariant under $\mathcal{T}$. $\square$

To summarize, we can think of $U(\mathfrak{g})$ as an algebraic analogue/substitute for the Lie group G integrating $\mathfrak{g}$. (The co-commutativity of $U\mathfrak{g}$ corresponds to the fact that $C(G, \mathbb{K})$ is a commutative algebra.) This point of view is taken in the definition of quantum groups, which are not really groups but are defined as suitable Hopf algebras.

DEFINITION 3.8. A derivation of an algebra $(\mathcal{A}, m, i)$ is a linear map $D \colon \mathcal{A} \to \mathcal{A}$ satisfying

$$D \circ m = m \circ (D \otimes 1 + 1 \otimes D).$$

A co-derivation of a co-algebra $(\mathcal{C}, \Delta, \epsilon)$ is a linear map $C \colon \mathcal{C} \to \mathcal{C}$ satisfying

$$\Delta \circ C = (C \otimes 1 + 1 \otimes C) \circ \Delta.$$

The space of co-derivations of a co-algebra is a Lie algebra under commutator, just as the space of derivations of an algebra.

PROPOSITION 3.9. *The symmetrization map* $\mathrm{sym} \colon S\mathfrak{g} \to U\mathfrak{g}$ *is a co-algebra homomorphism. For all* $\xi \in \mathfrak{g}$*, the left regular representation* $\rho^L(\xi) \colon U(\mathfrak{g}) \to U(\mathfrak{g}),\ x \mapsto \xi x$ *is a co-derivation of* $U\mathfrak{g}$*.*

PROOF. The symmetrization map is functorial with respect to Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$. Functoriality for the diagonal inclusion $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ shows that sym intertwines Δ, while functoriality relative to the projection $\mathfrak{g} \mapsto \{0\}$. implies that sym intertwines ϵ.

Next, since Δ is an algebra homomorphism,

$$\Delta(\xi x) = \Delta(\xi)\Delta(x) = (\xi \otimes 1 + 1 \otimes \xi)\Delta(x).$$

which shows that $\rho^L(\xi)$ is a co-derivation. $\square$

It is in fact possible to recover $\mathfrak{g}$ from $U(\mathfrak{g})$. For this we need the following

DEFINITION 3.10. An element x of a Hopf algebra $(\mathcal{A}, m, i, \Delta, \epsilon, s)$ is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. Let $P(\mathcal{A})$ denote the space of primitive elements.

LEMMA 3.11. *For any Hopf algebra* $\mathcal{A}$*, the space of primitives* $P(\mathcal{A})$ *is a Lie subalgebra under commutator. Left multiplication*

$$P(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}$$

is an action of this Lie algebra by co-derivations.

PROOF. Since Δ is an algebra homomorphism,

$$\begin{aligned}\Delta(xy - yx) &= \Delta(x)\Delta(y) - \Delta(y)\Delta(x)\\ &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x)\\ &= (xy - yx) \otimes 1 + 1 \otimes (xy - yx).\end{aligned}$$

For the second claim, we compute, for $\xi \in P(\mathcal{A})$ and $x \in \mathcal{A}$,

$$\Delta(\xi x) = \Delta(\xi)\Delta(x) = (\xi \otimes 1 + 1 \otimes \xi)\Delta(x).$$

$\square$

For any vector space E, we have $E \subset P(S(E))$ by definition of the coproduct. More generally, for any Lie algebra we have $\mathfrak{g} \subset P(U(\mathfrak{g}))$. Recall that $\mathbb{K}$ is any field of characteristic 0.

LEMMA 3.12. *For any finite-dimensional vector space E over $\mathbb{K}$, $P(S(E)) = E$.*

PROOF. Suppose x is a non-zero primitive element of degree $k > 1$. Let e_a be a basis of E, with dual basis e^a. The component of $\Delta(x)$ in $S^{k-1}(E) \otimes E$ is given by $\sum_a \iota_S(e^a)x \otimes e_a$, where $\iota_S(\mu)$ is the derivation of $S(E)$ given on generators by $\iota_S(\mu)(\xi) = \langle \mu, \xi \rangle$. Thus $\iota_S(e^a)x = 0$ for all a, which proves $x = 0$. Alternatively: Identify $S(E) = \mathrm{Pol}(E^*)$ with polynomial functions. For $f \in \mathrm{Pol}(E^*)$, the definition of Δ becomes $(\Delta f)(\mu, \nu) = f(\mu + \nu)$ (since the dual map to diagonal inclusion is the addition map). Hence f is primitive is and only if $f(\mu + \nu) = f(\mu) + f(\nu)$. If $f \in \mathrm{Pol}^k(E^*)$, taking $\mu = \nu$ in this condition gives $2^k f(\mu) = 2f(\mu)$ for all μ, so $f = 0$ or $k = 1$. □

THEOREM 3.13. *For any Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, $P(U(\mathfrak{g})) = \mathfrak{g}$.*

PROOF. The PBW theorem says that $\mathrm{sym} : S(\mathfrak{g}) \to U(\mathfrak{g})$ is an isomorphism of co-algebras. Since the definition of primitive elements only involves the co-multiplication, this implies $P(U(\mathfrak{g})) = \mathrm{sym}(P(S(\mathfrak{g}))) = \mathrm{sym}(\mathfrak{g}) = \mathfrak{g}$. □

4. Petracci's proof of the PBW theorem

4.1. Co-derivations of $S(E)$. It is easy to see that the space of derivations of a symmetric algebra $S(E)$ is isomorphic to the space $\mathrm{Hom}(E, S(E))$: Any such homomorphism extends uniquely to a derivation. Dually, one expects that the space of co-derivations of $S(E)$ is isomorphic to the space $\mathrm{Hom}(S(E), E)$.

THEOREM 4.1. *There is a canonical isomorphism between the space of co-derivations D of $S(E)$ and the space of formal vector fields $X \in \mathrm{Hom}(S(E), E)$, given by*

$$D = m \circ (1 \otimes X) \circ \Delta.$$

PROOF. Evaluated on e^{tv}, the above formula reads,

$$D(e^{tv}) = e^{tv} X(e^{tv}).$$

We first show that if D is a co-derivation, then $X(e^{tv}) := e^{-tv} D(e^{tv})$ lies in $E \subset S(E)$. Equivalently, we show that $X(e^{tv})$ is primitive:

$$\begin{aligned}
\Delta(X(e^{tv})) &= \Delta(e^{-tv})\Delta(D(e^{tv})) \\
&= (e^{-tv} \otimes e^{-tv}) \cdot (D \otimes 1 + 1 \otimes D)\Delta(e^{tv}) \\
&= (e^{-tv} \otimes e^{-tv}) \cdot (D(e^{tv}) \otimes e^{tv} + e^{tv} \otimes D(e^{tv})) \\
&= e^{-tv} D(e^{tv}) \otimes 1 + 1 \otimes e^{-tv} D(e^{tv}) \\
&= X(e^{tv}) \otimes 1 + 1 \otimes X(e^{tv}).
\end{aligned}$$

Conversely, if $X \in \mathrm{Hom}(S(E), E)$, a similar calculation shows that $D(e^{tv}) := e^{tv} X(e^{tv})$ defines a co-derivation:

$$\begin{aligned}
\Delta(D(e^{tv})) &= (e^{tv} \otimes e^{tv})\Delta(X(e^{tv})) \\
&= (e^{tv} \otimes e^{tv})(X(e^{tv}) \otimes 1 + 1 \otimes X(e^{tv})) \\
&= D(e^{tv}) \otimes e^{tv} + e^{tv} \otimes D(e^{tv}) \\
&= (D \otimes 1 + 1 \otimes D) \circ \Delta(e^{tv}).
\end{aligned}$$

□

The Lie bracket on $\mathrm{Hom}(S(E),E)$ induced by this isomorphism reads,

$$[X_1,X_2](e^{tv}) = X_1(e^{tv}X_2(e^{tv})) - X_2(e^{tv}X_1(e^{tv})).$$

From now on, we will simplify notation and omit the parameter t, and simply write $X(e^v)$ for the "value" of X at $e^v = \sum_{j=0}^{\infty}\frac{1}{j!}v^j$.

4.2. A $\mathfrak{g}$-representation by co-derivations. The main idea in Petracci's approach to the PBW theorem is to define a $\mathfrak{g}$-representation on $S(\mathfrak{g})$, which under symmetrization goes to the left-regular representation of $\mathfrak{g}$ on $U\mathfrak{g}$. This representation should be by coderivations, since $\rho^L(\xi)$ is a coderivation and sym is a co-algebra homomorphism. For $\zeta\in\mathfrak{g}$, and any formal power series $\phi\in\mathbb{K}[[z]]$, define $X^\zeta_\phi\in\mathrm{Hom}(S\mathfrak{g},\mathfrak{g})$ by

$$X^\zeta_\phi(e^\xi) = \phi(\mathrm{ad}_\xi)(\zeta).$$

THEOREM 4.2 (Petracci). *Let $X^\zeta = X^\zeta_\phi$ for $\phi(z) = j^R(z)^{-1} = \frac{z}{e^z-1}$. Then the map $\mathfrak{g}\to\mathfrak{X}_\mathfrak{g}$, $\zeta\mapsto X^\zeta$ is a Lie algebra homomorphism:*

$$[X^{\zeta_1},X^{\zeta_2}] = X^{[\zeta_1,\zeta_2]}.$$

Indeed, ϕ is the unique formal power series with $\phi(0)=1$ having this property.

Hence, $\zeta\mapsto X^\zeta\in\mathrm{Hom}(S(E),E)$ defines a $\mathfrak{g}$-representation on $S\mathfrak{g}$ by coderivations. Note that the formula for X^ζ is similar to our formula for the right-invariant vector field on a Lie group G, written in 'exponential coordinates. [2] However, the present formulation is completely algebraic, and works for any field of characteristic 0.

The proof of Theorem 4.2 requires the computation of commutators of vector fields of the form $X^\zeta_\phi(e^{t\xi}) = \phi(t\,\mathrm{ad}_\xi)\zeta$. Let us introduce the following notation:

$$\left(z_1^{k_1}z_2^{k_2} : [a,b]\right)_\xi = [\mathrm{ad}_\xi^{k_1}(a),\mathrm{ad}_\xi^{k_2}(b)].$$

Extend by linearity to formal power series in z_1,z_2. The following Lemma shows the usefulness of this notation:

LEMMA 4.3. *For any $\phi\in\mathbb{K}[[z]]$, and any $a,b\in\mathfrak{g}$,*

$$\phi(\mathrm{ad}_\xi)[a,b] = \Big(\phi(z_1+z_2):\ [a,b]\Big)_\xi.$$

PROOF. By the Jacobi identity, $\mathrm{ad}_\xi[a,b] = [\mathrm{ad}_\xi a,b]+[a,\mathrm{ad}_\xi b]$ and induction,

$$\mathrm{ad}_\xi^n[a,b] = \sum_{i=0}^{n}[\mathrm{ad}_\xi^i a,\mathrm{ad}_\xi^{n-i} b] = \big((z_1+z_2)^n:\ [a,b]\big)_\xi.$$

□

LEMMA 4.4. *For all $\phi\in\mathbb{K}[[z]]$, $\zeta\in\mathfrak{g}$, $Y\in\mathfrak{g}$ one has the following identity,*

$$X^\zeta_\phi(e^\xi Y) = \Big(\frac{\phi(z_1+z_2)-\phi(z_2)}{z_1}:\ [Y,\zeta]\Big)_\xi. \tag{27}$$

[2] Signs ?

PROOF. It suffices to consider the case $\phi(z) = z^n$, which is an induction on n: The cases $n = 0, 1$ are clear, while

$$\begin{aligned}
X^{\zeta}_{z^{n+1}}(e^{\xi}Y) &= \frac{\partial}{\partial s}|_{s=0}\,\mathrm{ad}^{n+1}(\xi + sY)\zeta \\
&= [Y, \mathrm{ad}^n(\xi)\zeta] + \mathrm{ad}(\xi)\frac{\partial}{\partial s}|_{s=0}\,\mathrm{ad}^{n+1}(\xi + sY)\zeta \\
&= \Big(z_2^n : [Y,\zeta]\Big)_{\xi} + \mathrm{ad}(\xi)\Big(\frac{(z_1+z_2)^n - z_2^n}{z_1} : [Y,\zeta]\Big)_{\xi} \\
&= \Big(z_2^n + (z_1+z_2)\frac{(z_1+z_2)^n - z_2^n}{z_1} : [Y,\zeta]\Big)_{\xi} \\
&= \Big(\frac{(z_1+z_2)^{n+1} - z_2^{n+1}}{z_1} : [Y,\zeta]\Big)_{\xi}.
\end{aligned}$$

□

LEMMA 4.5. *The commutator of vector fields $X^{\zeta_1}_{\phi_1}$ and $X^{\zeta_2}_{\phi_2}$ is given by the formula,*

$$[X^{\zeta_1}_{\phi_1}, X^{\zeta_2}_{\phi_2}](e^{\xi}) = \Big(\frac{\phi_1(z_1+z_2) - \phi_1(z_1)}{z_2}\phi_2(z_2) + \frac{\phi_2(z_1+z_2) - \phi_2(z_2)}{z_1}\phi_1(z_1) : [\zeta_2, \zeta_1]\Big)_{\xi}.$$

PROOF. From (27) with $\zeta = \zeta_1, \phi = \phi_1, Y = X^{\zeta_2}_{\phi_2}$, we obtain

$$X^{\zeta_1}_{\phi_1}(e^{\xi}X^{\zeta_2}_{\phi_2}(e^{\xi})) = \Big(\frac{\phi_1(z_1+z_2) - \phi_1(z_2)}{z_1}\phi_2(z_2) : \ [\zeta_2, \zeta_1]\Big)_{\xi},$$

and hence the formula for the commutator. □

PROOF OF PETRACCI'S THEOREM. Since

$$X^{[\zeta_1,\zeta_2]}_{\phi}(e^{\xi}) = \phi(\mathrm{ad}_{\xi})[\zeta_1, \zeta_2] = \big(\phi(z_1+z_2) : \ [\zeta_1, \zeta_2]\big)_{\xi},$$

we see that $\zeta \mapsto X^{\zeta}_{\phi}$ is a Lie algebra homomorphism if and only if ϕ satisfies the functional equation,

$$\phi(z_1+z_2) + \frac{\phi(z_1+z_2) - \phi(z_2)}{z_1}\phi(z_1) + \frac{\phi(z_1+z_2) - \phi(z_1)}{z_2}\phi(z_2) = 0$$

This functional equation has a unique solution for any given initial condition $\phi(0)$. Letting $z_2 \to 0$ this turns into a differential equation,

$$\phi(z) + \frac{\phi(z) - \phi(0)}{z}\phi(z) + \phi'(z)\phi(0) = 0.$$

If $\phi(0) = 0$ this gives $\phi(z) = -\frac{\phi(z)^2}{z}$, i.e. $\phi(z) = -z$. If $\phi(0) \neq 0$, introduce $\tilde{\phi}(z) = \frac{z}{\phi(z)} + 1$. It is straightforward to check that the resulting equation for $\tilde{\phi}$ is just the usual functional equation for the exponential function, $\tilde{\phi}(z_1+z_2) - \tilde{\phi}(z_1)\tilde{\phi}(z_2) = 0$. Hence $\tilde{\phi}(z) = e^{cz}$ for some constant c, and therefore $\phi(z) = \frac{z}{e^{cz}-1}$, with initial condition $\phi(0) = \frac{1}{c}$. In particular, there is a unique solution with $\phi(0) = 1$. □

This completes the construction of the representation $\rho\colon \mathfrak{g} \to \mathrm{Hom}(S\mathfrak{g}, S\mathfrak{g})$ by coderivations, $\rho(\zeta)(e^{\xi}) = e^{\xi}\phi(\mathrm{ad}_{\xi})\zeta$. Since $\frac{z}{e^z-1}$ is the generating function for

Bernoulli numbers, we have the explicit formula,

$$\rho(\zeta)(\xi^n) = \sum_{k=0}^{n} \binom{n}{k} B_k \, \xi^{n-k} \operatorname{ad}_\xi^k(\zeta).$$

where B_k are the Bernoulli numbers. For example,

$$\begin{aligned}
\rho(\zeta)(1) &= \zeta, \\
\rho(\zeta)(\xi) &= \xi\zeta - \tfrac{1}{2}[\xi, \zeta], \\
\rho(\zeta)(\xi^2) &= \xi^2\zeta - \xi[\xi, \zeta] + \tfrac{1}{6}[\xi, [\xi, \zeta]] \\
&\cdots
\end{aligned}$$

Extend to an algebra homomorphism $\rho\colon U\mathfrak{g} \to \operatorname{Hom}(S\mathfrak{g}, S\mathfrak{g})$. The *symbol map* for the enveloping algebra is defined as

$$\sigma\colon U\mathfrak{g} \to S\mathfrak{g}, \quad x \mapsto \rho(x).1$$

THEOREM 4.6. *The symbol map is an inverse to the symmetrization map.*

PROOF. Since we already know that the symmetrization map is onto, it suffices to show the symbol map is left inverse to the symmetrization map. That is, it is enough to show that $\rho(\zeta^n)(1) = \zeta^n$ for all n. Setting $\xi = \zeta$ in the formula for $\rho(\zeta)(\xi^n)$, only the term $k = 0$ contributes, and gives $\rho(\zeta)(\zeta^n) = \zeta^{n+1}$. Hence, by induction

$$\rho(\zeta^n)(1) = \rho(\zeta)^n(1) = \zeta^n.$$

□

REMARK 4.7. The representation given and by the co-derivation $\phi(u) = -u$ is just the adjoint representation of $\mathfrak{g}$ on $S\mathfrak{g}$.

5. The center of the enveloping algebra

We have already observed that the center of the enveloping algebra $U\mathfrak{g}$ is just the ad-invariant subspace, $(U\mathfrak{g})^{\mathfrak{g}}$. Elements of the center are also called *Casimir elements*. If $\mathfrak{g}$ admits an invariant quadratic form, then $\frac{1}{2} e_a e^a \in U\mathfrak{g}$ (with e_a, e^a B-dual bases of $\mathfrak{g}$) is an example of a Casimir element, called the *quadratic Casimir*.

Suppose $\rho\colon \mathfrak{g} \to \operatorname{End}(E)$ is a $\mathfrak{g}$-representation, and extend to a representation $\rho\colon U\mathfrak{g} \to \operatorname{End}(E)$. Then for all $x \in \operatorname{Cent}(U\mathfrak{g})$, the operator $\rho(x)$ commutes with all $\rho(\xi), \xi \in \mathfrak{g}$:

$$[\rho(x), \rho(\xi)] = \rho([x, \xi]) = 0.$$

If ρ is irreducible (i.e. ρ has no non-trivial sub-representations), and if our ground files is $\mathbb{K} = \mathbb{C}$, this implies (Schur's lemma) that $\rho(x)$ is a multiple of the identity. That is, for any irreducible representation one obtains an *algebra* homomorphism

$$\operatorname{Cent}(U\mathfrak{g}) \to \mathbb{K}, \; x \mapsto \rho(x).$$

For semi-simple Lie algebras, it is known that this algebra homomorphism characterizes ρ up to isomorphism. In fact, it suffices to know this map on a set of generators for $\operatorname{Cent}(U\mathfrak{g})$. For example, if $\mathfrak{g} = \mathfrak{su}(2)$ any irreducible representation is determined by the value of the quadratic Casimir in this representation.

In terms of the identification $U\mathfrak{g} \cong \mathfrak{D}^L(G)$, the center corresponds to the space $\mathfrak{D}^{L\times R}(G)$ of bi-invariant differential operators. For instance, if G is a quadratic Lie group, the Laplace operator

$$D = \frac{1}{2}\sum_{ij} B(e^i, e^j) e_i^L e_j^L$$

(where $e^i \in \mathfrak{g}$ is the B-dual basis to e_i) defined by B is an example of a bi-invariant differential operator.

These examples motivate that one would like to understand the structure of $\mathrm{Cent}(U\mathfrak{g})$ as an algebra. The symmetrization map $\mathrm{sym}\colon S\mathfrak{g} \to U\mathfrak{g}$ restricts an isomorphism on invariants, $(S\mathfrak{g})^{\mathfrak{g}} \to (U\mathfrak{g})^{\mathfrak{g}} = \mathrm{Cent}(U\mathfrak{g})$. Unfortunately this restricted map is not an algebra homomorphism: For example, if $p = \frac{1}{2}\sum_a e_a e^a \in S^2\mathfrak{g}$ is the quadratic polynomial, one usually has $\mathrm{sym}(p^2) \neq \mathrm{sym}(p)^2$.

Duflo's theorem says that this can be fixed by pre-composing sym with a certain operator on $S\mathfrak{g}$. For any $\mu \in \mathfrak{g}^*$, let $\widehat{\mu} = \iota_S(\mu) \in \mathrm{End}(S\mathfrak{g})$ denote the derivation such that $\iota_S(\mu)\xi = \langle \mu, \xi\rangle$ for $\xi \in \mathfrak{g} = S^1\mathfrak{g}$. If we identify $S\mathfrak{g}$ with polynomials on $\mathfrak{g}^*$, then $\widehat{\mu}$ is the first order differential operator defined by μ. The map $\mathfrak{g}^* \to \mathrm{End}(S\mathfrak{g})$ extends to an algebra homomorphism $S\mathfrak{g}^* \to \mathrm{End}(S\mathfrak{g})$, $p \mapsto \widehat{p}$, whose image are the constant coefficient differential operators on $\mathfrak{g}^*$. But it extends even further to an algebra homomorphism

$$\mathrm{Hom}(S\mathfrak{g}, \mathbb{K}) \to \mathrm{End}(S\mathfrak{g}),\ p \mapsto \widehat{p}$$

from the completion of the symmetric algebra,

$$\overline{S}\mathfrak{g}^* = \mathrm{Hom}(S\mathfrak{g}, \mathbb{K}) = \prod_{k=0}^{\infty} S^k\mathfrak{g}^*.$$

One may think of $\overline{S}\mathfrak{g}^*$ as infinite-order differential operators; their action on polynomials is well-defined.

Let $J\colon \mathfrak{g} \to \mathbb{R}$ denote the function, $J(\xi) = \det(j(\mathrm{ad}_\xi))$, with $j(z) = \frac{\sinh(z/2)}{z/2}$, and let $J^{1/2}$ its square root (defined at least in a neighborhood of 0.) Taking the Taylor expansion of $J^{1/2}$ at the origin, we obtain an element of $\overline{S}\mathfrak{g}^*$, which we will again denote by $J^{1/2}$. Let Let $\widehat{J}^{1/2}$ denote the corresponding infinite-order differential operator, given informally by $J^{1/2}(\frac{\partial}{\partial\mu})$.

Theorem 5.1 (Duflo). *The composition*

$$\mathrm{sym} \circ \widehat{J^{1/2}}\colon S\mathfrak{g} \to U\mathfrak{g}$$

restricts to an algebra isomorphism $(S\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cent}(U\mathfrak{g})$.

In the following sections, we present a proof of Duflo's theorem for the case that $\mathfrak{g}$ is quadratic. This proof will relate the appearance of the factor $J^{1/2}$ in Duflo's theorem with that in the theory of Clifford algebras.

CHAPTER 6

Weil algebras

1. Differential algebras

A *differential space* is a $\mathbb{Z}_2$-graded vector space $E = E^{\bar{0}} \oplus E^{\bar{1}}$, together with an odd operator $\mathrm{d}\colon E \to E$ such that $\mathrm{d} \circ \mathrm{d} = 0$, i.e. $\mathrm{im}(\mathrm{d}) \subset \ker(\mathrm{d})$. One calls

$$H(E, \mathrm{d}) = \frac{\ker(\mathrm{d})}{\mathrm{im}(\mathrm{d})}$$

the cohomology of the differential space (E, d). It inherits a $\mathbb{Z}_2$-grading from E. If (E, d) is a *graded differential space*, that is if E is a $\mathbb{Z}$-graded vector space and the differential raises degree by 1, then $H(E, \mathrm{d})$ is a graded vector space.

If *differential algebra* is a $\mathbb{Z}_2$-graded vector algebra $\mathcal{A} = \mathcal{A}^{\bar{0}} \oplus \mathcal{A}^{\bar{1}}$, together with an odd derivation $\mathrm{d}\colon E \to E$ such that $\mathrm{d} \circ \mathrm{d} = 0$. In this case, $H(\mathcal{A}, \mathrm{d})$ inherits a $\mathbb{Z}_2$-graded algebra structure. if the $\mathbb{Z}_2$-grading comes from a $\mathbb{Z}$-grading, and d raises the degree by 1, then the cohomology is a graded algebra. (In a similar way, one defines *differential Lie algebras.*)

EXAMPLE 1.1. Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{A} = \wedge \mathfrak{g}^*$ the exterior algebra over the dual space. Recall that the space of derivations is isomorphic to $\mathrm{Hom}(\mathfrak{g}^*, \wedge \mathfrak{g}^*)$, since any derivation is determined by its value on generators. Let $\iota(\xi)$ be the derivation of degree -1, given by contraction. For any $\xi \in \mathfrak{g}$, let $L(\xi)$ denote the derivation of degree 0, given on generators by the co-adjoint representation:

$$L(\xi)\mu = -\operatorname{ad}^*_\xi \mu.$$

The commutators of Lie derivatives with contractions are $[L(\xi_1), \iota(\xi_2)] = \iota([\xi_1, \xi_2])$. Define a derivation d of degree 1 by

$$\iota(\xi)\mathrm{d}\mu := L(\xi)\mu.$$

Then $[\iota(\xi), \mathrm{d}] = L(\xi)$ and $[L(\xi), \mathrm{d}] = 0$ as one easily checks on generators. From these two equations, we get $[\iota(\xi), [\mathrm{d}, \mathrm{d}]] = 0$, hence for all $\mu \in \mathfrak{g}^*$

$$\iota(\xi)[\mathrm{d}, \mathrm{d}]\mu = [\mathrm{d}, \mathrm{d}]\iota(\xi)\mu = [\mathrm{d}, \mathrm{d}]\langle \mu, \xi \rangle = 0.$$

This shows that $2\mathrm{d} \circ \mathrm{d} = [\mathrm{d}, \mathrm{d}] = 0$ i.e. d is a differential. For later reference, let us summarize the commutation relations between the operators $\mathrm{d}, L(\xi), \iota(\xi)$:

$$\begin{aligned}
[\mathrm{d}, \mathrm{d}] &= 0 \\
[\iota(\xi), \mathrm{d}] &= L(\xi) \\
[L(\xi), \mathrm{d}] &= 0 \\
[L(\xi), L(\zeta) &= L([\xi, \zeta]) \\
[L(\xi), \iota(\zeta)] &= \iota([\xi, \zeta]) \\
[\iota(\xi), \iota(\zeta)] &= 0
\end{aligned}$$

The formula

$$\iota(\xi_1)\iota(\xi_2)\mathrm{d}(\mu) = -\langle \mu, [\xi_1, \xi_2]\rangle,$$

shows that d is dual to the Lie bracket; the condition $\mathrm{d} \circ \mathrm{d} = 0$ may be traced back to the Jacobi identity for $\mathfrak{g}$. The cohomology algebra $H(\wedge(\mathfrak{g}^*), \mathrm{d})$ is called the (Chevalley-Eilenberg) *Lie algebra cohomology* of $\mathfrak{g}$, and is denoted $H(\mathfrak{g})$. Using dual bases $e_i \in \mathfrak{g}$ and $e^i \in \mathfrak{g}^*$, the Lie algebra differential may be written

$$\mathrm{d} = \tfrac{1}{2} \sum_i \epsilon(e^i) \circ L(e_i),$$

as always this is checked on generators. In particular, we see that the $\mathfrak{g}$-invariants $(\wedge \mathfrak{g}^*)^{\mathfrak{g}}$ are all cocycles. For $\mathfrak{g}$ semi-simple, one can show that the inclusion of invariants induces an isomorphism in cohomology.

EXAMPLE 1.2. More generally, suppose $L_V : \mathfrak{g} \to \mathrm{End}(V)$ is any $\mathfrak{g}$-representation. Let

$$C^\bullet = V \otimes \wedge^\bullet \mathfrak{g}^*$$

with grading induced from the grading on the exterior algebra. We obtain a $\mathfrak{g}$-representation on the tensor product, with generators $L(\xi) = L_V(\xi) \otimes 1 + 1 \otimes L_\wedge(\xi)$. Let $\iota(\xi) = 1 \otimes \iota_\wedge(\xi)$, and define

$$\mathrm{d} = \sum_i L_V(e_i) \otimes \epsilon(e^i) + 1 \otimes \mathrm{d}_\wedge.$$

Then $[\mathrm{d}, L(\xi)] = 0$ since each of the terms in d is ad-equivariant, and also $[\mathrm{d}, \iota(xi)] = L_V(\xi) \otimes 1 + 1 \otimes L_\wedge(\xi)$. This implies

$$[\iota(\xi), [\mathrm{d}, \mathrm{d}]] = 2[L(\xi), \mathrm{d}] = 0,$$

i.e. $\iota(\xi)$ commutes with $[\mathrm{d}, \mathrm{d}] = 2\mathrm{d} \circ \mathrm{d}$. By an easy induction on the degree, this implies $\mathrm{d} \circ \mathrm{d} = 0$, as one may also verify by direct calculation. Note that $\iota(\xi), L(\xi), \mathrm{d}$ have the same commutation relations as in the special case $V = \mathbb{K}$. Note also that C^0 consists of the invariants $V^{\mathfrak{g}} \subset V$, hence

$$H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}.$$

EXAMPLE 1.3. Suppose $\mathfrak{g}$ carries an invariant quadratic form B, used to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Then $\mathrm{ad}_\xi \in \mathfrak{o}(\mathfrak{g})$ for all $\xi \in \mathfrak{g}$. Let $\lambda(\xi) = \lambda(\mathrm{ad}_\xi) \in \wedge^2 \mathfrak{g}$ be as defined in (19). Thus, by definition

$$-\iota(\zeta)\lambda(\xi) = \{\lambda(\xi), \zeta\} = [\xi, \zeta].$$

That is, $L(\xi) = \{\lambda(\xi), \cdot\}$. Define $\phi \in \wedge^3 \mathfrak{g}$ in terms of contractions by

$$\iota(\xi)\phi \equiv \{\phi, \xi\} = \lambda(\xi).$$

The fact that the map $\lambda \colon \mathfrak{g} \to \wedge^2 \mathfrak{g}$ is ad-equivariant implies that ϕ is ad-invariant:

$$\iota(\xi)L(\zeta)\phi = L(\zeta)\iota(\xi)\phi - \iota([\zeta, \xi])\phi = L(\zeta)\lambda(\xi) - \lambda([\zeta, \xi]) = 0.$$

Hence $\mathrm{d}\phi = 0$. In fact, we have

$$\mathrm{d} = \{\phi, \cdot\}$$

as we may once again verify by checking on generators: $\mathrm{d}\xi = \lambda(\xi) = \{\phi, \xi\}$. The relations between $\iota(\xi), L(\xi), \mathrm{d}$ are now lifted to Poisson bracket relations between elements of $\wedge\mathfrak{g}$:

$$\begin{aligned}
\{\phi,\phi\} &= 0,\\
\{\phi,\xi\} &= \lambda(\xi),\\
\{\phi,\lambda(\xi)\} &= 0,\\
\{\lambda(\xi),\lambda(\zeta)\} &= \lambda([\xi,\zeta]),\\
\{\lambda(\xi),\zeta\} &= [\xi,\zeta],\\
\{\xi,\zeta\} &= B(\xi,\zeta).
\end{aligned}$$

These identities quantize to the Clifford algebra. The Clifford commutator $[q(\phi), q(\phi)]$ is a (possibly non-zero) scalar since $[\xi,[q(\phi),q(\phi)]] = 2[[\xi,q(\phi)],q(\phi)] = 2[\gamma(\xi),q(\phi)] = 0$. With some extra effort, one finds that this scalar is $-\frac{1}{24}\operatorname{tr}(\mathrm{Cas}_\mathfrak{g})$, where $\mathrm{Cas}_\mathfrak{g}$ is the Casimir element, and the trace is taken in the adjoint representation. (We will not explain the calculation here: at this point all that matters is that it is a constant.) One obtains,

$$\begin{aligned}
[q(\phi),q(\phi)]_{\mathrm{Cl}} &= -\frac{1}{24}\operatorname{tr}(\mathrm{Cas}_\mathfrak{g})\\
[q(\phi),\xi]_{\mathrm{Cl}} &= L(\xi),\\
[q(\phi),\gamma(\xi)]_{\mathrm{Cl}} &= 0,\\
[\gamma(\xi),\zeta]_{\mathrm{Cl}} &= [\xi,\zeta]_\mathfrak{g}\\
[\gamma(\xi),\gamma(\zeta)]_{\mathrm{Cl}} &= \gamma([\xi,\zeta]_\mathfrak{g}),\\
[\xi,\zeta]_{\mathrm{Cl}} &= B(\xi,\zeta).
\end{aligned}$$

As a consequence, $\mathrm{Cl}(\mathfrak{g})$ is a differential algebra with

$$\mathrm{d}_{\mathrm{Cl}} = [q(\phi),\cdot]$$

is a differential on $\mathrm{Cl}(\mathfrak{g})$, satisfying the same commutator relations with $L(\xi), \iota(\xi)$ as for the exterior algebra. (The cohomology of $(\mathrm{Cl}(\mathfrak{g}),\mathrm{d})$ is zero if $\operatorname{tr}(\mathrm{Cas}_\mathfrak{g}) \neq 0$. This follows because in this case, one can construct a homotopy operator using the invertibility of $q(\phi)$.)

EXAMPLE 1.4. If E is any differential space, the symmetric and tensor algebras $S(E), T(E)$ are differential algebras. Here $S(E)$ is defined using the super-sign convention: That is, $S(E) = S(E^{\bar 0}) \otimes \wedge(E^{\bar 1})$. If E is $\mathbb{Z}$-graded then $S(E)$, $T(E)$ inherit a $\mathbb{Z}$-grading. The differential d on E is extended to $S(E)$ respectively $T(E)$ as a derivation: The identity $\mathrm{d}\circ\mathrm{d}$ follows because $2\mathrm{d}\circ\mathrm{d} = [\mathrm{d},\mathrm{d}]$ is a derivation, hence determined by its values on generators.

Suppose V be a given (ungraded) vector space, and define a graded vector space E_V by $E_V^1 = V$, $E_V^2 = V$ and $E_V^i = 0$ for $i \neq 0$. For $v \in V$, use the same notation for the corresponding element in $E_V^{\bar 1}$, and write $\overline{v}$ for the element in $E_V^{\bar 0}$. Then

$$\mathrm{d}v = \overline{v},\ \ \mathrm{d}\overline{v} = 0$$

defines a differential space, with trivial cohomology. The differential algebra

$$S(E_V) = S(V) \otimes \wedge(V)$$

is called the *Koszul algebra*. It is characterized by a universal property: For any commutative differential algebra $(\mathcal{A},\mathrm{d})$ and any linear map $V \to \mathcal{A}^{\bar 1}$ (taking values in the odd part) there is a unique extension to a homomorphism of differential algebras $S(E_V) \to \mathcal{A}$.

One can also consider a non-commutative version of the Koszul algebra, using the tensor algebra $T(E_V)$ rather than the symmetric algebra. This then has a similar universal property for non-commutative differential algebras $(\mathcal{A}, \mathrm{d})$.

2. Cochain maps, homotopy operators

A *morphism of differential spaces* (E_1, d_1) and (E_2, d_2) is an even linear map $\phi\colon E_1 \to E_2$ such that $\phi \circ \mathrm{d}_1 = \mathrm{d}_2 \circ \phi$. Morphisms are also called *cochain maps.* A morphism of differential algebras $(\mathcal{A}_1, \mathrm{d}_1)$ and $(\mathcal{A}_2, \mathrm{d}_2)$ is a cochain map which is also an algebra homomorphism. In the $\mathbb{Z}$-graded case, one requires in addition that ϕ has degree 0.

Any morphism ϕ of differential spaces (resp. algebras) induces a morphism of vector spaces (resp. algebras) in cohomology, $H(\phi)\colon H(E_1, \mathrm{d}_1) \to H(E_2, \mathrm{d}_2)$.

Two morphisms $\phi, \phi'\colon E_1 \to E_2$ of differential spaces are called *homotopic* if there exists an odd linear map $h\colon E_1 \to E_2$ (called *homotopy operator*) with

$$h \circ \mathrm{d}_1 + \mathrm{d}_2 \circ h = \phi - \phi'.$$

(If E_i are $\mathbb{Z}$-graded, one usually requires that h has degree -1.)

Given a homotopy operator h, it follows in particular that $\phi - \phi'$ takes $\ker(\mathrm{d}_1)$ to $\mathrm{im}(\mathrm{d}_2)$. That is, $H(\phi) - H(\phi') = H(\phi - \phi') = 0$. Two morphisms $\phi\colon E_1 \to E_2$ and $\psi\colon E_2 \to E_1$ are called *homotopy inverses* if $\phi \circ \psi$ and $\psi \circ \phi$ are both homotopic to the identity maps of E_2, E_1, respectively. In this case, $H(\phi)$ induces an isomorphism in cohomology, with inverse $H(\psi)$. A morphism ϕ admitting a homotopy inverse is also called a *homotopy equivalence.*

EXAMPLE 2.1. The space E_V has trivial cohomology, since the identity map is homotopy equivalent to the zero map. Indeed, let $s\colon E_V \to E_V$ be the map of degree -1, $s(\overline{v}) = v$, $s(v) = 0$. Then $[\mathrm{d}, s] = \mathrm{id}$ on E_V. Consider next the Koszul algebra $S(E_V)$. We want to show that the inclusion of scalars $\iota\colon \mathbb{K} \to S(E_V)$ and the augmentation map $\pi\colon S(E_V) \to \mathbb{K}$ are homotopy inverses. Consider the derivation extension of $s \in \mathrm{End}(E_V)$ to $S(E_V)$. Then $[\mathrm{d}, s]$ is the derivation extension of the identity map $\mathrm{id}\colon E_V \to E_V$. Thus

$$[\mathrm{d}, s]\Big|_{S^k(E_V)} = k.$$

(here k should not be confused with our choice of grading on the Koszul algebra). It follows that the operator

$$[\mathrm{d}, s] + i \circ \pi$$

on $S(E)$ is invertible: Its inverse is equal to $\frac{1}{k}$ on $S^k(E_V)$ for $k > 0$, and equal to 1 on $S^0(V)$.

Note that $[\mathrm{d}, s]$ commutes with d (e.g. by check on generators), hence $[\mathrm{d}, s] + i \circ \pi$ is a morphism of differential spaces. The operator

$$h = s \circ ([\mathrm{d}, s] + i \circ \pi)^{-1}$$

is a homotopy equivalence between id and $i \circ \pi$, by the following calculation:

$$\begin{aligned}[h, \mathrm{d}] &= [s, \mathrm{d}] \circ ([\mathrm{d}, s] + i \circ \pi)^{-1} \\ &= \mathrm{id} - i \circ \pi \circ ([\mathrm{d}, s] + i \circ \pi)^{-1} \\ &= \mathrm{id} - i \circ \pi.\end{aligned}$$

Thus, $i\colon \mathbb{K} \to \mathcal{A}$ is a homotopy equivalence: the Koszul algebra is *acyclic*. A similar proof shows that the non-commutative Koszul algebra $T(E_V)$ is acyclic as well: Just replace 'S' with 'T' everywhere.

EXAMPLE 2.2 (Stokes formula). Let $\mathcal{S}$ be the commutative differential algebra, with generators t of degree 0 and $\mathrm{d}t$ of degree 1. A general element of $\mathcal{S}$ is a linear combination

$$y = \sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t \tag{28}$$

One can think of $\mathcal{S}$ as differential forms with polynomial coefficients. Let $\pi_0, \pi_1\colon \mathcal{S} \to \mathbb{K}$ be the two cochain maps, given on the element (28) by

$$\pi_0(y) = a_0, \quad \pi_1(y) = \sum_k a_k.$$

(Think of these as 'evaluations at $t = 0, 1$'.) Note that π_0, π_1 are both cochain maps. A homotopy between these two cochain maps is given by the "integration operator" $J\colon \mathcal{S} \to \mathbb{K}$,

$$J(y) = \sum_l \frac{b_l}{l+1}.$$

The homotopy identity $[\mathrm{d}, J] = \pi_1 - \pi_0$ is just the Stokes' theorem. Indeed,

$$[J, \mathrm{d}](y) = J(\mathrm{d}y) = J(\sum_{k>0} k a_k t^{k-1}\mathrm{d}t) = \sum_{k>0} a_k = (\pi_1 - \pi_0)(y).$$

We conclude by listing some further properties of homotopy operators. Their proof is straightforward.

PROPOSITION 2.3. *Let $h\colon E_1 \to E_2$ be a homotopy between two cochain maps $\phi, \phi'\colon E_1 \to E_2$.*

1. *For any differential space F, the map $h \otimes 1\colon E_1 \otimes F \to E_2 \otimes F$ is a homotopy between $\phi \otimes 1$ and $\phi' \otimes 1$.*
2. *For any cochain map $g\colon E_2 \to E_3$, the map $g \circ h\colon E_1 \to E_3$ is a homotopy between $g \circ \phi$, $g \circ \phi'$.*
3. *For any cochain map $f\colon E_0 \to E_1$, the map $h \circ f\colon E_0 \to E_2$ is a homotopy between $\phi \circ f$, $\phi' \circ f$.*

As an application we can prove:

THEOREM 2.4. *Let $\mathcal{A}$ be a commutative differential algebra. Then any two homomorphisms of differential algebras $\phi_0, \phi_1\colon S(E_V) \to \mathcal{A}$ are (canonically) homotopic. Similarly, for every differential algebra $\mathcal{A}$, any two homomorphisms of differential algebras $\phi_0, \phi_1\colon T(E_V) \to \mathcal{A}$ are (canonically) homotopic.*

PROOF. We present the proof for $T(E_V)$. (The proof for $S(E_V)$ is parallel.) let $j_0, j_1\colon V \to \mathcal{A}^{\bar{1}}$ be the restrictions of ϕ_0, ϕ_1 to $V \subset T(E_V)^{\bar{1}}$. Define a linear map

$$j\colon V \to \mathcal{S} \otimes \mathcal{A}, \ v \mapsto (1-t) \otimes j_0(v) + t \otimes j_1(v),$$

and extend to a homomorphism of differential algebras $\phi\colon T(E_V) \to \mathcal{S} \otimes \mathcal{A}$. Then

$$(\pi_0 \otimes 1) \circ j = j_0, \quad (\pi_1 \circ j) = j_1,$$

and hence $(\pi_0 \circ \phi) = \phi_0$, $(\pi_1 \circ \phi) = \phi_1$. Let $h = (J \otimes 1) \circ \phi$. Then

$$[\mathrm{d}, h] = ([\mathrm{d}, J] \otimes 1) \circ \phi = (\pi_1 - \pi_0) \otimes 1 \circ \phi = \phi_1 - \phi_0,$$

as required. □

3. $\mathfrak{g}$-differential algebras

Suppose that $\mathfrak{g}$ is a Lie algebra.

DEFINITION 3.1. A $\mathfrak{g}$-*differential space* is a differential space (E, d), together with the following extra structure: A linear map $L\colon \mathfrak{g} \to \mathrm{End}(E)$, where all $L(\xi)$ are even, a linear map $\iota\colon \mathfrak{g} \to \mathrm{End}(E)$, where all $\iota(\xi)$ are odd, such that

$$\begin{aligned}
[\mathrm{d}, \mathrm{d}] &= 0 \\
[\iota(\xi), \mathrm{d}] &= L(\xi) \\
[L(\xi), \mathrm{d}] &= 0 \\
[L(\xi), L(\zeta)] &= L([\xi, \zeta]) \\
[L(\xi), \iota(\zeta)] &= \iota([\xi, \zeta]) \\
[\iota(\xi), \iota(\zeta)] &= 0
\end{aligned}$$

The operators $\iota(\xi)$ are called contractions, the operators $L(\xi)$ are called Lie derivatives. A $\mathfrak{g}$-*differential algebra* is a $\mathbb{Z}_2$-graded algebra $\mathcal{A}$, with the structure of a $\mathfrak{g}$-differential space where $L(\xi), \iota(\xi)$ are derivations.

Sometimes we also consider graded $\mathfrak{g}$-differential spaces and algebras, where we assume in addition that E resp. $\mathcal{A}$ carries a grading and that $\mathrm{d}, L(\xi), \iota(\xi)$ have degrees $1, 0, -1$.

REMARKS 3.2. The first equation $[\mathrm{d}, \mathrm{d}] = 0$ just restates that d is a differential. $[L(\xi), L(\zeta)] = L([\xi, \zeta])$ says that L is a representation of $\mathfrak{g}$, and $[L(\xi), \mathrm{d}] = 0$ means that d is $\mathfrak{g}$-equivariant, or in other words that each $L(\xi)$ is a cochain map. The condition $[\iota(\xi), \mathrm{d}] = L(\xi)$ means that the contractions are homotopy operators for the chain maps $L(\xi)$: In particular, $L(\xi)$ induces the 0 action on cohomology.

Before giving examples, let us introduce one more concept:

DEFINITION 3.3. A *connection* on a $\mathfrak{g}$-differential algebra $\mathcal{A}$ is a $\mathfrak{g}$-equivariant linear map $\theta\colon \mathfrak{g}^* \to \mathcal{A}^{\bar{1}}$ with the property $\iota(\xi)\theta(\mu) = \langle \mu, \xi \rangle$. (If $\mathcal{A}$ is $\mathbb{Z}$-graded, we require that $\theta(\mu) \in \mathcal{A}^1$.)

Sometimes it is more convenient to view the equivariant map $\theta\colon \mathfrak{g}^* \to \mathcal{A}^{\bar{1}}$ as an invariant element $\theta \in \mathcal{A}^{\bar{1}} \otimes \mathfrak{g}$. The defining condition the reads, $(\iota(\xi) \otimes 1)\theta = 1 \otimes \xi$.

EXAMPLE 3.4. We have already seen that $\wedge\mathfrak{g}^*$ and $V \otimes \wedge\mathfrak{g}^*$ are $\mathfrak{g}$-differential spaces. If V is an algebra and $\mathfrak{g}$ acts by derivations, then $V \otimes \wedge\mathfrak{g}^*$ is a $\mathfrak{g}$-differential algebra. The map $\theta(\mu) = 1 \otimes \mu$ defines a connection.

EXAMPLE 3.5. If M is a manifold with an action of a Lie group G, the algebra $\mathcal{A} = \Omega(M)$ of differential forms is a $\mathfrak{g} - \mathrm{da}$, with d the de Rham differential and $\iota(\xi), L(\xi)$ the contractions and Lie derivatives for the generating vector fields. One can show that if G is compact, then $\mathcal{A}$ admits a connection if and only if the action is *locally free*, in the sense that for all $\xi \neq 0$ the generating vector field ξ_M is non-zero everywhere.

DEFINITION 3.6. Let E be a $\mathfrak{g}$-differential space. One defines the *basic subcomplex* E_{bas} to be the subspace of all $x \in E$ with $\iota(\xi)x = 0$ and $L(\xi)x = 0$ for all ξ.

Note that this is indeed a subcomplex: if $x \in E_{\mathrm{bas}}$ then $\mathrm{d}x \in E_{\mathrm{bas}}$ since $L(\xi)\mathrm{d}x = \mathrm{d}L(\xi)x = 0$ and $\iota(\xi)\mathrm{d}x = L(\xi)x - \mathrm{d}\iota(\xi)x = 0$. One calls $H_{\mathrm{bas}}(E) := H(E_{\mathrm{bas}})$ the *basic cohomology* of E. A morphism of $\mathfrak{g}$-differential spaces $E_1 \to E_2$ induces a morphism of differential spaces $(E_1)_{\mathrm{bas}} \to (E_2)_{\mathrm{bas}}$, hence of the basic cohomology.

EXAMPLE 3.7. Suppose V is a $\mathfrak{g}$-module, with a $\mathfrak{g}$-representation by derivations. An element of $V \otimes \wedge\mathfrak{g}^*$ is annihilated by all contractions if and only if it is contained in $V \otimes \wedge^0\mathfrak{g}^* = V$. Thus $(V \otimes \wedge\mathfrak{g}^*)_{\mathrm{bas}} = V^{\mathfrak{g}}$ are the $\mathfrak{g}$-invariants. The differential on this space is just zero, so $H_{\mathrm{basic}}(V \otimes \wedge\mathfrak{g}^*)_{\mathrm{bas}} = V^{\mathfrak{g}}$.

4. The Weil algebra

Consider the Koszul differential d_K on the tensor product $S(E_{\mathfrak{g}^*}) = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$. A before, we associate to each $\mu \in \mathfrak{g}^*$ the degree 1 generators $\mu \in \wedge^1\mathfrak{g}^*$ and the degree 2 generators $\overline{\mu} \in S^1\mathfrak{g}^*$. The Koszul algebra also carries a natural $\mathfrak{g}$-representation, given as the tensor product of the co-adjoint representations on $S\mathfrak{g}^*$ and $\wedge\mathfrak{g}^*$. Can we turn $S(E_{\mathfrak{g}^*})$ into a $\mathfrak{g}$-differential algebra?

To describe $\iota(\xi)$, we have to declare its action on generators $\overline{\mu}$ and μ. In fact, the action on $\overline{\mu}$ is determined since $\overline{\mu} = \mathrm{d}\mu$ we are forced to take

$$\iota(\xi)\overline{\mu} = \iota(\xi)\mathrm{d}\mu = L(\xi)\mu - \mathrm{d}\iota(\xi)\mu = L(\xi)\mu.$$

For the degree 1 generators, it is natural to take $\iota(\xi)\mu = \langle \mu, \xi\rangle$. It is straightforward to check the relations involving $\iota(\xi)$ on generators, so that we have turned $S(E_{\mathfrak{g}^*})$ into a $\mathbb{Z}$-graded $\mathfrak{g}$-differential algebra, with $\theta(\mu) = \mu \in \wedge^1\mathfrak{g}^*$ as a connection. A similar prescription turns the non-commutative Koszul algebra $T(E_{\mathfrak{g}^*})$ into a $\mathbb{Z}$-graded $\mathfrak{g}$-differential algebra.

DEFINITION 4.1. The Koszul algebra $S(E_{\mathfrak{g}^*})$ with this structure of a $\mathfrak{g}$-differential algebra is called the *Weil algebra*, and is denoted $W\mathfrak{g}$. Similarly $\tilde{W}\mathfrak{g} := T(E_{\mathfrak{g}^*})$ is called the noncommutative *Weil algebra*.[1]

THEOREM 4.2. *The basic subcomplex of the Weil algebra $W\mathfrak{g}$ is the space $(S\mathfrak{g}^*)^{\mathfrak{g}}$, where $S\mathfrak{g}^*$ is the symmetrix algebra generated by the variables*

$$\widehat{\mu} := \overline{\mu} - \lambda(\mu).$$

Here $\lambda(\mu) \in \wedge^2\mathfrak{g}^*$ is defined by $\iota(\xi)\lambda(\mu) = L(\xi)\mu$. The differential on the basic subcomplex is just 0, so

$$H_{\mathrm{bas}}(W\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}.$$

PROOF. Identify $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ where $S\mathfrak{g}^*$ is the symmetric algebra generated by the elements $\widehat{\mu}$, and $\wedge\mathfrak{g}^*$ is the exterior algebra generated by the variables μ. In terms of the new generators $\mu, \widehat{\mu}$ the formulas for contractions simplify to

$$\iota(\xi)\mu = \langle \mu, \xi\rangle, \quad \iota(\xi)\widehat{\mu} = 0.$$

It follows that the subalgebra annihilated by all contractions is $S\mathfrak{g}^* \otimes \wedge^0\mathfrak{g}^* = S\mathfrak{g}^*$, and hence the basic subcomplex is the invariant part, $(W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)^{\mathfrak{g}}$. Note that the basic subcomplex is entirely comprised of *even* elements. Since the differential d is odd, it must therefore vanish on $(S\mathfrak{g}^*)^{\mathfrak{g}}$. □

[1] The algebra $\tilde{W}\mathfrak{g}$ is different from the non-commutative Weil algebra of [?], which we will discuss below under a different name, the *quantized Weil algebra* $\mathcal{W}\mathfrak{g}$.

The computation of the algebra $H_{\mathrm{bas}}(\tilde{W}\mathfrak{g})$ is less obvious. We will prove later that the quotient map $\tilde{W}\mathfrak{g} \to W\mathfrak{g}$ induces an isomorphism in basic cohomology, so that $H_{\mathrm{bas}}(\tilde{W}\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$ as well.

Note that we did not have to compute the differential in the new variables, even though this is of course straightforward: [2]

THEOREM 4.3. *The differential d on $W\mathfrak{g}$ is a sum of two commuting differentials $d = d_1 + d_2$, where d_1 is the Koszul differential for the variables $\mu, \widehat{\mu}$, i.e.*

$$d_1\mu = \widehat{\mu}, \;\; d_1\widehat{\mu} = 0,$$

while d_2 is the (Chevalley-Eilenberg) Lie algebra differential for the $\mathfrak{g}$-module $S\mathfrak{g}^*$ generated by the variables $\widehat{\mu}$:

$$d_2\mu = \lambda(\mu), \quad d_2\widehat{\mu} = -\sum_i \widehat{L(e_i)\mu} \otimes e^i.$$

PROOF. On odd generators, $\mathrm{d}\mu = \overline{\mu} = \widehat{\mu} - \lambda(\mu) = \mathrm{d}_1\mu + \mathrm{d}_2\mu$, while on even generators,

$$\mathrm{d}\widehat{\mu} = \mathrm{d}\lambda(\mu) = -\tfrac{1}{2}\mathrm{d}\sum_i L(e_i)\mu \wedge e^i = -\sum_i \widehat{L(e_i)\mu} \otimes e_i = \mathrm{d}_2\widehat{\mu} = \mathrm{d}_1\widehat{\mu} + \mathrm{d}_2\widehat{\mu}.$$

The fact that $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2, \mathrm{d}_1, \mathrm{d}_2$ are all differentials implies that

$$2[\mathrm{d}_1, \mathrm{d}_2] = [\mathrm{d}, \mathrm{d}] - [\mathrm{d}_1, \mathrm{d}_1] - [\mathrm{d}_2, \mathrm{d}_2] = 0.$$

□

5. The quantized Weil algebra

Suppose $\mathfrak{g}$ carries an invariant quadratic form, used to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and hence $W\mathfrak{g} = S\mathfrak{g} \otimes \wedge\mathfrak{g}$. The contractions and Lie derivatives read

$$\iota(\xi)\zeta = B(\xi, \zeta), \;\; \iota(\xi)\overline{\zeta} = [\xi, \zeta]_{\mathfrak{g}},$$
$$L(\xi)\zeta = [\xi, \zeta]_{\mathfrak{g}}, \;\; L(\xi)\overline{\zeta} = \overline{[\xi, \zeta]_{\mathfrak{g}}},$$

while d is the Koszul differential, $\mathrm{d}\xi = \overline{\xi}$, $\mathrm{d}\overline{\xi} = 0$. We would like to introduce a Poisson bracket on $W\mathfrak{g}$, in such a way that

$$\iota(\xi) = \{\xi, \cdot\}, \;\; L(\xi) = \{\overline{\xi}, \cdot\}.$$

The formulas for contractions and Lie derivatives force,

$$\{\xi, \zeta\} = B(\xi, \zeta), \quad \{\overline{\xi}, \overline{\zeta}\} = \overline{[\xi, \zeta]_{\mathfrak{g}}}, \quad \{\overline{\xi}, \zeta\} = [\xi, \zeta]_{\mathfrak{g}}.$$

The Weil differential becomes a Poisson bracket as well:

PROPOSITION 5.1. *The Weil differential may be written $d = \{D, \cdot\}$ where $D \in (W\mathfrak{g})^3$ is defined by*

$$D = \sum_i \overline{e^i} e_i - 2\phi.$$

Here e_i is a basis of $\mathfrak{g}$, with dual basis e_i. The Poisson bracket of D with itself is

$$\{D, D\} = \sum_i \widehat{e_i}\widehat{e^i},$$

the quadratic element in the symmetric algebra $S\mathfrak{g}$ generated by the $\widehat{\xi}$.

[2] Check signs around here

PROOF. We have,

$$\begin{aligned}\{\sum_i \overline{e^i} e_i,\ \xi\} &= \overline{\xi} - \sum_i [e^i, \xi]_{\mathfrak{g}} e_i \\ &= \mathrm{d}\xi + 2\lambda(\xi).\end{aligned}$$

and

$$\begin{aligned}\{\sum_i \overline{e^i} e_i,\ \overline{\xi}\} &= -L(\xi) \sum_i \overline{e^i} e_i \\ &= 0\end{aligned}$$

since $\sum_i \overline{e^i} e_i$ is invariant. On the other hand, from the definition of ϕ,

$$\{\phi, \xi\} = \lambda(\xi),\ \{\phi, \overline{\xi}\} = -L(\xi)\phi = 0.$$

This shows $\mathrm{d}\xi = \{D, \xi\}$. On the other hand, $\{D, \overline{\xi}\} = -L(\xi)D = 0 = \mathrm{d}\overline{\xi}$ since D is invariant. Let us re-write the definition of D in terms of the variables $\xi, \widehat{\xi}$:

$$D = \sum_i \widehat{e^i} e_i + \phi.$$

(It suffices to note that $\iota(\xi) D = \widehat{\xi} + \lambda(\xi) = \overline{\xi}$.)

We have

$$\{D, D\} = \sum_{ij} \widehat{e^i} \widehat{e^j} B(e_i, e_j) + \ldots = \sum_i \widehat{e^i} \widehat{e_i} + \ldots$$

where the dots indicate terms in $S\mathfrak{g} \otimes \wedge^+ g$. But $\{D, D\}$ lies in the subspace annihilated by all the contractions, since

$$\iota(\xi)\{D, D\} = 2\{\overline{\xi}, D\} = 0.$$

Hence the $\ldots$ term all cancel. $\square$

Now let us try to quantize all this! We define a *quantum Weil algebra* $\mathcal{W}\mathfrak{g}$ as the algebra generated by elements ξ (odd) and $\overline{\xi}$ (even), with commutator relations

$$[\xi, \zeta]_{\mathcal{W}} = B(\xi, \zeta),\quad [\overline{\xi}, \zeta]_{\mathcal{W}} = [\xi, \zeta]_{\mathfrak{g}},\quad [\overline{\xi}, \overline{\zeta}]_{\mathcal{W}} = \overline{[\xi, \zeta]_{\mathfrak{g}}}.$$

(More formally, $\mathcal{W}\mathfrak{g}$ is defined as a quotient of the tensor algebra, $\tilde{W}(\mathfrak{g}) = T(E_{\mathfrak{g}})$.) Note that $U\mathfrak{g}$ is contained in $\mathcal{W}\mathfrak{g}$ as the subalgebra generated by the $\overline{\xi}$'s, and $\mathrm{Cl}(\mathfrak{g})$ is the subalgebra generated by the ξ's. In fact, $\mathcal{W}\mathfrak{g}$ is nothing but the *semi-direct product of* $U\mathfrak{g}$ *with* $\mathrm{Cl}(\mathfrak{g})$. We can introduce a filtration on $\mathcal{W}\mathfrak{g}$, by assigning filtration degree 1 to the generators ξ and filtration degree 2 to the generators $\overline{\xi}$. Using that $\mathrm{gr}(U\mathfrak{g}) = S\mathfrak{g}$ and $\mathrm{gr}(\mathrm{Cl}\mathfrak{g}) = \wedge\mathfrak{g}$, it is not hard to see that

$$\mathrm{gr}(\mathcal{W}\mathfrak{g}) = W\mathfrak{g}.$$

The element $\mathcal{D} = \sum_i \overline{e^i} e_i - 2q(\phi) \in \mathcal{W}\mathfrak{g}$ is called the *cubic Dirac operator.*

THEOREM 5.2. *We have the following commutator relations in $\mathcal{W}\mathfrak{g}$:*

$$\begin{aligned}
[\mathcal{D},\mathcal{D}]_{\mathcal{W}} &= \mathrm{Cas}_{\mathfrak{g}} - \frac{1}{6}\,\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}}),\\
[\overline{\xi},\mathcal{D}]_{\mathcal{W}} &= 0,\\
[\xi,\mathcal{D}]_{\mathcal{W}} &= \overline{\xi}\\
[\overline{\xi},\zeta]_{\mathcal{W}} &= [\xi,\zeta]_{\mathfrak{g}}\\
[\overline{\xi},\overline{\zeta}]_{\mathcal{W}} &= \overline{[\xi,\zeta]_{\mathfrak{g}}}\\
[\xi,\zeta]_{\mathcal{W}} &= B(\xi,\zeta)
\end{aligned}$$

Here $\mathrm{Cas}_{\mathfrak{g}} = \sum_i \overline{e_i}\,\overline{e^i} \in U\mathfrak{g}$ *is the Casimir element, and* $\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}})$ *its trace in the adjoint representation.*

PROOF. The last three relations hold by definition of the algebra structure. In particular, $L(\xi) = \overline{\xi}$ are the generators of the adjoint action, so $[\overline{\xi},\mathcal{D}]_{\mathcal{W}} = L(\xi)\mathcal{D} = 0$. The formula $[\xi,\mathcal{D}]_{\mathcal{W}} = \overline{\xi}$ follows easily from the definition of $\mathcal{D}$. This then implies $[\xi,[\mathcal{D},\mathcal{D}]_{\mathcal{W}}]_{\mathcal{W}} = 2[\overline{\xi},\mathcal{D}] = 0$, so that $[\mathcal{D},\mathcal{D}] \in (U\mathfrak{g})^{\mathfrak{g}} \subset \mathcal{W}\mathfrak{g}$. Hence we may compute $[\mathcal{D},\mathcal{D}]$, using that all terms in $U\mathfrak{g} \otimes q(\oplus_{i>0} \wedge^i \mathfrak{g})$ must cancel:

$$\begin{aligned}
[\mathcal{D},\mathcal{D}]_{\mathcal{W}} &= \sum_{ij} [\overline{e_i}e^i - 2q(\phi), \overline{e_j}e^j - 2q(\phi)]_{\mathcal{W}}\\
&= \sum_{ij} \overline{e_i e_j}[e^i,e^j]_{\mathrm{Cl}} + 4[q(\phi),q(\phi)] \bmod \mathfrak{g}\otimes q(\wedge^2\mathfrak{g})\\
&= \mathrm{Cas}_{\mathfrak{g}} - \frac{1}{6}\,\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}}).
\end{aligned}$$

□

Hence, setting

$$\iota(\xi) = [\xi,\cdot]_{\mathcal{W}},\ L(\xi) = [\overline{\xi},\cdot],\ \mathrm{d} = [\mathcal{D},\cdot]$$

defines the structor of a $\mathfrak{g}$-diiferential space on $\mathcal{W}\mathfrak{g}$, with connection $\theta\colon \mathfrak{g}^* \cong \mathfrak{g} \to \mathcal{W}\mathfrak{g}$ given by $\theta(\xi) = \xi$.

By the same argument as for the Weil algebra $W\mathfrak{g}$, the contraction operators simplify if we introduce new variables

$$\widehat{\xi} = \overline{\xi} + \gamma(\xi)$$

where $\gamma(\xi) = q(\lambda(\xi)) \in \mathrm{Cl}(\mathfrak{g})$:

$$\iota(\xi)\widehat{\zeta} = [\xi, \overline{\zeta} - \gamma(\zeta)]_{\mathcal{W}} = 0.$$

Note also that $[\widehat{\gamma}(\xi),\widehat{\zeta}]_{\mathcal{W}} = \widehat{[\xi,\zeta]_{\mathfrak{g}}}$. As a consequence, we have

$$\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$$

(the usual product of algebras), where $U\mathfrak{g}$ is the enveloping algebra generated by the variables $\widehat{\xi}$, and $\mathrm{Cl}(\mathfrak{g})$ is the Clifford algebra generated by the variables ξ. As for the usual Weil algebra, we find that its basic subcomplex is

$$(\mathcal{W}\mathfrak{g})_{\mathrm{bas}} = (U\mathfrak{g})^{\mathfrak{g}}$$

with the zero differential. Thus $H_{\mathrm{bas}}(W\mathfrak{g}) = (U\mathfrak{g})^{\mathfrak{g}}$.

6. Chern-Weil homomorphisms

Let $\theta_W, \theta_{\tilde{W}}$ denote the canonical connection for the Weil algebra $W\mathfrak{g}^*$, respectively for the non-commutative Weil algebra $\tilde{W}\mathfrak{g}$. The commutative and non-commutative Weil algebras are characterized by the following universal property:

Proposition 6.1. *The Weil algebra $W\mathfrak{g}$ is universal among commutative $\mathfrak{g}$-differential algebras with connection. That is, for any commutative $\mathfrak{g}$-differential algebra $\mathcal{A}$ with connection $\theta_{\mathcal{A}}$ there is a unique homomorphism of $\mathfrak{g}$-differential algebras*

$$c\colon W\mathfrak{g} \to \mathcal{A} \tag{29}$$

such that $c \circ \theta_W = \theta_{\mathcal{A}}$. Similarly $\tilde{W}\mathfrak{g}$ is universal among non-commutative $\mathfrak{g}$-differential algebras with connection.

Proof. Suppose $\mathcal{A}$ is a commutative $\mathfrak{g}$-differential algebra with connection. By the universal property of Koszul algebras, the map $\theta\colon \mathfrak{g}^* \to \mathcal{A}^{\bar{1}}$ extends uniquely to a homomorphism of differential algebras $c\colon W\mathfrak{g} \to \mathcal{A}$. The calculation

$$\begin{aligned}
\iota(\xi)c(\overline{\mu}) &= \iota(\xi)\mathrm{d}\theta(\mu)\\
&= L(\xi)\theta(\mu) - \mathrm{d}\iota(\xi)\theta(\mu)\\
&= \theta(L(\xi)\mu) - \mathrm{d}\langle\mu,\xi\rangle\\
&= c(L(\xi)\mu) = c(\iota(\xi)\overline{\mu}),
\end{aligned}$$

together with $\iota(\xi)c(\mu) = \iota(\xi)\theta(\mu) = \langle\mu,\xi\rangle = c(\iota(\xi)\mu)$, shows that c intertwines contractions. Since $L(\xi) = [\iota(\xi), \mathrm{d}]$, it intertwines Lie derivatives as well. ☐

Since θ may be recovered from c, we can directly think of a connection as a morphism of $\mathfrak{g}$-differential algebras, $\tilde{W}\mathfrak{g} \to \mathcal{A}$ (or $W\mathfrak{g} \to \mathcal{A}$ if $\mathcal{A}$ is commutative).

Definition 6.2. A homotopy h between morphisms $\phi, \phi'\colon E_1 \to E_2$ of $\mathfrak{g}$-differential spaces is called a $\mathfrak{g}$-homotopy if it intertwines the contraction operators: [3]

$$[h, \iota(\xi)] := h \circ \iota(\xi) + \iota(\xi) \circ h = 0.$$

This then implies that h intertwines Lie derivatives as well:

$$\begin{aligned}
[h, L(\xi)] &= [h, [\mathrm{d}, \iota(\xi)]]\\
&= [[h, \mathrm{d}], \iota(\xi)] - [\mathrm{d}, [h, \iota(\xi)]]\\
&= [\phi - \phi', \iota(\xi)] = 0.
\end{aligned}$$

Theorem 6.3. *Let $\mathcal{A}$ be a commutative differential algebra. Then any two morphisms of $\mathfrak{g}$-differential algebras $c_0, c_1\colon W\mathfrak{g} \to \mathcal{A}$ are $\mathfrak{g}$-homotopic. Similarly for non-commutative $\mathfrak{g}$-differential algebras.*

Proof. We had shown in Theorem 2.4 that any two morphisms from a Koszul algebra into a commutative differential algebra are canonically homotopic. By inspection (check on generators), this canonical homotopy intertwines contractions. ☐

[3] By mild abuse of notation, we write a commutator even though the two $\iota(\xi)$'s act on different spaces.

It follows that if $\mathcal{A}$ is a commutative $\mathfrak{g}$-differential algebra admitting a connection, then the algebra homomorphism in basic cohomology

$$(S\mathfrak{g}^*)^{\mathfrak{g}} = H_{\mathrm{bas}}(W\mathfrak{g}) \to H_{\mathrm{bas}}(\mathcal{A})$$

is independent of the connection. It is called the *Chern-Weil homomorphism.*

REMARK 6.4. This terminology comes from the differential geometry of principal G-bundles $P \to B$ with connections $\theta\colon \mathfrak{g}^* \to \Omega^1(P)$. In this case, the basis subcomplex $\Omega(P)_{\mathrm{bas}}$ is isomorphic to the de Rham complex $\Omega(B)$. The elements in $H_{\mathrm{deRham}}(B) = H(\Omega(B))$ obtained as images under the Chern-Weil homomorphism are called the *characteristic classes.*

Similarly, for a non-commutative $\mathfrak{g}$-differential algebra $\mathcal{A}$ with connection, the characteristic homomorphism $c\colon \tilde{W}\mathfrak{g} \to \mathcal{A}$ induces an algebra homomorphism

$$H_{\mathrm{bas}}(W\mathfrak{g}) \to H_{\mathrm{bas}}(\mathcal{A}).$$

We will see below that for $\mathcal{A} = W\mathfrak{g}$ this map is an algebra *isomorphism*, i.e. that $H_{\mathrm{bas}}(W\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$ *as algebras.* It follows that even in the non-commutative case, there is a canonical algebra homomorphism $(S\mathfrak{g}^*)^{\mathfrak{g}} \to H_{\mathrm{basic}}(\mathcal{A})$. In particular, taking $\mathcal{A} = \mathcal{W}\mathfrak{g}$ we obtain a canonical *algebra* homomorphism $(S\mathfrak{g})^{\mathfrak{g}} \to (U\mathfrak{g})^{\mathfrak{g}}$! To prove all this, we first of all have to understand the dependence of the characteristic homomorphism on the connection $\theta_{\mathcal{A}}$.

The following Proposition shows that tensoring a $\mathfrak{g}$-differential algebra $\mathcal{A}$ with a Weil algebra does not change the $\mathfrak{g}$-homotopy type, provided $\mathcal{A}$ admits a connection:

PROPOSITION 6.5. *If $\mathcal{A}$ is a commutative $\mathfrak{g}$-differential algebra with connection, the map $\phi\colon W\mathfrak{g} \otimes \mathcal{A} \to \mathcal{A}$, $w \otimes x \mapsto c(w)x$ is a $\mathfrak{g}$-homotopy equivalence, with $\mathfrak{g}$-homotopy inverse the inclusion, $\psi\colon \mathcal{A} \to W\mathfrak{g} \otimes \mathcal{A}$, $x \mapsto 1 \otimes x$. Similarly for non-commutative $\mathfrak{g}$-differential algebras (replacing $W\mathfrak{g}$ with $\tilde{W}\mathfrak{g}$).*

PROOF. Clearly, $\phi \circ \psi$ is the identity. The opposite composition is $\psi \circ \phi(w \otimes x) = 1 \otimes c(w)x$. It is enough to show that the two maps $W\mathfrak{g} \to W\mathfrak{g} \otimes \mathcal{A}$, taking w to $w \otimes 1$ and $1 \otimes c(w)$, are homotopic. But this follows because they are the characteristic homomorphism for the two natural connections on $W\mathfrak{g} \otimes \mathcal{A}$, given by $\theta_W \otimes 1$ and $1 \otimes \theta_{\mathcal{A}}$. □

7. Symmetrization

Suppose E is a $\mathbb{Z}_2$-graded vector space, $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra, and $\phi\colon E \to \mathcal{A}$ is a linear map preserving grading. Then ϕ admits a canonical extension to the symmetric algebra,

$$\mathrm{sym}(\phi)\colon S(E) \to \mathcal{A}$$

by symmetrization: For homogeneous elements $v_i \in E$,

$$v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{k \in S_k} (-1)^{N_s(v_1,\ldots,v_k)} \phi(v_{s^{-1}(1)}) \cdots \phi(v_{s^{-1}(k)}).$$

Here $N_s(v_1, \ldots, v_k)$ is the number of pairs $i < j$ such that v_i, v_j are odd elements and $s^{-1}(i) > s^{-1}(j)$. For the special case $\mathcal{A} = T(E)$, this becomes the inclusion as 'symmetric tensors', and the general case may be viewed as this inclusion followed by the algebra homomorphism $T(E) \to \mathcal{A}$.

Now suppose (E, d) is a differential space, $(\mathcal{A}, \mathrm{d})$ a differential algebra, and $\phi\colon E \to \mathcal{A}$ is a morphism of differential spaces. Then $\mathrm{sym}(\phi)$ is again a morphism of differential spaces. This is a special case of the following Lemma:

LEMMA 7.1. *Let E be a super vector space, $\mathcal{A}$ a super algebra, and $\phi\colon E \to \mathcal{A}$ be a homomorphism of super vector spaces. Suppose ϕ intertwines a given endomorphism $D_E \in \mathrm{End}(E)$ with a derivation $D_{\mathcal{A}}$. Let $D_{S(E)} \in \mathrm{End}(S(E))$ be the derivation extension of D_E^0. Then $\mathrm{sym}(\phi)$ intertwines D_E and $D_{\mathcal{A}}$.*

PROOF. It suffices to prove this for $\mathcal{A} = T(E)$, with $D_{\mathcal{A}} = D_{T(E)}$ the derivation extension of E. The action of the derivation $D_{T(E)}$ on $T^k(E)$ commutes with the action of the symmetric group S^k, and in particular preserves the invariant subspace. It therefore restricts to $D_{S(E)}$ on $S(E) \subset T(E)$. □

THEOREM 7.2. *The quotient map $\phi\colon \tilde{W}\mathfrak{g} \to W\mathfrak{g}$ is a $\mathfrak{g}$-homotopy equivalence, with homotopy inverse $\psi\colon W\mathfrak{g} \to \tilde{W}\mathfrak{g}$ given by symmetrization, $S(E_{\mathfrak{g}^*}) \to T(E_{\mathfrak{g}^*})$.*

PROOF. Clearly, $\phi \circ \psi$ is the identity. Let f be the cochain map,

$$f\colon \tilde{W}\mathfrak{g} \otimes W\mathfrak{g} \to \tilde{W}\mathfrak{g},\ w \otimes w' \mapsto w\psi(w')$$

and let $c_0, c_1\colon \tilde{W}\mathfrak{g} \to \tilde{W}\mathfrak{g} \otimes W\mathfrak{g}$ be the two cochain maps, $c_0(w) = w \otimes 1$, $c_1(w) = 1 \otimes \phi(w)$. Then $f \circ c_0$ is the identity map, while $f \circ c_1 = \psi \circ \phi$. Since c_0, c_1 are the characteristic homomorphisms for the two natural connections on $\tilde{W}\mathfrak{g} \otimes W\mathfrak{g}$, they are homotopic. Hence so are their compositions with f. □

8. Duflo's theorem

Consider now the quantized Weil algebra $\mathcal{W}\mathfrak{g}$. Define a quantization map

$$q\colon W\mathfrak{g} \to \mathcal{W}\mathfrak{g} \tag{30}$$

as a composition of the symmetrization map $W\mathfrak{g} \to \tilde{W}\mathfrak{g}$ with the quotient map (characteristic homomorphism) $\tilde{W}\mathfrak{g} \to \mathcal{W}\mathfrak{g}$. Thus q is a homomorphism of $\mathfrak{g}$-differential spaces, and hence induces an homomorphism of basic subcomplexes, $(W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g})^{\mathfrak{g}} \to (\mathcal{W}\mathfrak{g})_{\mathrm{bas}} = (U\mathfrak{g})^{\mathfrak{g}}$. Since the map $W\mathfrak{g} \to \tilde{W}\mathfrak{g}$ induces an algebra isomorphism in basic cohomology, it follows that q induces an algebra isomorphism in basic cohomology. We have thus obtained an *algebra* isomorphism,

$$(S\mathfrak{g})^{\mathfrak{g}} \to (U\mathfrak{g})^{\mathfrak{g}}. \tag{31}$$

The map q is given on generators by $q(\xi) = \xi$ and $q(\overline{\xi}) = \overline{\xi}$, extended to the symmetric algebra $W\mathfrak{g} = S(E_{\mathfrak{g}})$ by symmetrization. In particular, on the symmetric algebra $S\mathfrak{g}$ generated by the $\overline{\xi}'$ it coincides with the standard symmetrization map for the enveloping algebra, $S\mathfrak{g} \to U\mathfrak{g}$, while on the exterior algebra generated by the ξ's it coincides with the quantization map for the Clifford algebra, $q\colon \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$. However, the symmetric and enveloping algebras in (**??**) are generated by the variables $\widehat{\xi}$, and on these variables q is *not* just symmetrization!

We would therefore like to re-express the map q in terms of the symmetrization map

$$\mathrm{sym} = \mathrm{sym}_{U\mathfrak{g}} \otimes q_{\mathrm{Cl}}\colon S\mathfrak{g} \otimes \wedge\mathfrak{g} \to U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g}),$$

where now $S\mathfrak{g}, U\mathfrak{g}$ are generated by the $\widehat{\xi}$'s.

Recall that the formula relating exponentials in the exterior and in the Clifford algebra involved a smooth function

$$\mathcal{S}\colon \mathfrak{g} \to \wedge\mathfrak{g}$$

of the form $\mathcal{S}(\xi) = J^{1/2}(\xi)\exp(\mathfrak{r}(\xi))$ where $J^{1/2}$ is the 'Duflo factor' and $\mathfrak{r}$ is a certain meromorphic function with values in $\wedge^2\mathfrak{g}$. This function gives rise to an element

$$\mathcal{S} \in \overline{S}\mathfrak{g}^* \otimes \wedge\mathfrak{g},$$

where the first factor can be thought of as constant coefficient (infinite order) differential operators. This element acts on $W\mathfrak{g} = S\mathfrak{g} \otimes \wedge\mathfrak{g}$ in a natural way: The $\overline{S}\mathfrak{g}^*$ factor acts as an infinite order differential operator, while the second factor acts by contraction.

THEOREM 8.1. *The isomorphism of $\mathfrak{g}$-differential spaces $q\colon W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ is given in terms of the generators $\xi, \widehat{\xi}$ by*

$$q = \operatorname{sym} \circ \mathcal{S}\colon S\mathfrak{g} \otimes \wedge\mathfrak{g} \to U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g}).$$

In particular, its restriction to $S\mathfrak{g} \subset W\mathfrak{g}$ is the Duflo map,

$$\mathrm{Duf} = \operatorname{sym} \circ \widehat{J^{1/2}}\colon S\mathfrak{g} \to U\mathfrak{g}.$$

PROOF. By definition, q is the symmetrization map relative to the variables $\xi, \overline{\xi}$. It may be characterized as follows: For all odd variables $\nu^i \in \mathfrak{g}^*$ and all even variables $\mu^j \in \mathfrak{g}^*$, and all $N = 0, 1, 2\ldots$,

$$q\Big((\sum_i \nu^i e_i + \sum_j \mu^j \overline{e}_j)^N\Big) = (\sum_i \nu^i e_i + \sum_j \mu^j \overline{e}_j)^N.$$

These conditions may be summarized in a single condition,

$$q\Big(\exp_W(\sum_i \nu^i e_i + \sum_j \mu^j \overline{e}_j)\Big) = \exp_{\mathcal{W}}(\sum_i \nu^i e_i + \sum_j \mu^j \overline{e}_j).$$

We want to express q in terms of the generators $e_i, \widehat{e}_i = \overline{e}_i - \lambda(e_i)$ of $W\mathfrak{g}$ respectively $e_i, \widehat{e}_i = \overline{e}_i - \gamma(e_i)$ of $\mathcal{W}\mathfrak{g}$. Using that $\widehat{e}_i$ and e_j commute in $\mathcal{W}\mathfrak{g}$,

$$\exp_{\mathcal{W}}(\sum_i \nu^i e_i + \sum_j \mu^j \overline{e}_j) = \exp_{\mathrm{Cl}}(\sum_i \nu^i e_i + \sum_j \mu^j \gamma(e_j))\exp_U(\sum_j \mu^j \widehat{e}_j).$$

The second factor is $\operatorname{sym}\exp_S(\sum_j \mu^j \widehat{e}_j)$ by definition of the symmetrization map. The first factor is the Clifford exponential of a quadratic element, and is related to the corresponding exponential in the exterior algebra,

$$\exp_{\mathrm{Cl}}(\sum_i \nu^i e_i + \sum_j \mu^j \gamma(e_j)) = q_{\mathrm{Cl}}\Big(\iota(\mathcal{S}(\mu))\exp_\wedge(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j))\Big).$$

Hence

$$\begin{aligned}\exp_{\mathcal{W}}(\sum_i \nu^i e_i + \sum_j \mu^j \overline{e}_j) &= \operatorname{sym}_W\Big(\iota(\mathcal{S}(\mu))\exp_\wedge(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j))\exp_S(\sum_j \mu^j \widehat{e}_j)\Big)\\ &= \operatorname{sym}_W \circ \widehat{\mathcal{S}}\Big(\exp_W\big(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j) + \sum_j \mu^j \widehat{e}_j\big)\Big)\\ &= \operatorname{sym}_W \circ \widehat{\mathcal{S}}\Big(\exp_W\big(\sum_i \nu^i e_i + \sum_j \mu^j \overline{e}_j\big)\Big).\end{aligned}$$

∎