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ASYMPTOTIC PROPERTIES OF SOLUTIONS TO LINEAR NONAUTONOMOUS DELAY DIFFERENTIAL EQUATIONS THROUGH GENERALIZED CHARACTERISTIC EQUATIONS

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Abstract. We study some properties concerning the asymptotic behavior of solutions to nonautonomous retarded functional differential equations, depending on the knowledge of certain solutions of the associated generalized characteristic equation.

1. Introduction

We are interested in the study of the asymptotic behavior of solutions to the linear nonautonomous retarded functional differential equation (RFDE)

$$x'(t) = L(t)x_t, \quad t \geqslant t_0 \in \mathbb{R}, \tag{1.1}$$

where $L(t)$ is a family of bounded linear functionals on $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{C})$, with $r > 0$, depending on the knowledge of certain solutions of the associated generalized characteristic equation (1.3), introduced below. For a comprehensive introduction for RFDE see [5].

By the Riesz representation theorem, for each $t \geqslant t_0$, there exists a complex valued function of bounded variation $\eta(t, \cdot)$ on $[0, r]$, normalized so that $\eta(t, 0) = 0$ and $\eta(t, \cdot)$ is continuous from the right in $(0, r)$ such that

$$L(t)\varphi = \int_0^r d_\theta \eta(t, \theta)\varphi(-\theta). \tag{1.2}$$

Consider the *generalized characteristic equation*

$$\lambda(t) = \int_0^r d_\theta \eta(t, \theta) \exp\Big(-\int_{t-\theta}^t \lambda(s) ds\Big), \tag{1.3}$$

The solutions of the generalized characteristic equation (1.3) are continuous functions $\lambda(\cdot)$ defined in $[t_0 - r, \infty)$ which satisfy (1.3).

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One obtains the generalized characteristic equation (1.3) by looking for solutions of (1.1) the form

$$x(t) = \exp\Big[\int_0^t \lambda(s)ds\Big]. \tag{1.4}$$

For autonomous RFDE, the constant solutions of (1.3) are the roots of the so called characteristic equation.

This work is motivated by Dix, Philos and Purnaras [1]. These authors studied the asymptotic behavior of solutions of nonautonomous linear function differential equations with discrete delays

$$x'(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \quad t \geqslant 0 \tag{1.5}$$

where the coefficients $a(\cdot)$ and $b_j(\cdot)$ are continuous real-valued functions on $[0,\infty)$, $\tau_j > 0$ for $j = 1, 2, \dots, k$ by means of the knowledge of solutions $\lambda(t)$, defined for $t \geqslant -r$, of the generalized characteristic equation associated to (1.5)

$$\lambda(t) = a(t) + \sum_{j=1}^{k} b_j(t) \exp\Big[-\int_{t-\tau_j}^{t} \lambda(s)ds\Big], \quad t \geqslant 0. \tag{1.6}$$

We also find in [1] a description of the development of results of the type of Theorem 2.1. We would like to mention results of this type are found in [3, 4] too. Dix, Philos and Purnaras extended their results for neutral functional differential equations in [2].

Theorem 2.1 provides a generalization of [1, Thm. 2.3], as it can be applied for instance for RFDE with distributed delay or discrete variable delays, as far as the delays are unifomly bounded. In fact, RFDE (1.5) can be written in the form (1.1) letting

$$L(t)\varphi = a(t)\varphi(0) + \sum_{j=1}^{k} b_j(t)\varphi(\tau_j), \quad \varphi \in \mathcal{C}.$$

We acknowledge that Theorem 2.1 is obtained by an adaptation of the proof of [1, Thm. 2.3] for the more general case of RFDE (1.1), together with ideas from [3]. We observe that [1, Remarks 2.4, 2.5 and 2.6] can be restated here for RFDE (1.1) without modification.

2. Results

Theorem 2.1. *Assume that $\lambda(t)$ is a solution of (1.3) such that*

$$\limsup_{t\to\infty} \int_0^r \theta |e^{-\int_{t-\theta}^t \lambda(s)ds}| d_\theta |\eta|(t,\theta) < 1. \tag{2.1}$$

Then for each solution x of (1.1), we have that the limit

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^t \lambda(s)ds} \tag{2.2}$$

exists, and

$$\lim_{t\to\infty} \Big[x(t)e^{-\int_{t_0}^t \lambda(s)ds}\Big]' = 0. \tag{2.3}$$

Furthermore,

$$\lim_{t\to\infty} x'(t)e^{-\int_{t_0}^t \lambda(s)ds} = \lim_{t\to\infty} \lambda(t)x(t)e^{-\int_{t_0}^t \lambda(s)ds} \tag{2.4}$$

if there exists the limit in the right hand side of (2.4).

Proof. Hypothesis (2.1) implies that there exists $t_1 \geqslant t_0$ such that

$$\sup_{t\geqslant t_1}\int_0^r \theta|e^{-\int_{t-\theta}^t \lambda(s)ds}|d_\theta|\eta|(t,\theta) < 1.$$

Hence without loss of generality, if necessary translating the initial time to t_1, we may assume $t_0 = 0$ and

$$\mu_\lambda := \sup_{t\geqslant 0}\int_0^r \theta|e^{-\int_{t-\theta}^t \lambda(s)ds}|d_\theta|\eta|(t,\theta) < 1. \tag{2.5}$$

Let x be a solution of (1.1), and set

$$y(t) = x(t)e^{-\int_0^t \lambda(s)ds}, \quad t \geqslant -r.$$

Differentiating $y(t)$ when $t \geqslant 0$, using that $x(t)$ is a solution of (1.1), (1.3) and the fundamental theorem of calculus, we obtain

$$\begin{aligned}
y'(t) &= \Big(x'(t) - x(t)\lambda(t)\Big)e^{-\int_0^t \lambda(s)ds} \\
&= \Big(\int_0^r d_\theta\eta(t,\theta)x(t-\theta) - x(t)\int_0^r d_\theta\eta(t,\theta)e^{-\int_{t-\theta}^t \lambda(s)ds}\Big)e^{-\int_0^t \lambda(s)ds} \\
&= \int_0^r d_\theta\eta(t,\theta)x(t-\theta)e^{-\int_0^{t-\theta} \lambda(s)ds}e^{-\int_{t-\theta}^t \lambda(s)ds} \\
&\quad - x(t)e^{-\int_0^t \lambda(s)ds}\int_0^r d_\theta\eta(t,\theta)e^{-\int_{t-\theta}^t \lambda(s)ds} \\
&= \int_0^r d_\theta\eta(t,\theta)y(t-\theta)e^{-\int_{t-\theta}^t \lambda(s)ds} - y(t)\int_0^r d_\theta\eta(t,\theta)e^{-\int_{t-\theta}^t \lambda(s)ds} \\
&= \int_0^r d_\theta\eta(t,\theta)[y(t-\theta) - y(t)]e^{-\int_{t-\theta}^t \lambda(s)ds} \\
&= -\int_0^r d_\theta\eta(t,\theta)\Big[\int_{t-\theta}^t y'(s)ds\Big]e^{-\int_{t-\theta}^t \lambda(s)ds}, \quad t \geqslant 0.
\end{aligned} \tag{2.6}$$

As a characteristic of RFDE, we have that $y'(t)$ is continuous for $t \geqslant 0$, understanding the derivative at $t = 0$ as the derivative from the right. Let

$$M_x = \max_{t\in[0,r]}|y'(t)|. \tag{2.7}$$

Let $t^* \geqslant r$ arbitrary and suppose that for some $A \geqslant 0$ we have

$$|y'(t)| \leqslant A, \quad t^* - r \leqslant t \leqslant t^*. \tag{2.8}$$

Using (2.5) and (2.6), we estimate that

$$\begin{aligned}
|y'(t^*)| &\leqslant \Big|\int_0^r d_\theta\eta(t,\theta)\Big[\int_{t-\theta}^t y'(s)ds\Big]e^{-\int_{t-\theta}^t \lambda(s)ds}\Big| \\
&\leqslant \int_0^r d_\theta|\eta|(t,\theta)\Big|\int_{t-\theta}^t y'(s)ds\Big|\Big|e^{-\int_{t-\theta}^t \lambda(s)ds}\Big| \\
&\leqslant A\int_0^r \theta\Big|e^{-\int_{t-\theta}^t \lambda(s)ds}\Big|d_\theta|\eta|(t,\theta) \leqslant A\mu_\lambda.
\end{aligned}$$

Since $|y'(t^*)| \leqslant A\mu_\lambda < A$, the continuity of $|y'(t)|$ implies that

$$|y'(t)| \leqslant A, \quad t \in [t^* - r, t^* + \delta].$$

Reasoning as above, we show that

$$|y'(t)| \leqslant A\mu_\lambda, \quad t \in [t^*, t^* + \delta].$$

Since $t \mapsto |y'(t)|$ is uniformly continuous on compact intervals, we proceed in this way a finite number of steps and finally conclude that

$$|y'(t)| \leqslant A\mu_\lambda, \quad t \in [t^*, t^* + r]. \tag{2.9}$$

Taking $t^* = nr$, n a positive integer, considering (2.7) for $n = 1$ and using (2.9) with $A = M_x(\mu_\lambda)^{n-1}$ as induction step, we have proved that

$$|y'(t)| \leqslant M_x(\mu_\lambda)^n, \quad t \geqslant nr. \tag{2.10}$$

We observe that (2.10) allows us to conclude that

$$|y'(t)| \leqslant M_x(\mu_\lambda)^{t/r-1}, \quad t \geqslant 0. \tag{2.11}$$

Letting $t \to \infty$, using (2.11), we obtain (2.3).

We obtain (2.4) by a straight forward application of (2.3), differentiating the quantity in the limit and doing simple computations.

We proceed to prove (2.2). The cases $M_x = 0$ and $\mu_\lambda = 0$ are simple, where we have $y(t) \to x(0)$ and $y(t) \to y(r)$ as $t \to \infty$, respectively. For $0 < \mu_\lambda < 1$, for $0 \leqslant t \leqslant T$ we obtain that

$$\begin{aligned}
|y(T) - y(t)| &= \Big| \int_t^T y'(s)ds \Big| \\
&\leqslant M_x \int_t^T (\mu_\lambda)^{s/r-1} ds \\
&= \frac{M_x r}{\mu_\lambda \ln \mu_\lambda} [(\mu_\lambda)^{T/r} - (\mu_\lambda)^{t/r}] \to 0 \quad \text{as } t \to \infty.
\end{aligned}$$

By the Cauchy's criterion of convergence, we have that $y(t) \to L_x$, for some L_x. This shows (2.2) and completes the proof. □

Example 2.2. Consider the linear retarded equation with variable delay

$$x'(t) = \frac{x(t - \tau(t))}{t + c - \tau(t)}, \quad t \geqslant t_0. \tag{2.12}$$

where $c \in \mathbb{R}$ and $\tau : [0, \infty) \to [0, r]$ is a continuous function such that $t + c - \tau(t) > 0$ for $t \geqslant t_0$. FDE (2.12) is written in the form (1.1) letting $\eta(t, \cdot)$ be given by $\eta(t, \theta) = 0$ for $\theta < \tau(t)$, $\eta(t, \theta) = 1/(t + c - \tau(t))$ for $\theta \geqslant \tau(t)$. We have that $\theta \mapsto \eta(t, \theta)$ is increasing and then $|\eta| = \eta$.

The generalized characteristic equation associated to (2.12) is given by

$$\lambda(t) = \frac{1}{t + c - \tau(t)} \exp\Big[-\int_{t-\tau(t)}^t \lambda(s)ds\Big] \tag{2.13}$$

and we have that a solution of (2.13) is given by

$$\lambda(t) = \frac{1}{t + c}. \tag{2.14}$$

For (2.12) and $\lambda(t)$ in (2.14), the left hand side of (2.1) reads as

$$\limsup_{t\to\infty} \int_0^r \theta |e^{-\int_{t-\theta}^t \lambda(s)ds}| d_\theta|\eta|(t, \theta) = \limsup_{t\to\infty} \frac{\tau(t)}{t + c} = 0.$$

and hence the hypothesis (2.1) of Theorem 2.1 is fulfilled and herefore, for all solutions $x(t)$ of (2.12), we have that

$$\lim_{t\to\infty} \frac{x(t)}{t+c} \text{ exists, and } \lim_{t\to\infty} \Big[\frac{x(t)}{t+c}\Big]' = 0. \tag{2.15}$$

Manipulating further the limits in (2.15), we are able to state that $x(t) = O(t)$ and $x'(t) = o(t)$ as $t \to \infty$.

Example 2.3. Consider the linear FDE with distributed delay

$$x'(t) = \int_0^1 \frac{x(t-\theta)}{t-\theta}, \quad t > 1. \tag{2.16}$$

We write (2.16) in the form (1.1) by setting $\eta(t,\theta) = \ln t - \ln(t-\theta)$ for $t > 1$ and $\theta \in [0,1]$. Since $\theta \mapsto \eta(t,\theta)$ is an increasing function, $|\eta| = \eta$.

The generalized characteristic equation associated to (2.16) is given by

$$\lambda(t) = \int_0^1 \frac{1}{t-\theta} \exp\Big[-\int_{t-\theta}^t \lambda(s)ds\Big] d\theta \tag{2.17}$$

which has a solution given by

$$\lambda(t) = 1/t. \tag{2.18}$$

For this $\lambda(t)$ and for $t > 1$, the integral in (2.1) reads as

$$\int_0^1 \frac{\theta}{t-\theta} \exp\Big[-\int_{t-\theta}^t \frac{ds}{s}\Big] d\theta = \int_0^1 \frac{\theta}{t} d\theta = \frac{1}{2t} \to 0 \quad \text{as } t \to \infty.$$

Hence the hypothesis (2.1) of Theorem 2.1 is fulfilled. Again we obtain that

$$\lim_{t\to\infty} \frac{x(t)}{t} \text{ exists,} \quad \lim_{t\to\infty} \Big[\frac{x(t)}{t}\Big]' = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{x'(t)}{t} = 0. \tag{2.19}$$

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