Some peculiarities of the Henstock and Kurzweil integrals of Banach space-valued functions

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Abstract

Some examples, due to G. Birkhoff, are used to explore the differences and peculiarities of the Henstock and Kurzweil integrals in abstract spaces. We also include a proof, due to C. S. Hönig, of the fact that the Bochner-Lebesgue integral is equivalent to the variational Henstock-McShane integral.

Key words: Henstock-Kurzweil integral, Bochner-Lebesgue integral, McShane integral.
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1 Introduction

In 1988, Professor Stefan Schwabik came to Brazil on a visit to Professor Chaim Samuel Hönig and Professor Luciano Barbanti. On that occasion, Professor Schwabik gave a series of lectures on generalized ODE's which motivated Professor Hönig to deal with the Henstock-Kurzweil integration theory for some years. In 1993, in a course on the subject at the University of São Paulo, São Paulo, Brazil, Professor Hönig presented some examples borrowed from [1] in order to clarify the differences and peculiarities of the integrals defined by Henstock ([12]) and by Kurzweil ([19]) for Banach space-valued functions. The notes on such examples are contained here. We also include a proof, due to Hönig ([17]), of the fact that the Bochner-Lebesgue integral is equivalent to the variational Henstock-McShane integral.

2 Basic definitions and terminology

Let $[a,b]$ be a compact interval of the real line $\mathbb{R}$. A *division* of $[a,b]$ is any finite set of closed non-overlapping intervals $[t_{i-1},t_i] \subset [a,b]$ such that $\cup_i [t_{i-1},t_i] = [a,b]$. We write

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$(t_i) \in D_{[a,b]}$ in this case. When $(t_i) \in D_{[a,b]}$ and $\xi_i \in [t_{i-1}, t_i]$ for every i, then (ξ_i, t_i) is a *tagged division* of $[a, b]$. By $TD_{[a,b]}$ we mean the set of all tagged divisions of $[a, b]$.

A *gauge* of $[a, b]$ is any function $\delta : [a, b] \to]0, \infty[$. Given a gauge δ of $[a, b]$, we say $(\xi_i, t_i) \in TD_{[a,b]}$ is *δ-fine*, if $[t_{i-1}, t_i] \subset \{t \in [a, b] ; |t - \xi_i| < \delta(\xi_i)\}$ for every i.

In what follows X denotes a Banach space.

A function $f : [a, b] \to X$ is integrable in the sense of Kurzweil or *Kurzweil integrable* (we write $f \in K([a, b], X)$) and $I = (K) \int_a^b f = (K) \int_a^b f(t)dt \in X$ is its integral if given $\varepsilon > 0$, there is a gauge δ of $[a, b]$ such that for every δ-fine $(\xi_i, t_i) \in TD_{[a,b]}$,

$$\left\| (K) \int_a^b f - \sum_i f(\xi_i)(t_i - t_{i-1}) \right\| < \varepsilon.$$

As it should be expected, the Kurzweil integral is linear and additive over non-overlapping intervals. The basic literature on this subject includes [11], [14], [20], [21], [22], [23], [26].

We use the notation " ˜ " to indicate the indefinite integral of a function $f \in K([a, b], X)$, that is, $\tilde{f} : [a, b] \to X$ is given by $\tilde{f}(t) = (K) \int_a^t f(s) ds$ for all $t \in [a, b]$. We have $\tilde{f} \in \mathcal{C}([a, b], X)$ (see [6] for instance), where $\mathcal{C}([a, b], X)$ is the Banach space of all continuous functions $f : [a, b] \to X$ equipped with the usual supremum norm, $\|f\|_\infty$.

A function $f : [a, b] \to X$ is integrable in the sense of Henstock or *Henstock integrable* or even *variationally Henstock integrable* (we write $f \in H([a, b], X)$) if given $\varepsilon > 0$, there is a function $F : [a, b] \to X$ and a gauge δ of $[a, b]$ such that for every δ-fine $(\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_i \|F(t_i) - F(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})\| < \varepsilon.$$

In this case, we write $(H) \int_a^t f = F(t) - F(a)$, $t \in [a, b]$.

Let $R([a, b], X)$ be the space of abstract Riemann integrable functions $f : [a, b] \to X$ with integral $\int_a^b f$. It is immediate that

$$H([a, b], X) \subset K([a, b], X) \text{ and } R([a, b], X) \subset K([a, b], X),$$

and the integrals coincide when they exist.

Two functions $g, f \in K([a, b], X)$ are called *equivalent*, whenever $\tilde{g}(t) = \tilde{f}(t)$ for all $t \in [a, b]$. When this is the case, $K([a, b], X)_A$ denotes the space of all equivalence classes of functions of $K([a, b], X)$ endowed with the Alexiewicz norm

$$f \in K([a, b], X) \mapsto \|f\|_A = \left\|\tilde{f}\right\|_\infty = \sup_{t \in [a,b]} \left\| (K) \int_a^t f(s) ds \right\|.$$

In an analogous way, $H([a, b], X)_A$ denotes the space of all equivalence classes of functions of $H([a, b], X)$ endowed with the Alexiewicz norm.

If $g, f \in H([a,b], X)$ are equivalent, then $g = f$ almost everywhere in the sense of the Lebesgue measure ([7]). On the other hand, we may have $f \in R([a,b], X) \setminus H([a,b], X)$ (i.e., f belongs to $R([a,b], X)$ but not to $H([a,b], X)$) such that $\tilde{f} = 0$ but $f(t) \neq 0$ for almost every $t \in [a,b]$ (see Example 2.1). Thus $g, f \in R([a,b], X) \subset K([a,b], X)$ and f equivalent to g do not imply $g = f$ almost everywhere.

Let $I \subset \mathbb{R}$ be an arbitrary set and let E be a normed space. A family $(x_i)_{i \in I}$ of elements of E is *summable* with sum $x \in E$ (we write $\sum_{i \in I} x_i = x$) if for every $\varepsilon > 0$, there is a finite subset $F_\varepsilon \subset I$ such that for every finite subset $F \subset I$ with $F \supset F_\varepsilon$,

$$\|x - \sum_{i \in F} x_i\| < \varepsilon.$$

Let $l_2(I)$ be the set of all families $(x_i)_{i \in I}$, $x_i \in \mathbb{R}$, such that the family $(|x_i|^2)_{i \in I}$ is summable. We write

$$l_2(I) = \left\{ x = (x_i)_{i \in I},\, x_i \in \mathbb{R};\, \sum_{i \in I} |x_i|^2 < \infty \right\}.$$

The expression

$$\langle x, y \rangle = \sum_{i \in I} x_i y_i$$

defines an inner product and $l_2(I)$ equipped with the norm

$$\|x\|_2 = \left(\sum_{i \in I} |x_i|^2 \right)^{1/2}$$

is a Hilbert space. Moreover by the Basis Theorem $\{e_i;\, i \in I\}$, where

$$e_i(j) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases},$$

is a complete ortonormal system for $l_2(I)$. We refer to the relation

$$\|x\|_2^2 = \sum_{i \in I} |\langle x_i, e_i \rangle|^2 = \sum_{i \in I} |x_i|^2, \quad \forall\, x \in l_2(I),$$

as *the Bessel equality.*

Example 2.1 *Let $[a,b]$ be non-degenerate and $X = l_2([a,b])$ be equipped with the norm*

$$x \mapsto \|x\|_2 = \left(\sum_{i \in [a,b]} |x_i|^2 \right)^{1/2}.$$

Consider a function $f : [a,b] \to X$ given by $f(t) = e_t$, $t \in [a,b]$. Given $\varepsilon > 0$, there exists $\delta > 0$, with $\delta^{\frac{1}{2}} < \dfrac{\varepsilon}{(b-a)^{\frac{1}{2}}}$, such that for every $\left(\frac{\delta}{2}\right)$-fine $(\xi_j, t_j) \in TD_{[a,b]}$,

$$\left\|\sum_j f(\xi_j)(t_j - t_{j-1}) - 0\right\|_2 = \left\|\sum_j e_{\xi_j}(t_j - t_{j-1})\right\|_2 = \left[\sum_j |t_j - t_{j-1}|^2\right]^{\frac{1}{2}} <$$

$$< \delta^{\frac{1}{2}}\left[\sum_j (t_j - t_{j-1})\right]^{\frac{1}{2}} < \varepsilon$$

where we applied the Bessel equality. Thus $f \in R([a,b],X) \subset K([a,b],X)$ and $\tilde{f} = 0$, since $\int_a^t f(s)ds = 0$ for every $t \in [a,b]$.

If $f \in H([a,b],X)$, then $(H)\int_a^t f = 0$ for every $t \in [a,b]$, since $H([a,b],X) \subset K([a,b],X)$ and $(H)\int_a^t f = (K)\int_a^t f = \int_a^t f = 0$. But

$$\sum_i \|f(\xi_i)(t_i - t_{i-1}) - 0\|_2 = b - a$$

for every $(\xi_i, t_i) \in TD_{[a,b]}$. Hence $f \notin H([a,b],X)$. ■

Let $\mathcal{L}_1([a,b],X)$ be the space of Bochner-Lebesgue integrable functions $f : [a,b] \to X$ with finite absolute Lebesgue integral, that is, $(L)\int_a^b \|f\| < \infty$. We denote by $(L)\int_a^b f$ the Bochner-Lebesgue integral of $f \in \mathcal{L}_1([a,b],X)$ (and also the Lebesgue integral of $f \in \mathcal{L}_1([a,b],\mathbb{R})$). The inclusion

$$\mathcal{L}_1([a,b],X) \subset H([a,b],X)$$

always holds (see [4], [17] or the Appendix). In particular,

$$R([a,b],\mathbb{R}) \subset \mathcal{L}_1([a,b],\mathbb{R}) \subset H([a,b],\mathbb{R}) = K([a,b],\mathbb{R})$$

(see [23], for instance, for a proof of the equality). On the other hand, when X is a general Banach space it is possible to find a function $f : [a,b] \to X$ which is abstract Riemann integrable but not Bochner-Lebesgue integrable. Both Examples 2.1 and 3.1 in the sequel show functions $f \in R([a,b],X) \setminus H([a,b],X)$ (i.e., f belongs to $R([a,b],X)$ but not to $H([a,b],X)$). In particular, such functions belong to $R([a,b],X) \setminus \mathcal{L}_1([a,b],X)$ and also to $K([a,b],X) \setminus H([a,b],X)$.

When real-valued functions are considered only, the Lebesgue integral is equivalent to a modified version of the Kurzweil integral. The idea of slightly modifying Kurzweil's definition is due to E. J. McShane ([24], [25]). Instead of taking δ-fine tagged divisions, McShane considered what we call δ-fine *semi-tagged divisions* (ξ_i, t_i) of $[a,b]$, that is $(t_i) \in$

$D_{[a,b]}$ and $[t_{i-1}, t_i] \subset \{t \in [a,b]\,; |t - \xi_i| < \delta(\xi_i)\}$ for every i. In this case, we write $(\xi_i, t_i) \in STD_{[a,b]}$. Notice that in the definition of semi-tagged divisions, it is *not* required that $\xi_i \in [t_{i-1}, t_i]$ for *any* i. In this manner, McShane's modification of the Kurzweil integral gives an elegant characterization of the Lebesgue integral through Riemann sums (see the Appendix).

Let us denote by $KMS([a,b],\mathbb{R})$ the space of real-valued Kurzweil-McShane integrable functions $f : [a,b] \to \mathbb{R}$, that is, $f \in KMS([a,b],\mathbb{R})$ is integrable in the sense of Kurzweil with the modification of McShane. Formally, $f \in KMS([a,b],\mathbb{R})$ if and only if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that

$$\left| I - \sum_i f(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon.$$

whenever $(\xi_i, t_i) \in STD_{[a,b]}$ is δ-fine. This definition can be extended to Banach space-valued functions.

We have

$$R([a,b],\mathbb{R}) \subset \mathcal{L}_1([a,b],\mathbb{R}) = KMS([a,b],\mathbb{R}) \subset K([a,b],\mathbb{R}) = H([a,b],\mathbb{R}).$$

Furthermore, $K([a,b],\mathbb{R}) \setminus \mathcal{L}_1([a,b],\mathbb{R}) \neq \emptyset$. The next classical example exhibits an $f \in K([a,b],\mathbb{R}) \setminus \mathcal{L}_1([a,b],\mathbb{R})$.

Example 2.2 *Let $F(t) = t^2 \sin(t^{-2})$ for $t \in]0,1]$ and $F(0) = 0$. Let $f = \frac{d}{dt}F$. Because f is Riemann improper integrable, it follows that $f \in K([a,b],\mathbb{R}) = H([a,b],\mathbb{R})$, since the Kurzweil and the Henstock integrals contain their improper integrals (see [21], Cauchy Extension). However $f \notin \mathcal{L}_1([a,b],\mathbb{R})$ (see [28]).*

Example 2.2 says $K([a,b],\mathbb{R}) = H([a,b],\mathbb{R})$ is not an absolute integrable space. More generally, $H([a,b],X)$ and hence $K([a,b],X)$ are non-absolute integrable spaces (see Example 3.4 and Lemma 4.3 in the Appendix).

The generalization of the Riemannian characterization of the Banach space-valued Lebesgue-type integral, namely the Bochner-Lebesgue integral, is not straightforward. In fact, Example 3.1 shows that the modification of McShane applied to the abstract Kurzweil integral can give a more general space than that of Bochner-Lebesgue. On the other hand, if McShane's idea is used to modify the variational definition of Henstock, then we obtain a Riemannian definition of the Bochner-Lebesgue integral (see [4], [17] or the Appendix). Thus, if $HMS([a,b],X)$ denotes the space of Henstock-McShane integrable functions $f : [a,b] \to X$, that is, $f \in HMS([a,b],X)$ is integrable in the sense of Henstock with the modification of McShane, then

$$HMS([a,b],X) = \mathcal{L}_1([a,b],X).$$

In addition,

$$\begin{cases} HMS([a,b],X) \subset H([a,b],X), \\ KMS([a,b],X) \subset K([a,b],X) \text{ and} \\ RMS([a,b],X) \subset R([a,b],X), \end{cases}$$

where $KMS([a,b],X)$ and $RMS([a,b],X)$ denote, respectively, the spaces of Kurzweil-McShane and Riemann-McShane integrable functions $f:[a,b]\to X$.

For other interesting results, the reader may want to consult [5].

3 Birkhoff's examples

The first example of this section shows a Banach space-valued function which is integrable in the sense of Riemann-McShane, but not integrable in the variational sense of Henstock (and neither in the Bochner-Lebesgue sense).

Example 3.1 *Let $G([a,b],X)$ be the Banach space, endowed with the usual supremum norm, $\|\cdot\|_\infty$, of all* regulated functions $f:[a,b]\to X$ *(i.e., f has discontinuities of the first kind only - see [16], p. 16). Let $X = G^-([0,1],\mathbb{R})$, where*

$$G^-([0,1],\mathbb{R}) = \{f \in G([0,1],\mathbb{R});\ f \text{ is left continuous}\},$$

and consider the function

$$f : t \in [0,1] \mapsto f(t) = 1_{[t,1]} \in X,$$

where 1_A denotes the characteristic function of a set $A \subset [0,1]$. Since f is a function of weak bounded variation (we write $f \in BW([0,1],X)$ - see [16], p. 23) and $\phi(t)=t$, $t\in[0,1]$, is an element of $\mathcal{C}([0,1],\mathbb{R})$, it follows from [16], Theorem 4.6, p. 24, that the abstract Riemann-Stieltjes integral, $\int_0^1 df\,\phi$, exists. Moreover, the Riemann-Stieltjes integral, $\int_0^1 f\,d\phi$, exists and the integration by parts formula

$$\int_0^1 f(t)dt = \int_0^1 f\,d\phi = f(t)\cdot t|_0^1 - \int_0^1 df\,\phi$$

holds (see [16], Theorem 1.3, p. 18). Hence $f \in R([0,1],X) \subset K([0,1],X)$. The indefinite integral $\widetilde{f}(t) = \int_0^t f(r)dr$, $t\in[0,1]$, of f is given by $\widetilde{f}(t)(s) = t\wedge s = \inf\{t,s\}$, since

$$\left(\int_0^t f(r)dr\right)(s) = \left(\int_0^t 1_{[r,1]}dr\right)(s) = \int_0^t 1_{[r,1]}(s)dr = \int_0^{t\wedge s} dr = t\wedge s.$$

Hence $\widetilde{f}$ is absolutely continuous. However $\widetilde{f}$ is nowhere differentiable as we will show later. Then the Lebesgue Theorem implies $f \notin \mathcal{L}_1([0,1],X)$. More generally, $f \notin$

$H([0,1],X)$ by the Fundamental Theorem of Calculus for the Henstock integral (see [7]). Or we can prove directly that $f \notin H([0,1],X)$, since

$$\left\| f(\xi_i)(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} f \right\| \geq \frac{1}{2}\,(t_i - t_{i-1}),$$

for every $(\xi_i, t_i) \in TD_{[0,1]}$. Thus $f \in R([0,1],X) \setminus H([0,1],X)$ and, in particular, $f \in R([0,1],X) \setminus \mathcal{L}_1([0,1],X)$. Moreover, we assert that $f \in RMS([0,1],X)$, that is, f is Riemann-McShane integrable. It is enough to show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-fine $(\xi_i, t_i) \in STD_{[0,1]}$,

$$\left\| \tilde{f}(1) - \sum_i f(\xi_i)(t_i - t_{i-1}) \right\| < \varepsilon.$$

Given $\varepsilon > 0$, let $0 < \delta < \varepsilon$ and suppose $(\xi_i, t_i) \in STD_{[0,1]}$ is δ-fine. If $\xi_i \leq s$ and $t_i < \xi_i + \delta$, then $t_i < s + \delta$ which implies $\sum_{\xi_i \leq s}(t_i - t_{i-1}) < s + \delta$ and then

$$s - \sum_{\xi_i \leq s}(t_i - t_{i-1}) < \delta. \tag{1}$$

If $\xi_j > s$ and $t_{j-1} > \xi_j - \delta$, then $t_{j-1} > s - \delta$ and therefore $\sum_{\xi_j > s}(t_j - t_{j-1}) < 1 - (s - \delta) = \sum_i (t_i - t_{i-1}) - s + \delta$. Then $0 \leq \sum_{\xi_i \leq s}(t_i - t_{i-1}) + \delta - s$ which implies

$$s - \sum_{\xi_i \leq s}(t_i - t_{i-1}) < \delta. \tag{2}$$

By (1) and (2), we have

$$\left\| \tilde{f}(1) - \sum_i f(\xi_i)(t_i - t_{i-1}) \right\|_\infty = \sup_{0 \leq s \leq 1} \left| \tilde{f}(1)(s) - \sum_i f(\xi_i)(s)(t_i - t_{i-1}) \right| =$$

$$= \sup_{0 \leq s \leq 1} \left| s - \sum_{\xi_i \leq s}(t_i - t_{i-1}) \right| < \delta < \varepsilon$$

and the assertion follows.

Now we give a proof of the fact that $\tilde{f}$ is neither strongly nor weakly differentiable. We begin by showing that $\tilde{f}$ is not strongly differentiable in the sense that the limit

$$\lim_{\varepsilon_1 \to 0_+,\, \varepsilon_2 \to 0_+} \left[\frac{\tilde{f}(t + \varepsilon_2) - \tilde{f}(t)}{\varepsilon_2} - \frac{\tilde{f}(t + \varepsilon_1) - \tilde{f}(t)}{\varepsilon_1} \right], \quad t \in [0,1[,$$

does not exist. In an analogous way, it can be proved that the limit

$$\lim_{\varepsilon_1\to 0_-,\,\varepsilon_2\to 0_-}\left[\frac{\tilde f(t)-\tilde f(t+\varepsilon_2)}{\varepsilon_2}-\frac{\tilde f(t)-\tilde f(t+\varepsilon_1)}{\varepsilon_1}\right],\quad t\in\,]0,1],$$

does not exist.

For $0<\varepsilon_1<\varepsilon_2$, *we have*

$$\left\|\frac{\tilde f(t+\varepsilon_2)-\tilde f(t)}{\varepsilon_2}-\frac{\tilde f(t+\varepsilon_1)-\tilde f(t)}{\varepsilon_1}\right\| = \sup_{0\le s\le 1}\left|\frac{(t+\varepsilon_2)\wedge s-t\wedge s}{\varepsilon_2}-\frac{(t+\varepsilon_1)\wedge s-t\wedge s}{\varepsilon_1}\right|\ge$$

$$\ge\left|\frac{(t+\varepsilon_2)\wedge s-t\wedge s}{\varepsilon_2}-\frac{(t+\varepsilon_1)\wedge s-t\wedge s}{\varepsilon_1}\right|_{s=t+\varepsilon_1}=$$

$$=\left|\frac{t+\varepsilon_1-t}{\varepsilon_2}-\frac{t+\varepsilon_1-t}{\varepsilon_1}\right|=\left|\frac{\varepsilon_1}{\varepsilon_2}-1\right|\to 1,$$

as we suppose, without loss of generality, that ε_1 *goes faster than* ε_2 *to zero.*

Let us show that $\tilde f$ *is not weakly differentiable in the following sense: if* Y *is a Banach space and* Y' *is its topological dual, then* $g:[a,b]\to Y$ *is* weakly right differentiable *at a point* $t\in[a,b[$ *with* weak right derivative *denoted by* $\frac{d^{\sigma+}g(t)}{dt}$ *whenever for every* $y'\in Y'$,

$$\lim_{\varepsilon\to 0_+}\left\langle\frac{g(t+\varepsilon)-g(t)}{\varepsilon},y'\right\rangle=\left\langle\frac{d^{\sigma+}g(t)}{dt},y'\right\rangle.$$

Analogously we define the weak left derivative *of* g *at a point* $t\in\,]a,b]$.

Let $BV_0([0,1],\mathbb{R})$ *be the Banach space of all functions* $h:[0,1]\to\mathbb{R}$ *of bounded variation which vanish at* $t=0$ *equipped with the norm given by the variation of* h, $V(h)$. *Then* $BV_0([0,1],\mathbb{R})=G^-([0,1],\mathbb{R})'$ *(see [16], Theorem 4.12, p. 26). Besides, for every* $\alpha\in BV_0([0,1],\mathbb{R})$, *the Riemann-Stieltjes integral,* $\int_0^1\tilde f\,d\alpha$, *exists (see [16]), since* $\tilde f$ *is continuous. Given* $\alpha\in BV_0([0,1],\mathbb{R})$, *we will show that*

$$\lim_{\varepsilon\to 0_+}\left\langle\frac{1}{\varepsilon}\left[\tilde f(t+\varepsilon)-\tilde f(t)\right],\alpha\right\rangle=\lim_{\varepsilon\to 0_+}\int_0^1\frac{1}{\varepsilon}\left[\tilde f(t+\varepsilon)-\tilde f(t)\right](s)d\alpha(s)=\left[\alpha(1)-\alpha(t+)\right],$$

where $\alpha(t+)$ *denotes the right lateral limit of* α *at* $t\in[0,1[$. *We have*

$$\lim_{\varepsilon\to 0_+}\int_0^1\frac{1}{\varepsilon}\left[\tilde f(t+\varepsilon)-\tilde f(t)\right](s)d\alpha(s)=\lim_{\varepsilon\to 0_+}\int_0^1\frac{1}{\varepsilon}\left[(t+\varepsilon)\wedge s-t\wedge s\right]d\alpha(s)=$$

$$=\lim_{\varepsilon\to 0_+}\int_t^{t+\varepsilon}\frac{1}{\varepsilon}(s-t)d\alpha(s)+\lim_{\varepsilon\to 0_+}\int_{t+\varepsilon}^1\frac{1}{\varepsilon}\left[(t+\varepsilon)-t\right]d\alpha(s)=$$

$$= \lim_{\varepsilon\to 0_+} \int_t^{t+\varepsilon} \frac{1}{\varepsilon}(s-t)d\alpha(s) + \alpha(1) - \alpha(t+).$$

But

$$\lim_{\varepsilon\to 0_+} \int_t^{t+\varepsilon} \frac{1}{\varepsilon}(s-t)d\alpha(s) = \lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon}\left[\int_t^{t+\varepsilon} s\, d\alpha(s) - \int_t^{t+\varepsilon} t\, d\alpha(s)\right] =$$

$$= \lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon}\left[s\alpha(s)|_t^{t+\varepsilon} - \int_t^{t+\varepsilon} \alpha(s)ds - t\alpha(t+\varepsilon) + t\alpha(t)\right] =$$

$$= \alpha(t+) - \lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \alpha(s)ds = 0,$$

where we applied the integration by parts formula to obtain the second equality. Hence

$$\lim_{\varepsilon\to 0_+} \int_0^1 \frac{1}{\varepsilon}\left[\tilde{f}(t+\varepsilon) - \tilde{f}(t)\right](s)d\alpha(s) = \alpha(1) - \alpha(t+).$$

In a similar way, it can be proved that

$$\left\langle \frac{1}{\varepsilon}\left[\tilde{f}(t) - \tilde{f}(t-\varepsilon)\right], \alpha \right\rangle \longrightarrow \alpha(t-) - \alpha(1),$$

as $\varepsilon \to 0_+$, *where* $\alpha(t-)$ *denotes the left lateral limit of* α *at* $t \in]0,1]$. *Therefore, we showed that* $\tilde{f}$ *is not weakly differentiable.* ■

As we mentioned before, the inclusion $\mathcal{L}_1([a,b],X) \subset KMS([a,b],X)$ always holds. When $X = G^-([0,1],\mathbb{R})$, for instance, one can find a function $f \in KMS([a,b],X) \setminus \mathcal{L}_1([a,b],X)$ (see Example 3.1). In general, $KMS([a,b],X) \setminus \mathcal{L}_1([a,b],X) \neq \emptyset$ for X of infinite dimension as we show next.

Proposition 3.1 (Hönig) *If* X *is an infinite dimensional Banach space, then there exists* $f \in KMS([a,b],X) \setminus \mathcal{L}_1([a,b],X)$.

Proof. Let $\dim X$ denote the dimension of X. If $\dim X = \infty$, then the Theorem of Dvoretsky-Rogers implies there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X which is summable but not absolutely summable. Thus, if we define a function $f : [1,\infty] \to X$ by $f(t) = x_n$, whenever $n \leq t < n+1$, then $(KMS)\int_a^b f = \sum_n x_n \in X$ if the integral exists (here, $(KMS)\int$ denotes the KMS integral). On the other hand, $f \notin \mathcal{L}_1([a,b],X)$, since $(L)\int_a^b \|f\| = \|x_1\| + \|x_2\| + \|x_3\| \ldots = \infty$. ■

The next example exhibits a function which is integrable in the sense of Kurzweil but not in Henstock's sense. It also shows that the Monotone Convergence Theorem, which holds for monotone ordered normed space-valued Kurzweil integrals ([8]), may not be valid for Henstock integrals.

Example 3.2 *Consider the space*

$$Z = l_2(\mathbb{N} \times \mathbb{N}) = \left\{ z = (z_{ij})_{i,j \in \mathbb{N}},\ z_{ij} \in \mathbb{R};\ \sum_{i,j=1}^{\infty} |z_{ij}|^2 < \infty \right\}$$

equipped with the norm

$$z \mapsto \|z\|_2 = \left(\sum_{i,j=1}^{\infty} |z_{ij}|^2 \right)^{1/2}$$

and the function

$$f : [0,1] \to Z$$

given by $f = \sum_{i=1}^{\infty} f_i$, *where* $f_i(t) = 2^i e_{ij}$ *whenever* $\frac{j}{2^i} \leq t < \frac{j}{2^i} + \frac{1}{2^{2i}}$, $j = 0, 1, 2, \ldots, 2^i - 1$, *and* $f_i(t) = 0$ *otherwise. By* e_{ij} *we mean the doubly infinite set of orthonormal vectors of* Z. *We have*

$$f_1(t) = \begin{cases} 2e_{10};\ 0 \leq t < 1/4, \\ 2e_{11};\ 1/2 \leq t < 3/4, \\ 0;\ 1/4 \leq t < 1/2 \text{ or } 3/4 \leq t \leq 1. \end{cases}$$

Hence

$$\int_0^1 f_1 = \int_0^{\frac{1}{4}} 2e_{10} + \int_{\frac{1}{2}}^{\frac{3}{4}} 2e_{11} = \frac{1}{2} e_{10} + \frac{1}{2} e_{11}$$

and therefore

$$\|f_1\|_A = \sup_{0 \leq t \leq 1} \left\| \int_0^t f_1 \right\|_2 = \left\| \int_0^1 f_1 \right\|_2 = \left\| \frac{1}{2} e_{10} + \frac{1}{2} e_{11} \right\|_2 = \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \right]^{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}.$$

Also

$$f_2(t) = \begin{cases} 4e_{20};\ 0 \leq t < 1/16, \\ 4e_{21};\ 1/4 \leq t < 5/16, \\ 4e_{22};\ 1/2 \leq t < 9/16, \\ 4e_{23};\ 3/4 \leq t < 13/16, \\ 0;\ \textit{otherwise}. \end{cases}$$

Then

$$\int_0^1 f_2 = \frac{1}{4} e_{20} + \frac{1}{4} e_{21} + \frac{1}{4} e_{22} + \frac{1}{4} e_{23}$$

and

$$\|f_1 + f_2\|_A = \sup_{0 \leq t \leq 1} \left\| \int_0^t (f_1 + f_2) \right\|_2 = \left\| \int_0^1 f_1 + \int_0^1 f_2 \right\|_2 =$$

$$= \left\| \frac{1}{2} e_{10} + \frac{1}{2} e_{11} + \frac{1}{4} e_{20} + \frac{1}{4} e_{21} + \frac{1}{4} e_{22} + \frac{1}{4} e_{23} \right\|_2 = \left[\frac{1}{2} + \frac{1}{4} \right]^{\frac{1}{2}}.$$

By induction, it can be proved that

$$\|f_1 + f_2 + \ldots + f_n\|_A = \left[\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}\right]^{\frac{1}{2}} < 1.$$

for every $n \in \mathbb{N}$. *Thus, if we define* $g_n = \sum_{i=1}^n f_i$, *for every* $n \in \mathbb{N}$, *then the sequence* $(\|g_n\|_A)_{n\in\mathbb{N}}$ *is bounded. Besides,* $g_n(t) \leq g_{n+1}(t) \leq f(t)$ *for all* $n \in \mathbb{N}$ *and* $t \in [0,1]$. *Hence the Monotone Convergence Theorem (see [8]) implies* $f \in K([0,1], Z)$ *and* $\int_0^1 g_n \to (K)\int_0^1 f$ *as* $n \to \infty$. *Since the Monotone Convergence Theorem also holds for the Kurzweil-McShane integral with obvious adaptations, it follows that* $f \in KMS([0,1], Z)$.

On the other hand, although $g_n \in H([0,1], Z)$ *for every* $n \in \mathbb{N}$, *Birkhoff asserted in [1] that the indefinite integral* $\tilde{f}$ *of* f *is nowhere differentiable and, therefore,* $f \notin H([0,1], Z)$ *by the Fundamental Theorem of Calculus for the Henstock integral (see [7]).* ■

It is known that the space of all equivalence classes of real-valued Kurzweil (or Henstock) integrable functions, equipped with the Alexiewicz norm, is non-complete ([2]). More generally, $K([a,b], X)_A$ and $H([a,b], X)_A$ are non-complete spaces. However such spaces are ultrabornological ([9]) and, therefore, they have good functional analytic properties (see [18] for instance). The next example shows a Cauchy sequence, in the Alexiewicz norm, of Henstock integrable functions which is not convergent.

Example 3.3 *Consider functions*

$$f_n : [0,1] \longrightarrow l_2(\mathbb{N} \times \mathbb{N}), \;\; n \in \mathbb{N}$$

defined by $f_n = \sum_{i=1}^n g_i$, *where* $g_i(t) = e_{ij}$ *whenever* $\frac{j-1}{2^i} \leq t < \frac{j}{2^i}$, $j = 1, 2, \ldots, 2^i$, *and* $g_i(t) = 0$ *otherwise. We have*

$$g_1(t) = \begin{cases} e_{11}; \; 0 \leq t < 1/2, \\ e_{12}; \; 1/2 \leq t < 1, \\ 0; \; t = 1. \end{cases}$$

Hence

$$\|g_1\|_A = \sup_{0\leq t\leq 1} \left\| \int_0^t g_1 \right\|_2 = \left\| \int_0^1 g_1 \right\|_2 = \left\| \frac{1}{2} e_{11} + \frac{1}{2} e_{12} \right\|_2 = \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \right]^{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}.$$

Also

$$g_2(t) = \begin{cases} e_{21}; \; 0 \leq t < 1/4, \\ e_{22}; \; 1/4 \leq t < 1/2, \\ e_{23}; \; 1/2 \leq t < 3/4, \\ e_{24}; \; 3/4 \leq t < 1, \\ 0; \; t = 1. \end{cases}$$

Then

$$\int_0^1 g_2 = \int_0^{\frac{1}{4}} e_{21} + \int_{\frac{1}{4}}^{\frac{1}{2}} e_{22} + \int_{\frac{1}{2}}^{\frac{3}{4}} e_{23} + \int_{\frac{3}{4}}^{1} e_{24} = \frac{1}{4}\left(e_{21} + e_{22} + e_{23} + e_{24}\right).$$

and therefore

$$\|g_2\|_A = \sup_{0\leq t\leq 1} \left\| \int_0^t g_2 \right\|_2 = \left\| \int_0^1 g_2 \right\|_2 = \left(4\frac{1}{4^2}\right)^{\frac{1}{2}} = \left(\frac{1}{4}\right)^{\frac{1}{2}}.$$

By induction, one can show that

$$\|g_i\|_A = \left\| \sum_{j=1}^{2^i} \int_{\frac{j-1}{2^i}}^{\frac{j}{2^i}} e_{ij} \right\|_2 = \left[2^i \left(\frac{1}{2^i}\right)^2\right]^{\frac{1}{2}} = \frac{1}{2^{\frac{i}{2}}},$$

for every $i \in \mathbb{N}$. *Then*

$$\|f_n - f_m\|_A = \left\| \sum_{i=n+1}^{m} g_i \right\|_A \leq \sum_{i=n+1}^{m} \frac{1}{2^{\frac{i}{2}}}$$

which goes to zero for sufficiently large $n, m \in \mathbb{N}$, *with* $n > m$. *Thus* $(f_n)_{n\in\mathbb{N}}$ *is a* $\|\cdot\|_A$-*Cauchy sequence.*

On the other hand,

$$\|f_n(t)\|_2 = \|g_1(t) + g_2(t) + \ldots + g_n(t)\|_2 = \sqrt{n},$$

for every $t \in [0,1]$. *Hence there is no function* $f(t) \in l_2(\mathbb{N}\times\mathbb{N})$, $t \in [0,1]$, *such that* $\lim_{n\to\infty} \|f_n - f\|_A = 0$. ■

The next example presents a Banach space-valued function which is both Henstock and Kurzweil-McShane integrable but is not absolutely integrable.

Example 3.4 *Let* $f : [0,1] \to l_2(\mathbb{N})$ *be given by* $f(t) = \dfrac{2^i}{i} e_i$, *whenever* $\dfrac{1}{2^i} \leq t < \dfrac{1}{2^{i-1}}$, $i = 1, 2, \ldots$. *Then*

$$\int_{\frac{1}{2^i}}^{\frac{1}{2^{i-1}}} \frac{2^i}{i} e_i \, dt = \frac{1}{i} e_i$$

which is summable in $l_2(\mathbb{N})$. *Since the Henstock integral contains its improper integrals (and the same applies to the Kurzweil integral), we have* $f \in H([0,1], l_2(\mathbb{N}))$. *However,* $f \notin \mathcal{L}_1([0,1], l_2(\mathbb{N}))$ *because the sequence* $\left(\frac{1}{i} e_i\right)_{i\in\mathbb{N}}$ *is not summable in* $\mathcal{L}_1([0,1], l_2(\mathbb{N}))$. *By the Monotone Convergence Theorem for the Kurzweil-McShane integral (which follows the ideas of [8] with obvious adaptations),* $f \in KMS([0,1], l_2(\mathbb{N}))$. *But* $f \notin RMS([0,1], l_2(\mathbb{N}))$, *since* f *is not bounded.* ■

The example that follows shows a function of the unit square to $l_2(\mathbb{N}\times\mathbb{N})$ not satisfying the Fubini Theorem.

Example 3.5 *Consider the function $f : [0,1]\times[0,1] \to l_2(\mathbb{N}\times\mathbb{N})$ given by $f(s,t) = 2^i g_i(t)$ on $2^{-i} \leq s < 2^{-i+1}$, $i = 1,2,3,\ldots$, and $f(s,t) = 0$ where not otherwise defined, where $g_i(t) = e_{ij}$ whenever $\dfrac{j-1}{2^i} \leq t < \dfrac{j}{2^i}$, $j = 1,2,\ldots,2^i$, and $g_i(t) = 0$ otherwise. Then $f(s,t)$ is integrable over $[0,1]\times[0,1]$ with*

$$\int\int_{[0,1]\times[0,1]} f(s,t)ds\,dt = \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} \frac{1}{2^i}\, e_{ij}.$$

The integral with respect to s on a single line $t = \text{constant}$ exists, but the integral with respect to t on a single line $s = \text{constant}$ does not because

$$\int_0^1 f(s,t)dt = 2e_{1j_1} + 4e_{2j_2} + 8e_{3j_3} + \ldots$$

for some $j_1, j_2, j_3, \ldots$. ■

The next example presents a function $f : [0,1] \to l_2(\mathbb{N})$ such that $\|f(t)\|_2 = 1$ for every $t \in [0,1]$, but $\|f\|_A < \varepsilon$ for a given $\varepsilon > 0$.

Example 3.6 *Let $\varepsilon > 0$, $n \in \mathbb{N}$ and $f : [0,1] \to l_2(\mathbb{N})$ be defined by $f(t) = e_n$, whenever $\frac{k-1}{n^2} \leq t < \frac{k}{n^2}$, $k = 1,2,\ldots,n^2$, and $f(t) = 0$ otherwise. Hence*

$$\|f\|_A = \left\| (K)\int_0^1 f(t)dt \right\|_2 = \left\| \sum_{k=1}^{n^2} \int_{\frac{k-1}{n^2}}^{\frac{k}{n^2}} e_n\, dt \right\|_2 = \left\| \sum_{k=1}^{n^2} \frac{1}{n^2}\, e_k \right\|_2 = \left(\frac{1}{n^4}\cdot n^2 \right)^{\frac{1}{2}} = \frac{1}{n}.$$

Then taking $n > \frac{1}{\varepsilon}$, we have $\|f\|_A < \varepsilon$. ■

Example 3.7 in the sequel is a Birkhoff-type example due to Hönig. It gives a sequence of functions $f_n : [0,1] \to l_2(\mathbb{N})$ such that $\sup_n \|f_n\|_A < \infty$ but $\|f_n(t)\|_2 \uparrow \infty$, for every $t \in [a,b]$.

Example 3.7 *Let 1_D denote the characteristic function of a set $D \subset [0,1]$. We define a sequence of functions $f_n : [0,1] \to l_2(\mathbb{N})$, $n \in \mathbb{N}$, as follows: $f_n = \sum_{i=1}^{n} g_i$, where*

$$g_i = \sum_{j=1}^{2^{i-1}} 1_{\left[\frac{j-1}{2^{i-1}},\frac{j}{2^{i-1}}\right]} e_{2^{i-1}+j-1}, \quad i = 1,2,\ldots.$$

Then $\sup_{n\to\infty} \|f_n\|_A < \infty$ and, for every $t \in [a,b]$ and every $n \in \mathbb{N}$, $\|f_n(t)\|_2 < \|f_{n+1}(t)\|_2$ and $\|f_n(t)\|_2 \to \infty$. ■

4 Appendix

The integrals introduced by J. Kurzweil ([19]) and independently by R. Henstock ([12]) in the late fifties give a Riemannian definition of the Denjoy-Perron integral which em-compasses the Newton, Riemann and Lebesgue integrals. In 1969, McShane showed that a small change in this definition leeds to the Lebesgue integral.

The Kurzweil and Henstock integrals can be immediately extended to Banach space-valued functions. The extension of the McShane integral made by Gordon, [10], gives a more general integral than that of Bochner-Lebesgue. But the variational Henstock-McShane definition for functions defined on a compact interval of the real line and taking values in a Banach space gives precisely the Bochner-Lebesgue integral. This fact was proved by Congxin and Xiabo ([4]) and independently by Hönig ([17]). Later, Di Piazza and Musal generalized this result ([5]).

Because reference [17] is unavailable to the majority of the mathematicians, we include its results in this Appendix. Unlike the proof of Congxin and Xiabo ([4]), which is based on the Frechet differentiability of the Bochner-Lebesgue integral, the idea of Hönig ([17]) to proof the equivalence of the Bochner-Lebesgue and the Henstock-McShane integrals uses the fact that the indefinite integral of Henstock-McShane and absolutely Henstock integrable functions are of bounded variation. In this manner, the proof in ([17]) seems to be more simple.

We say that a function $f : [a,b] \to X$ is *Bochner-Lebesgue integrable* (we write $f \in \mathcal{L}_1([a,b],X)$, if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions, $f_n : [a,b] \to X$, $n \in \mathbb{N}$, such that

(i) $f_n \to f$ almost everywhere (i.e., $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost every $t \in [a,b]$), and

(ii) $\lim_{n,m\to\infty} (L) \int_a^b \|f_n(t) - f_m(t)\|\, dt = 0$.

We define $(L) \int_a^b f(t)dt = \lim_{n\to\infty} (L) \int_a^b f_n(t)dt$ and $\|f\|_1 = (L) \int_a^b \|f(t)\|\, dt$. The space of all equivalence classes of Bochner-Lebesgue integrable functions, equipped with the norm $\|f\|_1$, is complete.

We say that $f : [a,b] \to X$ is measurable, whenever there is a sequence of simple functions $f_n : [a,b] \to X$ such that $f_n \to f$ almost everywhere. When this is the case,

$$f \in \mathcal{L}_1([a,b],X) \quad \text{if and only if} \quad (L) \int_a^b \|f(t)\|\, dt < \infty \tag{3}$$

(see [29]).

Our aim in the following pages is to show that the integrals of Bochner-Lebesgue and Henstock-McShane coincide, that is, $\mathcal{L}_1([a,b],X) = HMS([a,b],X)$. In this manner, we will prove that the inclusions $\mathcal{L}_1([a,b],X) \subset HMS([a,b],X)$ and $HMS([a,b],X) \subset \mathcal{L}_1([a,b],X)$ hold and we will show that the integrals coincide when defined.

Recall that $(KMS)\int_a^b f$ denotes the integral of a function $f \in KMS([a,b],X)$.

Lemma 4.1 *Given a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $KMS([a,b],X)$ *and a function* $f:[a,b]\to X$, *suppose there exists* $\lim_{n\to\infty}(L)\int_a^b \|f_n(t)-f(t)\|\,dt = 0$. *Then* $f \in KMS([a,b],X)$ *and*

$$\lim_{n\to\infty}(KMS)\int_a^b f_n(t)dt = (KMS)\int_a^b f(t)dt.$$

Proof. Given $\varepsilon > 0$, take n_ε such that for $m,n \geq n_\varepsilon$,

$$(KMS)\int_a^b \|f_n(t)-f_m(t)\|\,dt < \varepsilon$$

and take a gauge δ of $[a,b]$ such that for every δ-fine $(\xi_i,t_i)\in STD_{[a,b]}$,

$$\sum_i \|f_{n_\varepsilon}(\xi_i)-f(\xi_i)\|\,(t_i-t_{i-1}) < \varepsilon. \tag{4}$$

The limit $I=\lim_{n\to\infty}(KMS)\int_a^b f_n(t)dt$ exists, since for $m,n\geq n_\varepsilon$,

$$\left\|(KMS)\int_a^b f_n(t)dt - (KMS)\int_a^b f_m(t)dt\right\| \leq$$

$$\leq (KMS)\int_a^b \|f_n(t)-f(t)\|\,dt + (KMS)\int_a^b \|f(t)-f_m(t)\|\,dt \leq 2\varepsilon.$$

Hence, if $I_n = (KMS)\int_a^b f_n(t)dt$, then

$$\left\|\sum_i f(\xi_i)(t_i-t_{i-1}) - I\right\| \leq \left\|\sum_i \left[f(\xi_i)-f_{n_\varepsilon}(\xi_i)\right](t_i-t_{i-1})\right\| +$$

$$+\left\|\sum_i f_{n_\varepsilon}(\xi_i)(t_i-t_{i-1}) - I_{n_\varepsilon}\right\| + \|I_{n_\varepsilon}-I\| \leq$$

$$\leq \sum_i \|f(\xi_i)-f_{n_\varepsilon}(\xi_i)\|\,(t_i-t_{i-1}) + \left\|\sum_i f_{n_\varepsilon}(\xi_i)(t_i-t_{i-1}) - I_{n_\varepsilon}\right\| + \|I_{n_\varepsilon}-I\|. \tag{5}$$

Then the first summand in (5) is smaller than ε by (4), the third summand is smaller than ε by the definition of n_ε and, if we refine the gauge δ we may suppose, by the definition of I_{n_ε}, that the second summand is smaller than ε and the proof is complete. ■

We show next that Lemma 4.1 remains valid if we replace KMS by HMS.

Lemma 4.2 *Consider a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $HMS([a,b],X)$ *and let* $f:[a,b]\to X$. *If there exists* $\lim_n (L)\int_a^b \|f_n(t)-f(t)\|\,dt = 0$, *then* $f\in HMS([a,b],X)$ *and*

$$\lim_n (KMS)\int_a^b f_n(t)dt = (KMS)\int_a^b f(t)dt.$$

Proof. By Lemma 4.1, $f \in KMS([a,b],X)$ and we have the convergence of the integrals. It remains to prove that $f\in HMS([a,b],X)$, that is, for every $\varepsilon>0$ there exists a gauge δ of $[a,b]$ such that for every δ-fine $(\xi_i,t_i)\in STD_{[a,b]}$,

$$\sum_i \left\|(KMS)\int_{t_{i-1}}^{t_i} f(t)dt - f(\xi_i)(t_i-t_{i-1})\right\| \le \varepsilon.$$

But

$$\sum_i \left\|(KMS)\int_{t_{i-1}}^{t_i} f(t)dt - f(\xi_i)(t_i-t_{i-1})\right\| \le \sum_i \left\|(KMS)\int_{t_{i-1}}^{t_i} [f(t)-f_n(t)]\,dt\right\| +$$

$$+\sum_i \left\|(KMS)\int_{t_{i-1}}^{t_i} f_n(t)dt - f_n(\xi_i)(t_i-t_{i-1})\right\| + \sum_i \|f_n(\xi_i)-f(\xi_i)\|\,(t_i-t_{i-1}). \quad (6)$$

Because $\int_a^b \|f_n(t)-f(t)\|\,dt \to 0$, there exists $n_\varepsilon>0$ such that the first summand in (6) is smaller than $\varepsilon/3$ for all $n\ge n_\varepsilon$. Choose an $n\ge n_\varepsilon$. Then we can take δ such that the third summand is smaller than $\varepsilon/3$, since it approaches $\int_a^b \|f_n(t)-f(t)\|\,dt$. Also, because $f_n\in HMS([a,b],X)$, we may refine δ so that the second summand becomes smaller than $\varepsilon/3$ and we finished the proof. ■

Lemma 4.3 $\mathcal{L}_1([a,b],X)\subset KMS([a,b],X)$.

For a proof of Lemma 4.3, see Theorem 16 in [10] for instance.

Now we are able to prove the inclusion

Theorem 4.1 $\mathcal{L}_1([a,b],X)\subset HMS([a,b],X)$.

Proof. By Lemma 4.3, $\mathcal{L}_1([a,b],X)\subset KMS([a,b],X)$. Then, following the steps of the proof of Lemma 4.3 and using Lemma 4.2, we obtain the result. ■

Let $BV([a,b],X)$ denote the space of all functions $f:[a,b]\to X$ of bounded variation. We show next that the indefinite integral of any function of $HMS([a,b],X)$ belongs to $BV([a,b],X)$.

Lemma 4.4 *If* $f\in HMS([a,b],X)$, *then* $\tilde{f}\in BV([a,b],X)$.

Proof. It is enough to show that every $\xi \in [a,b]$ has a neighborhood where $\tilde{f}$ is of bounded variation. By hypothesis, given $\varepsilon > 0$, there exists a gauge δ of $[a,b]$ such that for every δ-fine semi-tagged division $d = (\xi_i, t_i)$ of $[a,b]$,

$$\sum_i \left\| \tilde{f}(t_i) - \tilde{f}(t_{i-1}) - f(\xi_i)(t_i - t_{i-1}) \right\| < \varepsilon. \tag{7}$$

Since $g = f$ almost everywhere implies $g \in HMS([a,b],X)$ and $\tilde{g} = \tilde{f}$ (this fact follows by straightforward adaptation of [11], Theorem 9.10 for Banach space-valued functions; see also [7]), we may change f on a set of measure zero and its indefinite integral does not change. We suppose, therefore, that $f(\xi) = 0$.

Let $s_0 < s_1 < \ldots < s_m$ be any division of $[\xi - \delta(\xi), \xi + \delta(\xi)]$. If we take $\xi_j = \xi$ for $j = 1, 2, \ldots, m$, then (ξ_j, s_j) is a δ-fine semi-tagged division of $[\xi - \delta(\xi), \xi + \delta(\xi)]$ and therefore from (7) and fact that $f(\xi_j) = f(\xi) = 0$ for all j, we have

$$\sum_{j=1}^{m} \left\| \tilde{f}(s_j) - \tilde{f}(s_{j-1}) \right\| \le \varepsilon$$

and the proof is complete. ■

Lemma 4.5 *Suppose $f \in H([a,b],X)$. The following properties are equivalent:*

(i) f is absolutely integrable;

(ii) $\tilde{f} \in BV([a,b],X)$.

Proof. (i) ⇒ (ii). Suppose f is absolutely integrable. Since the variation of $\tilde{f}$, $V(\tilde{f})$, is given by

$$V(\tilde{f}) = \sup \left\{ \sum_i \left\| \tilde{f}(t_i) - \tilde{f}(t_{i-1}) \right\| ; \, (t_i) \in D_{[a,b]} \right\}$$

we have

$$\sum_i \left\| \tilde{f}(t_i) - \tilde{f}(t_{i-1}) \right\| = \sum_i \left\| (K) \int_{t_{i-1}}^{t_i} f(t)dt \right\| \le$$

$$\le \sum_i (K) \int_{t_{i-1}}^{t_i} \|f(t)\| \, dt = (K) \int_a^b \|f(t)\| \, dt.$$

(ii) ⇒ (i). Suppose $\tilde{f} \in BV([a,b],X)$. We will prove that the integral $(K) \int_a^b \|f(t)\| \, dt$ exists and $(K) \int_a^b \|f(t)\| \, dt = V(\tilde{f})$. Given $\varepsilon > 0$, we need to find a gauge δ of $[a,b]$ such that

$$\left| \sum_i \|f(\xi_i)\| \, (t_i - t_{i-1}) - V(\tilde{f}) \right| < \varepsilon,$$

whenever $(\xi_i, t_i) \in TD_{[a,b]}$ is δ-fine. But

$$\left|\sum_i \|f(\xi_i)\| (t_i - t_{i-1}) - V(\tilde{f})\right| \leq$$

$$\leq \sum_i \left| \|f(\xi_i)\| (t_i - t_{i-1}) - \left\|(K)\int_{t_{i-1}}^{t_i} f(t)dt\right\| \right| + \left|\sum_i \left\|(K)\int_{t_{i-1}}^{t_i} f(t)dt\right\| - V(\tilde{f})\right| \leq$$

$$\leq \sum_i \left\| f(\xi_i)(t_i - t_{i-1}) - (K)\int_{t_{i-1}}^{t_i} f(t)dt\right\| + \left|\sum_i \left\|\tilde{f}(t_i) - \tilde{f}(t_{i-1})\right\| - V(\tilde{f})\right|. \qquad (8)$$

By the definition of $V(\tilde{f})$, we may take $(t_i) \in D_{[a,b]}$ such that the last summand in (8) is smaller than $\varepsilon/2$. Because $f \in H([a,b], X)$, we may take a gauge δ such that for every δ-fine $(\xi_i, t_i) \in TD_{[a,b]}$, the first summand in (8) is also smaller than $\varepsilon/2$ (and we may suppose that the points chosen for the second summand are the points of the δ-fine tagged division (ξ_i, t_i)). ■

The next result is a consequence of the fact that $HMS([a,b], X) \subset H([a,b], X)$ and Lemmas 4.4 and 4.5.

Corollary 4.1 *All functions of $HMS([a,b], X)$ are absolutely integrable.*

Lemma 4.6 *All functions of $H([a,b], X)$ are measurable.*

For a proof of Lemma 4.6, see Theorem 9 in [3] for instance.
Finally, we can prove the inclusion

Theorem 4.2 $HMS([a,b], X) \subset \mathcal{L}_1([a,b], X)$.

Proof. The result follows from the facts that all functions of $H([a,b], X)$ and hence of $HMS([a,b], X)$ are measurable (Lemma 4.6) and all functions of $HMS([a,b], X)$ are absolutely integrable (Corollary 4.1) (see [29]). ■

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