A note on binary quintic forms and lines of principal curvature on surfaces in $\mathbb{R}^5$

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Abstract

Asymptotic curves are well defined on a smooth surface M in the Euclidean space $\mathbb{R}^5$ and are solutions of a binary quintic differential equation. We propose in this paper a definition of the lines of principal curvature on M that uses covariants of binary quintic forms as well as the metric structure on M.

1 Introduction

Our aim is to define lines of principal curvature on a smooth surface M embedded in the Euclidean space $\mathbb{R}^5$. We want these curves to form an orthogonal net away from isolated points (umbilics) on M, so their equation should be a binary quadratic differential equation. Asymptotic curves on M are well defined and are given by a binary quintic differential equation ([10, 15]). This equation defines at each point on M a binary quintic form. There are three independent covariant of this form of degree 2 and order 2, 6 and 8 ([17]). These are denoted in [17] by 2.2, 6.2 and 8.2 and we shall denote them here by $C_{2,2}, C_{6,2}$ and $C_{8,2}$. Each one of these can be viewed as a binary quadratic form and represented by a point in the projective plane. The polar lines of these points with respect to the conic of degenerate quadratic forms contain a unique quadratic form $P_{2,2}$, $P_{6,2}$ and $P_{8,2}$, whose roots are orthogonal [16] (this is where the metric on M appears). We define the $C_{i,2}$-lines of principal curvature on M, $i = 2, 6, 8$, as the solution curves of the quadratic differential equation associated to $P_{i,2}$.

Lines of principal curvature on smooth orientable surfaces in $\mathbb{R}^3$ are classical objects and are defined as the curves whose tangents at each point are parallel to an eigenvector

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of the shape operator (Weingarten map). They form an orthogonal net away from umbilic points. Their configurations at umbilics and their structural stability are studied more recently in [3, 14]. For surfaces in $\mathbb{R}^n$, $n \geq 4$, shape operators are defined along normal vector fields. For a given normal vector field μ on M, one can define, as is done in [13] when $n = 4$, μ-lines of principal curvature. These are the curves whose tangents at each point are parallel to an eigenvector of the shape operator along μ. They form an orthogonal net on M away from μ-umbilic points. Our aim is to define a distinguished pair of orthogonal foliations on M and show that they are μ-lines of principal curvature of some normal vector field μ on M. We did this for surfaces in $\mathbb{R}^4$ in [16] using the fact that the asymptotic directions are given by a binary quadratic differential equation; see Theorem 4.1. (The construction in [16] follows from that in [5] and is valid for self-adjoint operators on surfaces endowed with a metric g which could have varying signature [7, 12].) We deal here with the case of surfaces in $\mathbb{R}^5$.

We recall in §4 some concepts on the extrinsic geometry of surfaces in $\mathbb{R}^5$ and in §3 results on binary forms that are needed in this paper. We define the lines of principal curvature in §4.

2 Surfaces in $\mathbb{R}^5$

Let M be a smooth surface in the Euclidean space $\mathbb{R}^5$ defined locally by an embedding $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^5$, and TM and NM its tangent and normal bundles. Let $\bar{\nabla}$ denote the Riemannian (Levi-Civita) connection of $\mathbb{R}^5$ and ∇ the induced Riemannian connection on M (see [6]). Given a normal field μ on M, the second fundamental form along μ, is the bilinear symmetric map $\mathrm{II}_\mu : TM \times TM \to \mathbb{R}$, given by $\mathrm{II}_\mu = \langle \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y, \mu\rangle$, where $\bar{X}$ (resp. $\bar{Y}$) denotes a local extension of X (resp. Y) to $\mathbb{R}^5$. To II_μ is associated a unique self-adjoint operator $S_\mu : TM \to TM$, given by $\mathrm{II}_\mu(X, Y) =< S_\mu(X), Y >$. The map S_μ is referred to as the *shape operator along* μ and is given by $S_\mu(X) = -(\bar{\nabla}_X \bar{\mu})^T$.

Let $p \in M$ and $\mathbf{e} = \{e_1, e_2, e_3, e_4, e_5\}$ be a frame in a neighbourhood of p, such that $\{e_1, e_2\}$ is a tangent frame and $\{e_3, e_4, e_5\}$ is a normal orthonormal frame in this neighbourhood. Consider the linear map $L_p : N_pM \to \mathcal{Q}_2$, where $\mathcal{Q}_2$ denotes the space of quadratic forms in two variables and $L_p(\mu)$ is the quadratic form associated to II_μ at q. If $\mu = (\mu_3, \mu_4, \mu_5) \in N_pM$ with respect to the basis $\{e_3, e_4, e_5\}$, then $L_p(\mu) = \mu_3(d^2\phi \cdot e_3) + \mu_4(d^2\phi \cdot e_4) + \mu_5(d^2\phi \cdot e_5)$. Let

$$\Lambda_p = \begin{bmatrix} \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_4 & \beta_4 & \gamma_4 \\ \alpha_5 & \beta_5 & \gamma_5 \end{bmatrix},$$

with $\alpha_i = \langle \phi_{uu}, e_i\rangle$,$\beta_i = \langle \phi_{uv}, e_i\rangle$ and $\gamma_i = \langle \phi_{vv}, e_i\rangle$, $i = 3, 4, 5$ (subscripts denote partial differentiation and (u, v) denote the parameters in U). The matrix Λ_p is referred to as the matrix of the second fundamental form with respect to the frame $\mathbf{e}$.

Asymptotic directions on surfaces in $\mathbb{R}^3$ are characterised in Differential Geometry textbooks in terms of the normal curvature. However, this approach does not generalise easily to manifolds immersed in higher dimensional spaces. A better approach is to define these directions in terms of certain singularities of maps associated to the contact of the surface with flat objects (k-planes). For example, given a generic smooth surface $M \subset \mathbb{R}^3$, the projection along a tangent direction ξ at $p \in M$ to a transverse plane is right-left equivalent to a cusp $(u, uv + v^3)$ if and only if ξ is an asymptotic direction. (The projections measure the contact of the surface with lines.) Following this approach, asymptotic directions on surfaces in $\mathbb{R}^4$ (resp. $\mathbb{R}^5$) are studied in [1, 9] (resp. [10, 15]). Let $\alpha = (\alpha_3, \alpha_4, \alpha_5)$, $\beta = (\beta_3, \beta_4, \beta_5)$, $\gamma = (\gamma_3, \gamma_4, \gamma_5)$ be the coefficients of the second fundamental form written as vectors and let $[,,]$ denote a 3×3-determinant. We have the following result from [15]. The formulae for the coefficients are simplified here using the identities $\alpha_v = \beta_u$ and $\beta_v = \gamma_u$.

Theorem 2.1 ([15]) *There is at least one and at most five asymptotic curves passing through any point on a generic immersed surface in $\mathbb{R}^5$. These curves are solutions of the implicit differential equation*

$$l_0 dv^5 + l_1 du dv^4 + l_2 du^2 dv^3 + l_3 du^3 dv^2 + l_4 du^4 dv + l_5 du^5 = 0, \tag{1}$$

where the coefficients l_i, depend on the coefficients of the second fundamental form and their first order partial derivatives, and are given by

$$l_0 = [\tfrac{\partial\gamma}{\partial v}, \beta, \gamma],$$

$$l_1 = 3[\tfrac{\partial\gamma}{\partial u}, \beta, \gamma] + [\tfrac{\partial\gamma}{\partial v}, \alpha, \gamma],$$

$$l_2 = 3[\tfrac{\partial\alpha}{\partial v}, \beta, \gamma] + 3[\tfrac{\partial\gamma}{\partial u}, \alpha, \gamma] + [\tfrac{\partial\gamma}{\partial v}, \alpha, \beta],$$

$$l_3 = [\tfrac{\partial\alpha}{\partial u}, \beta, \gamma] + 3[\tfrac{\partial\alpha}{\partial v}, a, \gamma] + 3[\tfrac{\partial\gamma}{\partial u}, \alpha, \beta],$$

$$l_4 = [\tfrac{\partial\alpha}{\partial u}, \alpha, \gamma] + 3[\tfrac{\partial\alpha}{\partial v}, \alpha, \beta],$$

$$l_5 = [\tfrac{\partial\alpha}{\partial u}, \alpha, \beta].$$

3 Binary forms

Binary forms have a rich history. We give a brief review of results that are of interest here. These are taken from [8, 11]. A binary form $f(x, y)$ of degree n in the two variables x and y is a homogeneous polynomial of degree n in x and y, that is,

$$f(x, y) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a_k x^k y^{n-k}.$$

The coefficients a_k of f belong to a field $\mathbb{K}$ of characteristic zero (in this paper $\mathbb{K} = \mathbb{R}$). Consider the action of $c \in GL_2(\mathbb{K})$ on the variables x and y

$$x = c_{11}\bar{x} + c_{12}\bar{y}, \qquad y = c_{21}\bar{x} + c_{22}\bar{y}.$$

The binary form f is transformed to another binary form $\bar{f}$ of degree n in the new variables $\bar{x}$ and $\bar{y}$, with coefficients $\bar{a}_k$,

$$\bar{f}(\bar{x}, \bar{y}) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \bar{a}_k \bar{x}^k \bar{y}^{n-k}.$$

A polynomial I in the variables $A_0, A_1, \ldots, A_n, X, Y$ is said to be a *covariant of index s* of binary forms of degree n if for any binary form $f(x, y)$ of degree n and any $c \in GL_2(\mathbb{K})$, the following holds $I(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_n, \bar{x}, \bar{y}) = (\det c)^s I(a_0, a_1, \ldots, a_n, x, y)$. We denote $I(a_0, a_1, \ldots, a_n, x, y)$ by $I(f)$.

A covariant in which the variables x and y do not occur is said to be an *invariant.*

Let I be a homogeneous polynomial covariant of a binary form. The *degree* of I is its degree in the variables x and y. The *order* of I is its degree in the coefficients $a_0, \ldots, a_n$. Let

$$\Omega_{\alpha\beta} = \frac{\partial^2}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2}{\partial x_\beta \partial y_\alpha}$$

The r^{th} *order transvectant* of a pair of smooth functions $Q(x, y)$ and $R(x, y)$ is the function

$$(Q, R)^{(r)} = (\Omega_{\alpha\beta})^r [Q(x_\alpha, y_\alpha) R(x_\beta, y_\beta)] \Big|_{\begin{array}{l} x = x_\alpha = x_\beta \\ y = y_\alpha = y_\beta \end{array}}.$$

We have

$$(Q, R)^{(r)} = \sum_{i=0}^{r} (-1)^i \left(\begin{array}{c} r \\ i \end{array} \right) \frac{\partial^r Q}{\partial x^{r-i} \partial y^i} \frac{\partial^r R}{\partial x^i \partial y^{r-i}}.$$

We are seeking covariants of binary quintics that have degree 2. There are three of them ([17]) and they are represented by the following transvectants (see for example Theorem 5.4 in [11]):

$$\begin{array}{rcl} C_{2,2} & = & \frac{1}{28800}(f, f)^{(4)} \\ C_{6,2} & = & \frac{1}{115200}((f, C_{2,2})^{(2)}, (f, C_{2,2})^{(2)})^{(2)} \\ C_{8,2} & = & \frac{1}{3317760000}((f, (f, C_{2,2})), (f, (f, C_{2,2})^{(2)}))^{(4)}. \end{array}$$

The covariants $C_{6,2}$ and $C_{8,2}$ have lengthy expressions and $C_{2,2}$ is given by

$$C_{22} = (a_0 a_4 + 3a_2^2 - 4a_1 a_3) y^2 + (a_0 a_5 - 3a_1 a_4 + 2a_2 a_3) xy + (a_1 a_5 - 4a_2 a_4 + 3a_3^2) x^2.$$

Remark 3.1 The covariant $C_{6,2}$ has an interpretation in terms of forms apolar to the quintic form f (see §5 in [8] for terminology and results on apolarity). The set of cubic forms apolar to f is, under some condition, a one dimensional vector space generated by the covariant $J = (f, C_{2,2})^{(2)}$, and $C_{6,2}$ is a scalar multiple of the Hessian of J.

4 Lines of principal curvature on surfaces in $\mathbb{R}^5$

We start by giving the motivation behind our definition of lines of principal curvature of surfaces in $\mathbb{R}^5$. We recall some notions from [7, 12, 16]. Let M be a two dimensional manifold (i.e., a surface) and suppose that M is endowed with a non degenerate metric g. Let $\phi : U \to M$ be a local parametrisation of M, where U is an open subset of $\mathbb{R}^2$. Then the first fundamental form of M (or the metric g) is the quadratic form

$$g = Gdv^2 + 2Fdudv + Edu^2 \tag{2}$$

with coefficients $E = g(\phi_u, \phi_u)$, $F = g(\phi_u, \phi_v)$, $G = g(\phi_v, \phi_v)$.

Suppose given on (M, g) a self-adjoint operator $\mathbb{A}$, that is, a smooth map $TM \to TM$ with the property that its restriction $\mathbb{A}_p : T_pM \to T_pM$ is a linear map satisfying $g(\mathbb{A}_p(X), Y) = g(X, \mathbb{A}_p(Y))$ at any $p \in M$ and for any $X, Y \in T_pM$. Let

$$l = g(\mathbb{A}(\phi_u), \phi_u), \quad m = g(\mathbb{A}(\phi_u), \phi_v) = g(\mathbb{A}(\phi_v), \phi_u), \quad n = g(\mathbb{A}(\phi_v), \phi_v).$$

We refer to these as the coefficients of $\mathbb{A}$ (they determine $\mathbb{A}$ in $\phi(U)$). A direction $X \in T_pM$ is called $\mathbb{A}$-*asymptotic* if $g(\mathbb{A}_p(X), X) = 0$. It follows that the $\mathbb{A}$-asymptotic curves (whose tangents at all points are $\mathbb{A}$-asymptotic directions) are solutions of the binary quadratic differential equation

$$ndv^2 + 2mdvdu + ldu^2 = 0. \tag{3}$$

Points where $\mathbb{A}_p$ is a multiple of the identity are called *umbilic points*. When $\mathbb{A}_p$ has real eigenvalues, we call them the $\mathbb{A}$-principal curvatures and we call their associated eigenvectors the $\mathbb{A}$-principal directions. The integral curves of the $\mathbb{A}$-principal directions are labelled the lines of $\mathbb{A}$-principal curvature. When these exist, they are orthogonal and are the solution curves of

$$(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)du^2 = 0. \tag{4}$$

Equations (3) and (4) are binary quadratic differential equations in the form

$$a(u, v)dv^2 + 2b(u, v)dvdu + c(u, v)du^2 = 0, \tag{5}$$

where a, b, c are smooth functions on U. Following [5], equation (5) determines a family of quadratic forms $ay^2 + 2bxy + cx^2$ (parametrised by (u, v)) which is represented at each point (u, v) by the point $Q = (a : 2b : c)$ in the projective plane $\mathbb{R}P^2$. We say that Q has orthogonal roots if the solutions of equation (5) are orthogonal. We denote by $\hat{Q}$ the polar line of Q with respect the conic of degenerate forms (which are forms with $b^2 - ac = 0$).

We identify a self-adjoint operator $\mathbb{A}$ with $\lambda\mathbb{A}$, where λ is a nowhere vanishing smooth function on M. Then, $\mathbb{A}$ is represented at each point $(u, v) \in U$ by the point

$A = (l : 2m : n)$ in $\mathbb{R}P^2$ (which also represents the equation of the $\mathbb{A}$-asymptotic curves). The metric g (2) is represented by the point $L = (G : 2F : E)$ in $\mathbb{R}P^2$. The polar line $\hat{L}$ of L is the set of quadratic forms whose roots are orthogonal ([12]). We denote by $Jac(A, g)$ the Jacobian of the quadratic forms $ny^2 + 2mxy + lx^2$ and $Gy^2 + 2Fxy + Ex^2$.

Theorem 4.1 ([5, 12, 16]) *The polar line $\hat{A}$ of A intersects $\hat{L}$ at $Jac(A, g)$. This point is the unique quadratic form on $\hat{A}$ which has orthogonal roots and is precisely the quadratic form representing the binary quadratic differential equation* (4) *of the lines of $\mathbb{A}$-principal curvature.*

The asymptotic curves on surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ are given by a binary quadratic differential equation. One can extract a self-adjoint operator from each of these equations and define the lines of principal curvature as the solutions of the quadratic differential equation associated to the quadratic form $\hat{A} \cap \hat{L}$ in Theorem 4.1; see [16]. (This can also be done for timelike surfaces in the Minkowski spaces $\mathbb{R}^3_1$ and $\mathbb{R}^4_1$ [7].) However, the asymptotic curves on a surface M in $\mathbb{R}^5$ are given by a binary quintic differential equation (Theorem 2.1), so we cannot use Theorem 4.1 directly to define the lines of principal curvature on M. We proceed as follows.

Let $\phi : U \to M$ be a local parametrisation of M. Equation (1) determines a family of binary quintic forms parametrised by points in U. If we denote this family by A, then its coefficients are smooth functions on U and are given by

$$A = (a_0, a_1, a_2, a_3, a_4, a_5) = (l_0, \frac{1}{5}l_1, \frac{1}{10}l_2, \frac{1}{10}l_3, \frac{1}{5}l_4, l_5),$$

with $l_0, \dots, l_5$ as in Theorem 2.1. We denote by $C_{2,2}(A), C_{6,2}(A), C_{8,2}(A)$ the (families of) transvectants associated to A (see §3). These are binary quadratic forms with coefficients depending smoothly on those of A. For instance,

$$C_{2,2}(A) = (20l_0l_4 - 8l_1l_3 + 3l_2^2)y^2 + 2(50l_0l_5 - 6l_1l_4 + l_2l_3)xy + (20l_1l_5 - 8l_2l_4 + 3l_3^2)x^2.$$

We can now apply Theorem 4.1 to define lines of principal curvature on M in terms of the above transvectants.

Definition 4.2 *Let M be a smooth surface embedded in $\mathbb{R}^5$. The $C_{i,2}$-lines of principal curvature on M, $i = 2, 6, 8$, are the solution curves of the binary quadratic differential equation associated to the quadratic form $Jac(g, C_{i,2}(A))$, which is the unique quadratic form on the polar line of $C_{i,2}(A)$ which has orthogonal roots.*

The equation of the $C_{i,2}$-lines of principal curvature that has the simplest expression is that of $C_{2,2}$-lines of principal curvature. It is given by

$$\begin{aligned}
&((8l_3l_1 - 3l_2^2 - 20l_4l_0)F + (-6l_4l_1 + 50l_5l_0 + l_3l_2)G)dv^2 + \\
&((8l_3l_1 - 3l_2^2 - 20l_4l_0)E + (-8l_4l_2 + 20l_5l_1 + 3l_3^2)G)dudv + \\
&((6l_4l_1 - 50l_5l_0 - l_3l_2)E + (-8l_4l_2 + 20l_5l_1 + 3l_3^2)F)du^2 = 0.
\end{aligned}$$

Proposition 4.3 *The $C_{i,2}$-lines of principal curvature, $i = 2, 6, 8$, do not depend on the choice of the local parametrisation of M. They also do not depend on the choice of the orthonormal frame $\mathbf{e} = \{e_1, e_2, e_3, e_4, e_5\}$, where $\{e_1, e_2\}$ (resp. $\{e_3, e_4, e_5\}$) is a tangent (resp. normal) orthonormal frame.*

Proof The asymptotic directions do not depend on the choice of the parametrisation (they are determined by the contact of M with k-dimensional planes, $k = 1, 2, 3, 4$ ([10, 15]). Let h be a change of parameter. Its differential map Dh_p determines, at each point p, a linear transformation in T_pM. We follow the notation in §3 for the action of Dh_p on binary forms. Let A be the binary quintic form associated to the equation of the asymptotic directions at p. Because $C_{i,2}(A)$ is a covariant binary form ($i = 2, 6, 8$), $C_{i,2}(\bar{A})$ is simply $\overline{C_{i,2}(A)}$. Therefore, the solutions of $Jac(\bar{g}, \overline{C_{i,2}(A)})$ are the same as those of $\overline{Jac(g, C_{i,2}(A))}$.

Another choice of a normal frame does not affect the coefficients of equation of the asymptotic curves nor those the metric. The argument for the tangent frame follows from the general argument above about changes of the parametrisation. $\square$

The set $M_2 = \{p \in M \,|\, \operatorname{rank} \Lambda_p = i\}$, with Λ_p as in §2, is defined in [10]. It is shown there that for a generically immersed surface in $\mathbb{R}^5$, M_2 is either empty or is a regular curve on M.

Proposition 4.4 *There exists a smooth normal vector field μ on $M \setminus M_2$ such that the $C_{i,2}$-lines of principal curvature ($i = 2, 6, 8$) on $M \setminus M_2$ as defined in Definition 4.2 are the lines of S_μ-principal curvature of the shape operator S_μ on $M \setminus M_2$.*

Proof Using the notation in §2, the shape operator S_μ along $\mu = (\mu_3, \mu_4, \mu_5)$ has coefficients $l = \mu_3\alpha_3 + \mu_4\alpha_4 + \mu_5\alpha_5$, $m = \mu_3\beta_3 + \mu_4\beta_4 + \mu_5\beta_5$, $n = \mu_3\gamma_3 + \mu_4\gamma_4 + \mu_5\gamma_5$. We are seeking μ so that $l = X$, $m = Y/2$, $n = Z$, where X, Y, Z are the coefficients of $C_{i,2}(A)$. We write $C_{i,2}(A) = (X, Y/2, Z)^T$ and obtain a linear system $\Lambda_p.\mu = C_{i,2}(A)$ which has a solution $\Lambda_p^{-1}.C_{i,2}(A)$ if Λ_p is invertible, i.e., if $p \notin M_2$. (We may not be able to extend $\Lambda_p^{-1}.C_{i,2}(A)$ to M_2.) $\square$

Remarks 4.5 1. It follows from the proof of Proposition 4.4 that any point on the polar line $\hat{P}$ of P (varying smoothly with $p \in M \setminus M_2$) can represent the coefficients of a shape operator S_μ for some normal vector field μ. (For $p \in M \setminus M_2$, there is a great circle in the unit sphere in the normal space N_pM such that the coefficients of the shape operators along this circle trace the polar line $\hat{P}$.) The μ-lines of curvature are the same for all such shape operators and are the solutions of the binary quadratic differential equation determined by P. Therefore, the normal vector field in Proposition 4.4 is not unique and cannot be unique. (For surfaces in $\mathbb{R}^4$ it is unique up to multiplication by nowhere vanishing functions on M [16].)

2. The point $L = (G : 2F : E)$ which represents the metric g is on $\hat{P}$. The matrix of the shape operator $S_{\Lambda_p^{-1}.L}$ is the identity matrix. Therefore, $M \setminus M_2$ is totally $\Lambda_p^{-1}.L$-umbilic (i.e., all points on $M \setminus M_2$ are $\Lambda_p^{-1}.L$-umbilics). In fact, L is the unique point on $\hat{P}$ with this property. (The vector $\Lambda_p^{-1}.L$ determines, in a sense, a north and a south pole on the unit sphere in N_pM.)

3. The concept of an asymptotic direction is an affine property of the surface (or submanifold) and is defined, for instance, in terms of the contact of the surface with lines. We can therefore define them on surfaces in the Minkowski space $\mathbb{R}^5_1$. When the surface $M \subset \mathbb{R}^5_1$ is timelike/Lorentzian, i.e., the induced metric on M is non-degenerate and has signature 1, the equation of the asymptotic curves on M is the same as the one given in Theorem 2.1. Hence, all the results here apply to these surfaces too.

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