A visual methodology to assess spatial graph vertex ordering algorithms

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***Abstract*—Graph vertex ordering is crucial for various graph-related applications, especially in spatial and urban data analysis where graphs represent real-world locations and their connections. The task is to arrange vertices along a single axis while preserving spatial relationships, but this often results in distortions due to the complexity of spatial data. Existing methods mostly assess ordering quality using a global metric, which may not capture specific use case needs or localized variations. This work proposes a new methodology to visually evaluate and compare vertex ordering techniques on spatial graphs. Two quantitative comparison mechanisms are proposed. Experiments on urban data from various cities demonstrate the methodology's effectiveness in tuning hyperparameters and comparing well-known vertex ordering techniques. The visual approach reveals nuanced spatial patterns that global metrics might miss, providing deeper insights into the behavior of different vertex ordering methods.**

I. Introduction

Graph vertex ordering is a fundamental task with significant impact on various applications, including graph visualization [1], finite element nodal ordering [2], word cloud construction [3], and CPU cache optimization [4]. In spatial data analysis, where graphs represent real-world locations and their connections, vertex ordering plays a critical role, particularly in visualization tasks. Many visualization techniques rely on one-dimensional graph node embedding to portray the spatial relationships between locations effectively, enabling tasks like spatio-temporal visualization where changes are shown alongside spatial distribution [5]–[7]. By carefully arranging graph nodes in a single axis, one can create clear and informative visualizations that reveal patterns and trends that might be hidden in the data. This paves the way for deeper understanding and better decision-making based on spatial information.

Several vertex ordering methods have been proposed in the literature (see Sec. II). Most of them are designed to exploit the graph's topological structure and edge weights to re-index the graph nodes to avoid large jumps in the indices of neighbor vertices. However, when the graph includes geometric information, meaning the vertices have spatial coordinates (spatial graphs), dimensionality reduction methods can be applied to organize the nodes according to their geometric proximity. This is the case of graphs derived from geolocated entities such as street map graphs [8], census tract graphs [9], and mobility graphs [10], which are widely used in urban data analysis tasks.

The current diversity of existing methods to perform such ordering naturally raises an important question: *which vertex ordering method does perform better in spatial graphs, mainly in terms of properly index neighbor nodes?*

While numerous metrics to assess the ordering quality exist in the literature, they primarily focus on computing a global quality measure to rank and compare different ordering methods. However, analyzing the ordering quality locally is also of great importance to assess where a provided ordering is not reliable and how distortions are distributed across the spatial domain. Fig. 1 illustrates such a situation, where the nodes corresponding to the corners of a street map graph are ordered by the Fiedler technique. Notice that the nodes within the green window are properly ordered, keeping neighbor corners closely indexed. In contrast, the nodes in the red window are also closely indexed but they are spatially far apart from each other, indicating that the ordering method is not working properly for those nodes. This example shows clearly the importance of performing local analysis.

This work presents a visual methodology to evaluate and compare graph vertex ordering methods on spatial graphs. The methodology is comprised of two local quantitative measures and suitable visualizations that can reveal relevant patterns of the orderings, enabling a detailed analysis of the methods' performances at a local level. The effectiveness of the proposed methodology is attested in a comprehensive set of experiments, making it possible to assess vertex ordering methods' quality in different scenarios and parameter settings. In the presented results, we demonstrate how relevant findings may be extracted from the visualizations, allowing not only a qualitative comparison among different ordering methods, but also providing a deeper understanding of how they behave in specific datasets.

In summary, the main contributions of this work are:

- a methodology to visually assess vertex ordering;
- two local spatial metrics for evaluating and comparing vertex ordering techniques;
- a series of experiments that demonstrate how the proposed visual methodology enhances the assessment of vertex ordering methods in different scenarios.

II. Related Work

This section discusses existing vertex ordering methods and metrics to assess their quality.

Vertex Ordering. Vertex ordering methods date back to the 1950s [11], initially for matrix reordering [12], [13]. In the context of geolocated graphs, space-filling curves (SFC) have been extensively used to traverse the spatial area containing the graph, ordering the nodes according to a one-dimensional (1D) curve [14]–[16]. Hierarchical clustering for vertex ordering relies on similarity measures to arrange vertices based on their hierarchical merging into clusters [17], [18]. Spectral methods, particularly those using the Fiedler vector, are notable for their effectiveness [19] and have been applied for seriation [20] and ranking [21]. Cauchy graph embedding [22] is a variant that preserves local topology but is computationally expensive for large graphs. Heuristic methods iterate through adjacency matrix rows/columns [23] or transform them into simpler representations [24]. Graph embedding methods first embed graphs in high-dimensional space and use PCA to project vertices into 1D space for ordering [25], [26]. Alternatively, dimensionality reduction methods [27] such as MDS [28], t-SNE [29], and UMAP [30] can directly project geolocated graphs onto a 1D space, which can then be post-processed to infer an ordering. Recent developments include Graph Neural Networks (GNNs) [31] and matrix completion [32].

Comparison Metrics. Several metrics have been proposed to compare vertex ordering techniques. These metrics range from the local and global aggregation of index jumps between neighboring vertices [1], to statistical tests [33], and entropy measures [23]. In [34], the total length of the line formed by connecting consecutive vertices according to a specific ordering, referred to as *minimal path length*, is examined as a quality measure. The topographic product [35] was proposed to measure the preservation or violation of neighborhood relations. To investigate various measures of neighborhood preservation for graphs embedded in two different metric spaces, Goodhill and Sejnowski [36] introduced a measure of discrepancy called *C measure*, which is essentially the correlation coefficient between similarities in the two spaces. Dimensionality reduction methods such as MDS, PCA, t-SNE and UMAP have been evaluated using various distortion measures [27], but these are not specifically designed for assessing vertex ordering. Barik et al. [37] proposed *gap* measures to evaluate eleven ordering schemes. However, those measures are global and do not consider geometric information. There are also metrics designed to evaluate orderings provided by graph-based clustering [38] and GNNs [31].

In the context of geolocated graphs, Guo and Gahegan [39] emphasized the importance of preserving locality between the original space and the ordering space. They argued that most prior work focused solely on one direction of this preservation so they proposed two groups of global measures: Key Similarity (KS) and Spatial Similarity (SS), which utilize kNN graphs with different edge weighting schemes. In this paper, we also introduce metrics aimed at assessing locality in both directions, forward and inverse. KS and Forward both focus on preserving spatial proximity but differ in their metrics and granularity, while SS and Inverse aim to avoid artificial proximity but differ in how they measure and normalize spatial relationships, making a direct comparison unfeasible.

Besides, our metrics are specifically designed to evaluate neighborhood preservation visually, focusing on the compactness of covered regions in both spaces as criteria. The proposed metrics are defined individually for each vertex rather than globally for the entire graph. We also propose a visual evaluation methodology that builds upon the geometric realization of spatial graphs, enabling visual qualitative analysis that reveals where the ordering technique performs better or worse. This methodology also supports quantitative evaluation, allowing analyses similar to those performed with other existing metrics.

III. Comparing Vertex Ordering Algorithms

We propose a visual methodology that is aimed at visually assessing vertex ordering methods on undirected connected spatial (UCS) graphs. Let $\mathcal{G} = ((V, s), E)$ be an UCS graph where V is the set of vertices, $s\colon V \to S$ assigns spatial coordinates to each vertex, and E is the set of edges. Here, S represents the space of geographic coordinates of the vertices in V (in our context S is $\mathbb{R}^2$).

Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of n vertices of $\mathcal{G}$. An ordering of V is a bijection $\phi(v_k) \to \mathbb{R}$ such that the index of each vertex is defined by the order imposed by ϕ. In other words, if $\phi(v_{i_1}) < \phi(v_{i_2}) < \cdots < \phi(v_{i_n})$, where $i_k \in \{1, 2, \ldots, n\}$, then index 1 is assigned to $\phi(v_{i_1})$, index 2 to $\phi(v_{i_2})$, and so on.

Given an UCS graph, an ordering technique is expected to assign nearly consecutive indices to vertices spatially close to each other and vice-versa. This motivates the proposed *forward* and *inverse* assessment approaches described in Section III-A and Section III-B, respectively. Section III-C presents the visualization mechanisms we propose to enable an intuitive visual analysis of vertex ordering techniques.

A. Forward approach

Let $I = \{i_1, i_2, \ldots, i_n\}$ be the ordered indices of the nodes in $\mathcal{G}$ given by a vertex ordering technique. In addition, let $I^m_{i_k}$ be a window of consecutive indices in I, centered at i_k, with m being the window size (we assume m is an odd number). The quality of the ordering for nodes within $I^m_{i_k}$ can be assessed from the bounding box B_{i_k} of the vertices in the spatial domain. More specifically, let $I^m_{i_k} = \{i_{k-\lfloor m/2 \rfloor}, \ldots, i_k, \ldots, i_{k+\lfloor m/2 \rfloor}\}$ be the ordered indices within a window centered at i_k and B_{i_k} be the bounding box of the vertices $V_{i_k} = \{v_{i_{k-\lfloor m/2 \rfloor}}, \ldots, v_{i_k}, \ldots, v_{i_{k+\lfloor m/2 \rfloor}}\}$ in the spatial domain (see Fig. 1). For a small value of m, a good vertex ordering should give rise to a locally compact region in the spatial domain, thus making B_{i_k} small. V_{i_k} compactness can be measured from the B_{i_k} diagonal length. When the vertices coordinates are given as latitude-longitude pairs, the Haversine distance between the two opposite corners of B_{i_k} can be applied to estimate the diagonal length. However, the spatial resolution of a graph can vary across different regions. To address this issue, the diagonal length of each B_{i_k} is normalized by that of its "optimal" bounding box $\tilde{B}_{i_k}$ consisting of the m nearest neighbors of i_k within the spatial domain. To compute measures over the entire graph,

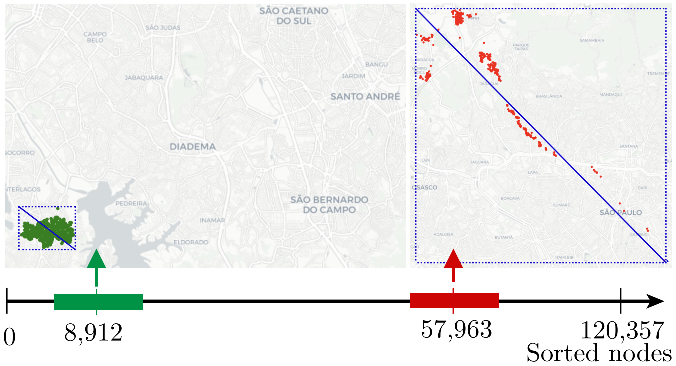


Fig. 1: Forward approach applied in the street map of São Paulo: nodes (street corners) ordered by the Fiedler ordering method are locally evaluated through sliding windows (bottom). The green window corresponds to the indices of a set of vertices, which are depicted on the map, resulting in a small bounding box with a 3.34km diagonal length. The red window gives rise to a much larger (and undesired) bounding box with 28.93km diagonal length.

a sliding window approach is adopted, where all possible windows $I^m_{i_k}$, $k = \lceil m/2 \rceil, \ldots, n - \lfloor m/2 \rfloor$ are considered, for a fixed odd value m, resulting in a set D with $n - m + 1$ diagonal values. Smaller values in D indicate better ordering.

B. Inverse approach

Let $R_i(r) = \{v \in V \mid d_{\mathcal{G}}(v, v_i) < r\}$, where $d_{\mathcal{G}}(v, v_i)$ is the length of a shortest path (in terms of graph edges and weights) from vertex v to a given vertex v_i. In other words, $R_i(r)$ is the set of vertices in the neighborhood of v_i whose graph distance to v_i is smaller than r. Let $I_{R_i(r)} \subset I$ be the set of indices of the nodes in $R_i(r)$ (ordered by a vertex ordering method). The vertex ordering local quality in the spatial neighborhood of v_i is measured by the diameter of $I_{R_i(r)}$ normalized by the number of nodes in $R_i(r)$, that is:

$$d_i^{inv} = \frac{max(I_{R_i(r)}) - min(I_{R_i(r)})}{|R_i(r)|}\,.$$

where $I_{R_i(r)}$ is the window comprising the indices of all vertex in $R_i(r)$, $max(I_{R_i(r)})$ and $min(I_{R_i(r)})$ the left and right most indices in the window and $|R_i(r)|$ the number of nodes in $R_i(r)$. Fig. 2 shows an example of a vertex in the city of São Paulo street map (black dot) and its neighbor vertices with $r = 0.5km$. The diameter of $I_{R_i(r)}$ is 726 and the normalized measure is $d_i^{inv} = 14.26$. A set of diameters, denoted by D^{inv}, may be computed by iterating through all graph nodes v_i and computing the corresponding values d_i^{inv} for a fixed r value.

C. Visualizations

The sets of local metrics D and D^{inv} provide rich information on the quality of vertex ordering methods. We propose four different visualizations designed to assess an ordering method. The first two provide an analysis of the distribution of the local quality through histograms or boxplots. Fig. 3 illustrates the local measures D distribution for three different techniques. In this example, the t-SNE ordering quality distribution is better than a random ordering and the original OpenStreetMap (OSM, [40]) ordering on the street map of São Paulo. Specifically, the local measures of t-SNE-based ordering concentrate in a small range of low values,

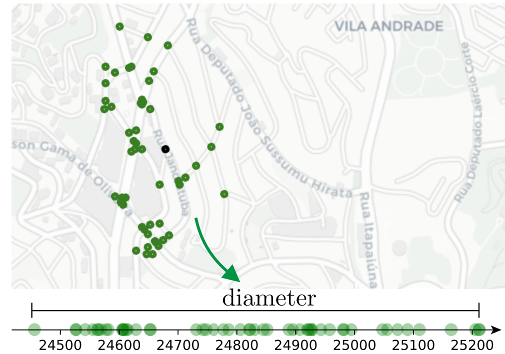


Fig. 2: Inverse approach: green nodes are the 0.5km neighbors of the black node. The quality of the ordering is assessed from the diameter of the range of the corresponding index set, normalized by the number of nodes in the range (bottom).

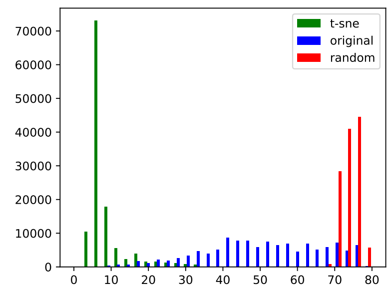


Fig. 3: Histograms built using the forward approach to compare t-SNE ordering to the original OSM and a random ordering, on the street map of São Paulo. The horizontal axis represents diagonal lengths in kilometers and the vertical axis their frequencies.

meaning that the bounding boxes of the sliding windows tend to be small. When a comparison among numerous orderings is required, boxplots may be more suitable, as shown in Fig. 5 for both D and D^{inv}. However, these visualizations provide a global quality view.

To assess spatial patterns, we propose two color-coding schemes applied to the vertices of the spatial graph. The first coloring scheme codes the nodes according to their ordered indices, as depicted in Fig. 4, where vertex orderings generated by t-SNE using different perplexity hyperparameters are visually analyzed. The second coloring scheme depicts the local measures on their corresponding vertices, as illustrated in Fig. 8 for both the forward and the inverse approach. In this example, generated on the street map of Maceió, the forward and inverse approaches reveal where discontinuities in the ordering provided by t-SNE mostly occur.

The visualizations, developed in Python using the matplotlib and folium libraries within Jupyter Notebook, are available for use and integration on GitHub: https://github.com/giva-lab/vertex_ordering. These tools enhance graph-based analytics by uncovering complex patterns and facilitating the identification of clusters, hierarchies, and trends, thus supporting data scientists, analysts, and visualization experts in deriving faster insights and making informed decisions.

IV. Experimental Evaluation and Discussion

We conducted experiments to demonstrate the assessment capabilities of the proposed methodology on three vertex ordering techniques: Fiedler, t-SNE, and UMAP, which are briefly described in Section IV-A. Our evaluation framework incorporates both forward and inverse approaches, complemented by visual qualitative analysis. To ensure a diverse and representative experimental setup, we selected street map graphs from various cities with distinct characteristics. Specifically, our experiments utilize street graphs from the cities of São Paulo, Maceió, Barcelona, Busan, Mumbai, Nairobi, and Bogotá, which were obtained from OpenStreetMap (OSM, [40]). We employed a window size of $n/100$ for the forward approach and a radius of $0.5km$ for the inverse approach. For benchmarking, we also compared the results of the vertex ordering techniques against two baselines: the original OSM vertex order and a random vertex order.

A. Vertex Ordering Algorithms

In the following we shortly describe the three ordering techniques assessed in our experiments.

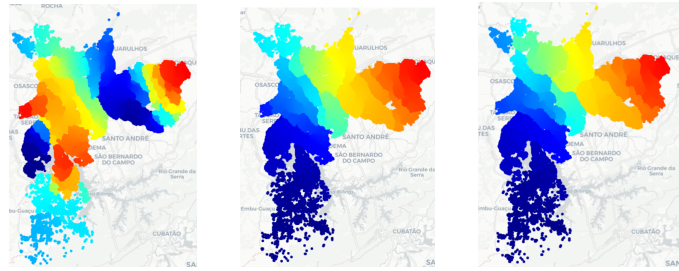
Fiedler. The Fiedler vector arranges graph nodes in a 1D space using the Laplacian matrix spectrum. The Laplacian matrix $\mathbf{L}$ of a connected graph $\mathcal{G}$ is defined as $L_{ij} = -1/l_{ij}$ if i and j are adjacent, $L_{ij} = 0$ otherwise, and $L_{ii} = \sum_{j \neq i} |L_{ij}|$, where l_{ij} is the length of the edge connecting vertices i and j. The Fiedler vector is the eigenvector of $\mathbf{L}$ associated with the smallest non-zero eigenvalue (see [41] for details). Each entry in the Fiedler vector is associated with a vertex, and its value gives the 1D embedding of the corresponding vertex.

t-SNE. Dimensionality reduction technique maps nearby points closer together in a lower-dimensional space, with performance heavily influenced by the number of neighbors, or perplexity [29]. It computes pairwise similarities in the original space using a Gaussian kernel, then projects points in the lower-dimensional space by minimizing the Kullback-Leibler (KL) divergence between the computed similarity distribution and a Student's t-distribution in the lower-dimensional space. For vertex ordering, the lower-dimensional space is 1D.

UMAP. UMAP is a dimensionality reduction technique aimed at preserving both local and global structures [30]. The algorithm constructs a weighted k-nearest neighbor graph for each input point x_i, resulting in a local graph. The edge weight function is determined by the distance between the node x_i and its neighbors, with normalization factors influencing the weights. The local graphs are combined into a unified topological representation using the probabilistic t-conorm. UMAP constructs a low-dimensional layout of the entire graph using a force-directed graph layout mechanism.

B. Hyperparameters tuning

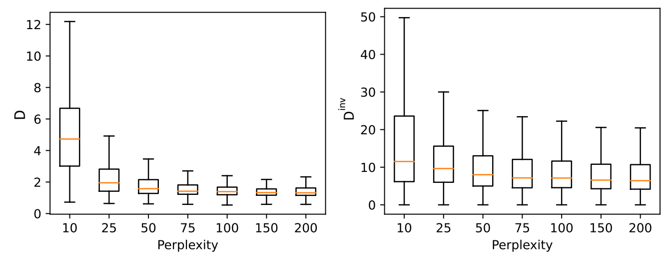
We first evaluate our methodology for assessing the impact of the perplexity hyperparameter of t-SNE on vertex ordering using the São Paulo street graph. Seven perplexity values were examined: 10, 25, 50, 75, 100, 150, and 200. Fig. 4 visually demonstrates the qualitative results of the resulting ordering, where city nodes (corners) are color-coded according to their indices. The visualization reveals distinct patterns for different perplexity values: lower perplexity values lead to scattered and disconnected regions with similar colors (distant vertices with close indices), whereas higher perplexity values achieve



(a) Perplexity 10 (b) Perplexity 75 (c) Perplexity 200

(d) Color scale of the index ordering

Fig. 4: Visualization of vertex orderings on the street graph of São Paulo in three distinct vertex orderings generated by t-SNE using different perplexity values: each node of the graph is color-coded according to its index (top), using a colormap (bottom).



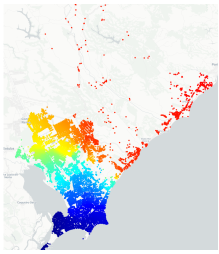
(a) Forward approach (b) Inverse approach

Fig. 5: Boxplots of the forward approach (Figure 5a) and inverse approach (Figure 5b) illustrating the performance of t-SNE across varying perplexity values on the São Paulo's street graph. The x-axis represents different perplexity values, while the y-axis denotes the diagonal length values for the forward approach and the normalized diameters for the inverse approach.

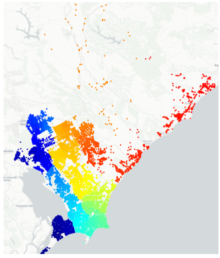
smoother and more continuous index distributions. Boxplots in Fig. 5a illustrate that t-SNE's performance improves with increasing perplexity (indicated by shorter bounding box diagonals), following an exponential trend. This finding is further supported by Figure 5b where the boxplots of the inverse approach show reduced dispersion with higher perplexity. The stability observed in the boxplots for both bounding box diagonals (forward measure) and interval diameter (inverse measure) beyond a perplexity value of 100 suggests an optimal perplexity range. This experiment demonstrated the value of the proposed approach for fine-tuning hyperparameters in vertex ordering methods, as the perplexity for t-SNE.

C. Vertex ordering techniques comparison

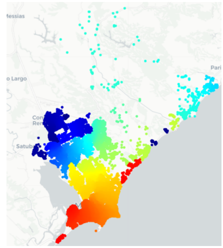
Figures 6 and 7 show, respectively, the qualitative and quantitative comparison of hyperparameters-tuned Fiedler, t-SNE, UMAP, and the two baseline vertex orderings, using Maceió street graph as a comparison basis. The color-coded vertex ordering visualization (Fig. 6) reveals that each ordering technique gives rise to a particular pattern. The Fiedler ordering produces smoother transitions from southeast to northwest. However, vertices on the far east and far west of the map tend to have similar colors, indicating that close indices are assigned to spatially distant vertices. In contrast, t-SNE displays a more clustered arrangement with clear color (and index) discontinuities (see the yellow and blue regions). The



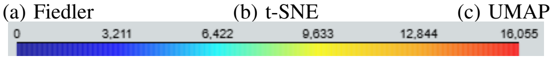
(a) Fiedler



(b) t-SNE

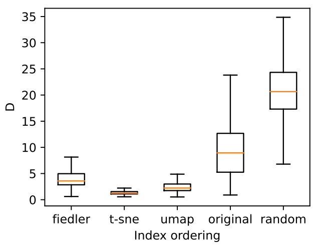


(c) UMAP

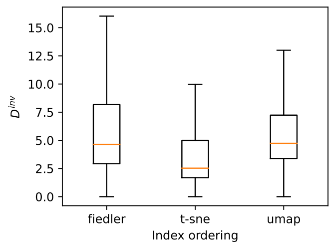


(d) Color scale of the index ordering

Fig. 6: Visual analysis of vertex orderings computed with different techniques on the street map of Maceió.



(a) Forward approach

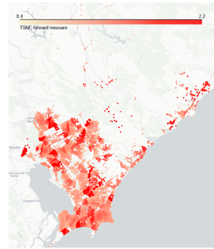


(b) Inverse approach

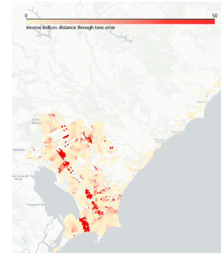
Fig. 7: Box plots of the forward approach (Figure 7a) and inverse approach (Figure 7b) applied to different vertex ordering techniques on the city of Maceió. The vertical axis represent the diagonal lengths for the forward approach, and the normalized diameters for the inverse approach. We omitted the original and random ordering for the inverse approach due to their significantly inferior performance, allowing for a clearer visualization of the best techniques.

dark blue regions indicate that close indices are assigned to vertices distant from each other. Similarly to Fiedler, UMAP shows smooth transitions in the densest part of the map but introduces several discontinuities in the less dense areas. The boxplots of the forward metric, depicted in Fig. 7a, show that t-SNE outperformed the other techniques globally, a trend confirmed by the inverse measure (see the median value in Fig. 7b). This evaluation illustrates the effectiveness of our methodology in distinguishing the performance of different vertex ordering techniques and identifying their characteristics. This experiment highlights the importance of performing local and global analysis. Analyzing only the global information provided by the boxplots could misleadingly lead to the conclusion that t-SNE performs well across the entire graph. However, the color-coded visualization reveals that discontinuities are locally present.

Figure 8 illustrates the performance of t-SNE ordering for both measures (forward and inverse approach) on each graph node, showcasing how t-SNE maintains local orders regarding the forward and inverse approaches. Figure 8a highlights regions where neighbors are far apart on the map, indicating areas of poor local ordering according to the forward measure. Conversely, Figure 8b highlights the discontinuities observed in Figure 6b, reflecting areas of poor local ordering according to the inverse measure. Interestingly, although Figure 6b suggests a strong discontinuity in the middle of the frontier between blue and yellow regions, the inverse measure indicates a weaker discontinuity. Upon further investigation, we found a gap of vertices in that region due to a green area, which could



(a) t-SNE according to forward approach



(b) t-SNE according to inverse approach

Fig. 8: Visual analysis of forward and inverse approach for each node according to vertex orderings computed by t-SNE (Figure 6b) on the street map of Maceió. Colors close to beige indicate low values, while colors close to red represent high values

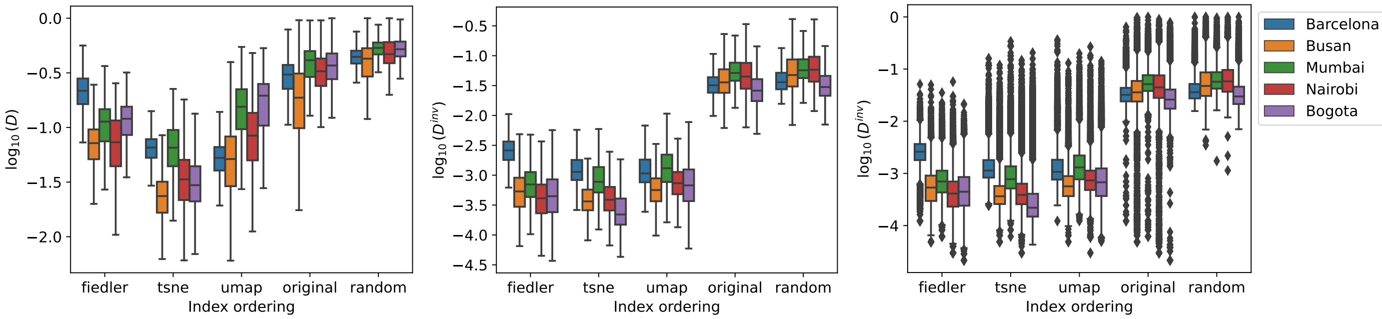
have led to a misinterpretation of discontinuity if only the index colormap was considered. Additionally, we can observe that regions performing well on the forward approach do not necessarily perform well on the inverse approach, and vice-versa. Our evaluation method reveals vertex ordering reliability from multiple perspectives, providing complementary insights in terms of the ordering method's local behavior.

We have extended the results obtained for Maceió to other cities (Barcelona, Busan, Mumbai, Nairobi, and Bogotá). Figure 9 shows that t-SNE outperformed other techniques for both forward and inverse measures in three out of five cities, although exhibiting high variability in terms of outliers in the inverse measure (Fig. 9c). On the other hand, Fiedler ordering presents much less variability in the inverse measure, although it does not achieve a global performance comparable to t-SNE for most of the cities. UMAP's performance was consistent across cities based on the inverse measure, presenting high variability in terms of the forward measure. Fig. 10 displays the histograms of the three evaluated methods in comparison to the random and OSM vertex orders, clearly demonstrating the superior performance of t-SNE. These histograms were constructed by aggregating data from all cities.

The evaluations above pinpoint the importance of the proposed measures in analyzing the behavior and performance of ordering methods. Moreover, they demonstrate that t-SNE outperformed other techniques, although it exhibits high variability in the inverse approach metric. In contrast, the Fiedler ordering showed discrepancies between geolocation and indexing, assigning close indices for spatially distant vertices. However, in the inverse approach, Fiedler showed fewer outliers and thus lower variability compared to t-SNE. UMAP was the least efficient, presenting smooth boxes like Fiedler and high variability in the inverse measure, similar to t-SNE. All these comparisons were possible due to the local visual analysis enabled by the proposed methodology, underscoring their value in evaluating the performance of vertex ordering techniques.

V. Conclusion

We introduced a visual methodology and two local measures designed to assess and compare vertex ordering techniques for spatial graphs. Our experiments demonstrated that the proposed methodology effectively uncovers regions where ordering methods present poor performance, enabling local



(a) Forward approach (b) Inverse approach (c) Inverse approach with outliers

Fig. 9: Analysis of the performance of different techniques across multiple cities. The x-axes represent the techniques, the y-axes (in log scale) the corresponding measures normalized per city.

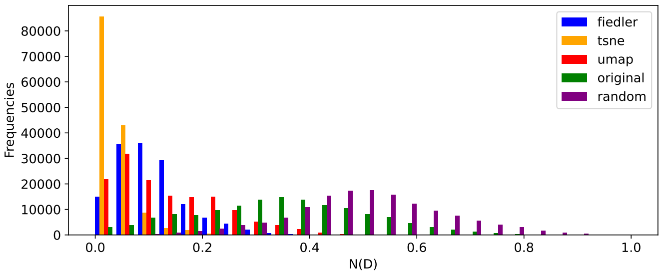


Fig. 10: Forward approach: comparison of the three methods against baselines (original OSM and random vertex order) for all cities (Barcelona, Busan, Mumbai, Nairobi and Bogotá) combined into a single histogram. The horizontal axis represents the normalized diagonal lengths, while the vertical axis indicates their frequencies.

and global analysis. Additionally, the methodology supports hyperparameter tuning, valuable for many applications.

Acknowledgment

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