Graph Fourier Transform

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A major obstacle to the development of a graph signal processing theory is the irregular and coordinate-free nature of a graph domain.

For instance, signal translation is basic operation for signal processing. However, that operation is not naturally implemented in graph domains.

Let $G = (V, E)$ be a weighted graph, $\mathbf{L}$ be its corresponding graph Laplacian, and $f : V \to \mathbb{R}$ a function defined on the vertices of G.

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Graph Fourier Transform

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Inverse Graph Fourier Transform

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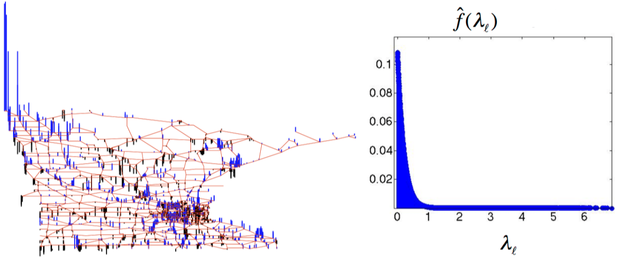
The graph Fourier transform (and its inverse) gives a way to represent a signal in two different domains: the vertex domain and the graph spectral domain.



Graph Fourier Transform: Synthesizing Signals

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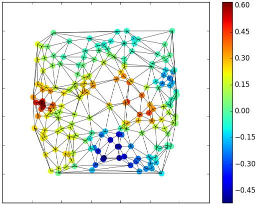

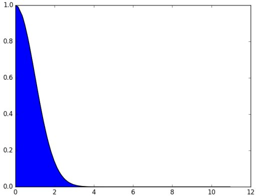

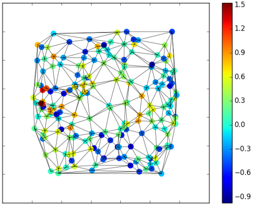

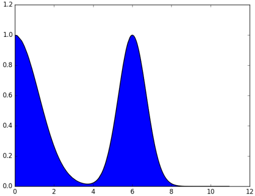

Graph Fourier Transform: Filtering

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Taking the IGFT

$$\mathcal{L}[f](i) = \sum_{l=0}^{n-1} \hat{f}(\lambda_l)\hat{h}(\lambda_l)u_l(i) \quad (1)$$

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Replacing $\hat{h}(\lambda_l)$ and the definition of $\hat{f}(\lambda_l)$ in Eq. (1) we get

$$\mathcal{L}[f](i) = \sum_{j=1}^{n} f(j) \sum_{k=0}^{K} a_k \sum_{l=1}^{n} \lambda_l^k u_l(j) u_l(i)$$

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Therefore, when the filter is a polynomial in the spectral domain, the filtered signal in each node i is a linear combination of the original signal in the neighborhood of i.

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Recalling that $\mathcal{F}[\delta(t-a)](\lambda) = e^{-\mathrm{i}2\pi\lambda a}$, we define translation to a vertex j as

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The normalizing constant $\sqrt{n}$ ensures that the translation operator preserves the mean of a signal, i.e., $\sum_{i=1}^{n}(T_j f)(i) = \sum_{i=1}^{n} f(i)$

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- $\sum_{i=1}^{n} (T_j f)(i) = \sqrt{n}\hat{f}(0) = \sum_{i=1}^{n} f(i)$
- $\|T_j f\| \neq \|f\|$ (the energy of the signal is not preserved)

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Notice that $\hat{f}(s \cdot \lambda_l)$ might not be in the interval $[0, \lambda_n]$. Therefore, dilation can only be used when f is generated from a kernel defined in the whole spectral domain.

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In contrast to the continous case, modulation in the graph setting does not correspond to a translation in the spectral domain.
However, if f is generated from a kernel localized in 0, than $\widehat{M_{\lambda_l} f}(i)$ is concentrated in λ_l.