Surfaces in $\mathbb{R}^4$ and their projections to 3-spaces

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Abstract

We derive geometrical information on smooth surfaces in $\mathbb{R}^4$ from the geometry of their images under linear projections to 3-spaces.

AMS Classification: 53A05, 58K05.
Key words: asymptotic directions; singularities; surfaces.

1 Introduction

Our aim in this paper is to obtain geometric information on smooth surfaces in $\mathbb{R}^4$ from the geometry of their images under linear projections to 3-spaces. Given a point p on a surface M and a linear projection π_v along a direction v to a transverse 3-space, we relate the geometry of $\pi_v(M)$ at $\pi_v(p)$ governed by its contact with planes to the geometry of M at p.

It is shown in [19] that if v is a non-asymptotic tangent direction at p then the projection π_v has a singularity of type cross-cap, that is, π_v can be written locally in the form (x, y^2, xy) after smooth changes of coordinates in the source and target. (The projection is then said to be $\mathcal{A}$-equivalent to (x, y^2, xy)). However, this equivalence relation preserves the singularities of $\pi_v(M)$ but not its affine geometry. The differential geometry of the cross-cap is studied in [8] and [26]. It is shown there that for an open and dense set of parameterisations of the cross-cap, the image falls into two types: the elliptic cross-cap whose parabolic set, in the parameter space, has an A_1^--singularity (a pair of transverse curves), and the hyperbolic cross-cap whose parabolic set has an A_1^+-singularity (an isolated point). (We changed here the way the two types are labelled in [26].) The passage from one type to another is realised at a parabolic cross-cap whose parabolic set has an A_2-singularity (a cusp), see Figure 2. This classification of cross-caps is applied to $\pi_v(M)$ in §4 to obtain geometric information about the surface M at p.

The projection π_v can of course be a submersion or have a singularity worse than a cross-cap. We deal here with all the generic cases and study the singularities of what

*The first author was partially supported by the DGICYT grant BFM2003–02037.

we call the *v-pre-parabolic set* (v-PPS for short). This is the pre-image on M of the parabolic set of $\pi_v(M)$. We show that one can recover all the geometry of M related to its contact with lines by studying the v-PPS.

The approach of associating a singular variety $X(M)$ to a smooth sub-manifold M in an Euclidean space and recovering the geometry of M from that of $X(M)$ is at the essence of applications of singularity theory to differential geometry (see the survey articles [2, 24]). Usually, the geometrical information on M is derived from the singular set of $X(M)$, which is invariant under diffeomorphisms. There are only few cases where the affine geometry of $X(M)$ (properties that are invariant under the affine group) has been exploited to recover information about M. The established results in these cases are derived from the duality results in [4, 6, 9, 22, 27]. They concern curves in $\mathbb{R}^n$ and surfaces in $\mathbb{R}^3$ with $X(M)$ being the dual of M or its focal set. For example, in the case where $X(M)$ is the focal set, the pre-image on M of the parabolic set of $X(M)$ is labelled the sub-parabolic set and is the locus of the geodesic inflections of the lines of curvatures of M ([20]). The focal set can be described as the bifurcation set of the family of distance squared functions. The structure of sub-parabolic set is obtained by exploiting the duality result in [9] between the bifurcation sets of the family of folding maps and that of distance squared functions. In the cases investigated in this paper, the image $\pi_v(M)$ does not fall into this category (i.e. is neither a bifurcation set nor a discriminant), so we proceed by analysing the singularities of the Gaussian curvature of $\pi_v(M)$.

The paper is organised as follows. In §2 we recall some results on the flat geometry of surfaces in $\mathbb{R}^4$ and give the expression of the Gaussian curvature of $\pi_v(M)$. In §3 (resp. §4) we study the cases where the direction v is not tangent (resp. is tangent) to M at p. In §5 we look at the way the v-PPS bifurcates as v changes in T_pM near the initial direction of the projection.

2 Preliminaries

The geometry of surfaces in $\mathbb{R}^4$ has been studied in [4, 7, 11, 12, 14, 15, 16, 17, 21, 23]. Given a point $p \in M$ consider the unit circle in T_pM parametrised by $\theta \in [0, 2\pi]$. The set of the curvature vectors $\eta(\theta)$ of the normal sections of M by the hyperplane $\langle\theta\rangle \oplus N_pM$ form an ellipse in the normal plane N_pM, called the curvature ellipse ([15]). Points on the surface are classified according to the position of the point p with respect to the ellipse (N_pM is viewed as an affine plane through p). The point p is called *elliptic/parabolic/hyperbolic* if it is inside/on/outside the ellipse.

The curvature ellipse is the image of the unit circle in T_pM by a map formed by a pair of quadratic forms (Q_1, Q_2). This pair of quadratic forms is the 2-jet of the 1-flat map $F : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ (i.e. without constant or linear terms) whose graph, in orthogonal co-ordinates, is locally the surface M. As the flat geometry of surfaces is affine invariant [5], an alternative approach for studying the geometry of surfaces in $\mathbb{R}^4$ is given in [4]. It uses the pencil of the binary forms determined by the pair (Q_1, Q_2). Each point on the surface determines a pair of quadratics $(Q_1, Q_2) = (ax^2 + 2bxy + cy^2, lx^2 + 2mxy + ny^2)$. A binary form $Ax^2 + 2Bxy + Cy^2$ is represented by its coefficients $(A, B, C) \in \mathbb{R}^3$, where the cone $B^2 - AC = 0$ corresponds to perfect squares. If the forms Q_1 and Q_2 are independent, they determine a line in the projective plane $\mathbb{R}P^2$ and the cone a conic. This line meets the conic in 0,1,2 points according as $\delta(p) < 0, = 0, > 0$, with

$$\delta(p) = (an - cl)^2 - 4(am - bl)(bn - cm).$$

A point p is said to be *elliptic/parabolic/hyperbolic* if $\delta(p) < 0/ = 0/ > 0$. The set of points (x, y) where $\delta = 0$ is called the *parabolic set* of M and is denoted by Δ. If Q_1 and Q_2 are dependent, the rank of the matrix $\begin{pmatrix} a & b & c \\ l & m & n \end{pmatrix}$ is 1 provided either of the forms is non-zero; the corresponding points on the surface are referred to as *inflection* points. (All the above notions agree with those defined using the curvature ellipse.)

Since the flat geometry is affine invariant, we consider the action of $\mathcal{G} = GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$ on the pairs of binary forms (Q_1, Q_2). The $\mathcal{G}$-orbits (see for example [13]) and the characterisation of the corresponding point on the surface are as follows:

(x^2, y^2)	hyperbolic point
$(xy, x^2 - y^2)$	elliptic point
(x^2, xy)	parabolic point
$(x^2 \pm y^2, 0)$	inflection point
$(x^2, 0)$	degenerate inflection
$(0, 0)$	degenerate inflection

The geometrical characterisation of points on M using singularity theory is first carried out in [16] via the family height functions. Recall that the family of height functions on M is given by

$$\begin{array}{rcl} h : M \times S^3 & \rightarrow & \mathbb{R} \\ (p, v) & \mapsto & h(p, v) = < p, v > \end{array}$$

where S^3 denotes the unit sphere in $\mathbb{R}^4$. The height function h_v is singular if and only if $v \in N_pM$. It is shown in [16] that elliptic points are non-degenerate critical points of h_v for any $v \in N_pM$. At a hyperbolic point, there are exactly two directions in N_pM, labelled *binormal directions*, such that p is a degenerate critical point of the corresponding height functions. The two binormal directions coincide at a parabolic point. A hyperplane orthogonal to a binormal direction is called an *osculating hyperplane*.

The direction of the kernel of the Hessian of the height functions along a binormal direction is an *asymptotic direction* associated to the given binormal direction ([16]). The asymptotic directions are labelled conjugate directions in [15], and are defined as the directions along θ such that the curvature vector $\eta(\theta)$ is tangent to the curvature ellipse (see also [11, 16]). So if p is not an inflection point, there are 2/1/0 asymptotic directions at p depending on p being a hyperbolic/parabolic/elliptic point. If p is an inflection point, then every direction in T_pM is asymptotic ([16]). The configurations of the asymptotic curves at inflection points of imaginary type (where Δ has an A_1^+-singularity) are given in [11], and the configurations at inflection points of real type (where Δ has an A_1^--singularity) and at other points on the curve Δ are given in [7].

Asymptotic directions can also be described as in [19] and [4] via the singularities of the projections of M to hyperplanes. The family of projections is given by

$$\begin{array}{rcl} \Pi : M \times S^3 & \rightarrow & TS^3 \\ (p, v) & \mapsto & (v, p- < p, v > v). \end{array}$$

For v fixed, the projection can be viewed locally at a point $p \in M$ as a map germ $\pi_v : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^3, 0$. If we allow smooth changes of coordinates in the source and target (i.e. consider the action of the Mather group $\mathcal{A}$) then the generic singularities of π_v are those that have $\mathcal{A}_e$-codimension ≤ 3 (which is the dimension of S^3). These are listed in Table 1 (see [18]).

Table 1: Generic local singularities of the projection of M to a 3-space ([18]).

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	$(x, y, 0)$	0
Crosscap	(x, y^2, xy)	0
$B_k^\pm$	$(x, y^2, x^2y \pm y^{2k+1}), k = 2, 3$	k
$S_k^\pm$	$(x, y^2, y^3 \pm x^{k+1}y), k = 2, 3$	k
$C_k^\pm$	$(x, y^2, xy^3 \pm x^k y), k = 3$	k
H_k	$(x, xy + y^{2k+2}, y^3), k = 2, 3$	k

The projection π_v is singular at p if and only if $v \in T_pM$. The singularity is a cross-cap unless v is an asymptotic direction at p. The codimension 2 singularities occur generically on curves on the surface and the codimension 3 ones at special points on these curves (see Figure 1 for their configurations at non inflection points). The H_2-curve coincides with the Δ-set ([4]). The B_2-curve of π_v, with v asymptotic, is also the A_3-set of the height function along the binormal direction associated to v ([4]). This curve meets the Δ-set tangentially at isolated points ([7]) and intersects the S_2-curve transversally at a C_3-singularity. At inflection points the Δ-set has a Morse singularity and the configuration of the B_2 and S_2-curves there is given in [4].

Definition 2.1 We say that a point $p \in M$ is *v-pre-parabolic* if $\pi_v(p)$ is a parabolic point of $\pi_v(M)$. The set of v-pre-parabolic points is called the *v-pre-parabolic set* (v-PPS for short).

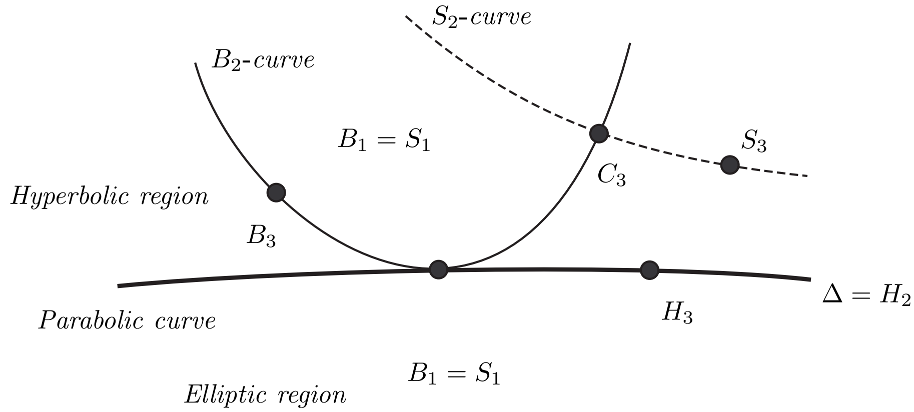

Figure 1: Special curves and points on a surface in $\mathbb{R}^4$, away from inflection points.

Suppose that M is parametrised locally at a point p by a smooth map $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^4$. We denote by (x, y) the coordinates in $\mathbb{R}^2$ and use subscripts for partial differentiation. Then the expression for the v-PPS is as follows.

Lemma 2.2 *The v-pre-parabolic set is given by*

$$P_v(x,y) = (\det(v, \phi_x, \phi_y, \phi_{xx}) \det(v, \phi_x, \phi_y, \phi_{yy}) - \det(v, \phi_x, \phi_y, \phi_{xy})^2)(x,y) = 0.$$

Proof Given three vectors X, Y, Z in $\mathbb{R}^4$, we have

$$\det(v, X, Y, Z) = \det(\pi_v(X), \pi_v(Y), \pi_v(Z)),$$

where the determinant in the right hand side is taken with respect to the orientation induced by v in its normal hyperplane. The image $\pi_v(M)$ is parametrised by $\psi = \pi_v \circ \phi$ and hence its parabolic set in U is given by the vanishing of

$$\begin{aligned} P_v(x,y) &= \det(\psi_x, \psi_y, \psi_{xx}) \det(\psi_x, \psi_y, \psi_{yy}) - \det(\psi_x, \psi_y, \psi_{xy})^2 \\ &= \det(v, \phi_x, \phi_y, \phi_{xx}) \det(v, \phi_x, \phi_y, \phi_{yy}) - \det(v, \phi_x, \phi_y, \phi_{xy})^2. \end{aligned}$$

□

Changing the parametrisation of the surface and making affine changes of coordinates in $\mathbb{R}^4$ transforms the surface $\pi_v(M)$ to one whose pre-parabolic set has the same structure as that of v-PPS. Let $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ be a local parametrisation of M, A an affine transformation in $\mathbb{R}^4$ and $h : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ a germ of a diffeomorphism. (We also denote by A the associated linear transformation to A.) We consider the Av-PPS of the surface M' parametrised by $A(\phi \circ h)$. If we denote this set by Q_{Av}, then we have the following relation, where $\mathcal{K}$ denotes the contact group (see for example [24] for a definition).

Lemma 2.3 *The germs P_v and Q_{Av} are $\mathcal{K}$-equivalent.*

Proof The proof follows by using Lemma 2.2 and observing that

$$Q_{Av}(x,y) = det(A)^2 Jac(h)^4 P_v(h(x,y)).$$

$\square$

So we are interested in the $\mathcal{K}$-singularities of the function $P_v(x,y)$ in Lemma 2.2. The $\mathcal{R}$-singularities (smooth changes of coordinates in the source) are classified by Arnold (see [1]). The simple $\mathcal{R}$ and $\mathcal{K}$-singularities coincide and are as follows.

Table 2. $\mathcal{K}$-simple singularities of functions ([1]).

Name	A_k	D_k	E_6	E_7	E_8
Normal form	$x^2 \pm y^{k+1}, k \geq 0$	$x^2y \pm y^{k-1}, k \geq 4$	$x^3 + y^4$	$x^3 + xy^4$	$x^3 + y^5$

We also need the following unimodal $\mathcal{K}$-singularities from [25] which are also unimodal $\mathcal{R}$-singularities ([1]. The notation in the table below are from [1]).

Table 3. $\mathcal{K}$-unimodular singularities of functions ([1, 25]).

Name	J_{10}	$X_{1,0}$	$X_{1,1}$	$Y^1_{1,1}$
Normal form	$x^3 + ax^2y^2 + y^6$ $4a^3 + 27 \neq 0$	$x^4 + ax^2y^2 + y^4$ $a^2 - 4 \neq 0$	$x^4 + x^2y^2 + ay^5$ $a \neq 0$	$x^5 + ax^2y^2 + y^5$ $a \neq 0$

The above singularities can also be written as $T_{p,q,r} : x^p + y^q + \lambda x^2y^2$ with $r = 2$, $(p,q) = (3,6), (4,4), (4,5), (5,5)$ (see [25]).

Remark 2.4 The flat geometry of a smooth submanifold in $\mathbb{R}^n$ (i.e. the geometry related to its contact with k-dimensional planes) and that of its projection to a subspace of $\mathbb{R}^n$ are clearly related in some cases. Let X be a smooth m-dimensional manifold in $\mathbb{R}^n$ and $\pi_1 : \mathbb{R}^n \to \mathbb{R}^p$ be a linear projection with $\pi_1|_X$ of maximal rank at $p \in X$. Let $\pi_2 : \mathbb{R}^p \to \mathbb{R}^q$ be another linear projection. Then the contact of $\pi_1(X)$ with $\ker \pi_2$ at $\pi_1(p)$ is the same as that of X with $\ker \pi_1 \oplus \ker \pi_2$ at p. (If ϕ is a local parametrisation of X, then the first contact is described by the singularities of $\pi_2 \circ (\pi_1 \circ \phi)$ and the second by those of $(\pi_2 \circ \pi_1) \circ \phi$.)

We shall use the following notation in the rest of the paper. The point p is chosen to be the origin in $\mathbb{R}^4$ and the surface M is given locally at p in Monge form $\phi(x,y) = (x, y, f^1(x,y), f^2(x,y))$ with

$$\begin{array}{rcl} f^1(x,y) & = & Q_1(x,y) + \sum_{i=3}^n c_{3i}x^{3-i}y^i + \sum_{i=4}^n c_{4i}x^{4-i}y^i + \cdots \\ f^2(x,y) & = & Q_2(x,y) + \sum_{i=3}^n d_{3i}x^{3-i}y^i + \sum_{i=4}^n d_{4i}x^{4-i}y^i + \cdots, \end{array}$$

and where the pair of quadratics (Q_1, Q_2) are taken in normal forms as in §2. The surface $\pi_v(M)$ is considered in a 3-dimensional affine space through the origin in $\mathbb{R}^4$ and transverse to v. Some calculations in this paper are carried out using the computer algebra packages Maple and Mathematica.

3 Projecting along a non-tangent direction

We consider in this section the case where $v \in S^3$ is not a tangent direction at the origin. We write $v = v_T + v_N$ where v_T is the orthogonal projection of v to the tangent space T_pM and v_N is its orthogonal projection to the normal space N_pM. Since $v_N \neq 0$, the image $\pi_v(M)$ is a smooth surface at $\pi_v(0)$.

Proposition 3.1 *A point $p \in M$ is v-pre-parabolic if and only if it is a non-elliptic point and $\langle v\rangle \oplus T_pM$ is an osculating hyperplane of M at p, that is, if and only if v_N is a binormal direction at p.*

Proof Given a local parametrisation ϕ of M at p, the tangent space T_pM is generated by $\phi_x(p)$ and $\phi_y(p)$, therefore $w = v \wedge \phi_x(p) \wedge \phi_y(p) \in N_pM \setminus \{0\}$. Let h_w be the corresponding height function, so that for any $X \in \mathbb{R}^4$, we have

$$\det(v, \phi_x(p), \phi_y(p), X) = \langle w, X\rangle = h_w(X).$$

By Lemma 2.2, p is v-pre-parabolic if and only if the determinant of the Hessian of h_w is zero, if and only if w is binormal. The result follows by observing that the orthogonal hyperplane to w is precisely $\langle v\rangle \oplus T_pM$. □

Proposition 3.2 *Suppose that p is a hyperbolic point and v_N is one of the binormal directions at p.*

(i) *For a generic surface M and a generic point p in its hyperbolic region, the v-PPS is smooth at p. The tangent line to the v-PPS at p can be in any direction except the asymptotic direction not associated to v_N.*

(ii) *The v-PPS is tangent to the asymptotic direction not associated to v_N if and only if p is on the B_2-curve, if and only if $\pi_v(p)$ is a cusp of Gauss of $\pi_v(M)$.*

(iii) *For a generic surface, the v-PPS can have singularities of type A_1, A_2 or A_3 on the B_2-curve.*

Proof We take the surface in Monge form as in §2, with $(Q_1, Q_2) = (x^2, y^2)$ and $v_N = (0, 0, 1, 0)$, so $v = (\alpha, \beta, 1, 0)$. The asymptotic direction associated to v_N is $(0, 1, 0, 0)$ and the other asymptotic direction is $(1, 0, 0, 0)$. The 1-jet of the v-PPS is given, after scaling, by

$$j^1P_v(x, y) = 2d_{30}x + (2\beta + d_{31})y.$$

We have $d_{30} = 0$ if and only if the height function along the other binormal direction has an A_3-singularity (i.e. the projection along $(1, 0, 0, 0)$ has a $B_{\geq 2}$-singularity, [4]). Following Remark 2.4, this means that the image $\pi_v(M)$ has a cusp of Gauss at $\pi_v(p)$. It is clear that when $d_{30} \neq 0$, the tangent line to the v-PPS can be along any direction except $(1, 0, 0, 0)$. (When $\beta = -d_{31}/2$, the v-PPS is tangent to the asymptotic direction associated to v_N.)

When $d_{30}=0$ and $\beta=-d_{31}/2$ the v-PPS becomes singular. Then $j^2P_v(x,y)$ is given, after scaling, by

$$3(4d_{40}-d_{31}^2)x^2-6(6d_{31}c_{30}+d_{32}d_{31}-d_{41})xy+2(d_{42}-d_{31}c_{31}+d_{32}\alpha-\frac{3}{2}d_{33}d_{31}-d_{32}^2)y^2.$$

The discriminant of the above quadratic form is

$$-24(4d_{40}-d_{31}^2)d_{32}\alpha+36(d_{31}c_{30}+d_{32}d_{31}-d_{41})^2-24(4d_{40}-d_{31}^2)(d_{42}-d_{31}c_{31}-\frac{3}{2}d_{33}d_{31}-d_{32}^2).$$

If $d_{32}(4d_{40}-d_{31}^2)\neq 0$, the singularity of the v-PPS is of type A_1 for all values of α, except one. For the exceptional value of α and for generic points on the B_2-curve the singularity of the v-PPS becomes an A_2. At special points on the B_2-curve it can degenerate further to an A_3.

If $d_{32}(4d_{40}-d_{31}^2)=0$, that is, if p is a C_3-singularity of the projection along $(1,0,0,0)$ (when $d_{32}=0$) or an A_4-point of the height function along $(0,0,0,1)$ (when $4d_{40}-d_{31}^2=0$), then the singularity of the v-PPS is generically of type A_1. □

Remark 3.3 A calculation shows that for generic surfaces, the family $P(x,y,v)=P_v(x,y)$, with v near the initial direction, is a versal unfolding of all the singularities of the v-PPS in Proposition 3.2 (see for example [10] for a definition of a versal unfolding).

Proposition 3.4 *Suppose that p is a parabolic point (i.e. $p\in\Delta$) and v_N is the unique binormal direction at p.*

(i) *Away from the inflection points, the v-PPS is a smooth curve tangent to the Δ-curve at p. Furthermore, its tangent direction is independent of v. At a B_2-point on Δ, there are two possible generic configurations of the triple Δ, the B_2-curve and the v-PPS. The B_2-curve lies in the region delimited by the v-PPS and Δ or the v-PPS lies in the region delimited by the B_2-curve and Δ.*

(ii) *The v-PPS has a Morse singularity at an inflection point (and so does Δ). The singularity type (A_1^+ or A_1^-) is independent of that of Δ. When both sets have an A_1^--singularity, their branches are generically transverse.*

Proof Here we take $(Q_1,Q_2)=(x^2,xy)$. Then the 1-jet of the v-PPS is given, after scaling, by $d_{32}x+3d_{33}y$. This coincides, up to scalar multiple, with the 1-jet of the Δ-set. The remaining statements follow by analysing the 2-jets of the appropriate curves. □

Remark 3.5 We observe that the family $P(x,y,v)=P_v(x,y)$ with v near the initial direction is not a versal unfolding of the Morse singularity of v-PPS at an inflection point. We have cone sections as v varies in S^3 (see §5 for more details).

4 Projecting along a tangent direction

We consider now the case where v is a tangent direction at the origin. We first assume that v is not an asymptotic direction, so the image $\pi_v(M)$ is a cross-cap.

Theorem 4.1 *Suppose that $v \in T_pM$ but is not an asymptotic direction (in particular, p is not an inflection point).*

(i) *The v-PPS has a Morse singularity if $p \notin \Delta$. Furthermore, the singularity is of type A_1^-, i.e. $\pi_v(M)$ is an elliptic cross-cap, if p is a hyperbolic point and of type A_1^+, i.e. $\pi_v(M)$ is a hyperbolic cross-cap, if p is an elliptic point (Figure 2).*

(ii) *At a generic point on Δ the v-PPS has an A_2-singularity, i.e. $\pi_v(M)$ is a parabolic cross-cap (Figure 2). At the point of tangency of the B_2-curve with Δ, the singularity of the v-PPS becomes an A_3.*

(iii) *The tangent directions to the v-PPS are along the asymptotic directions to M at p.*

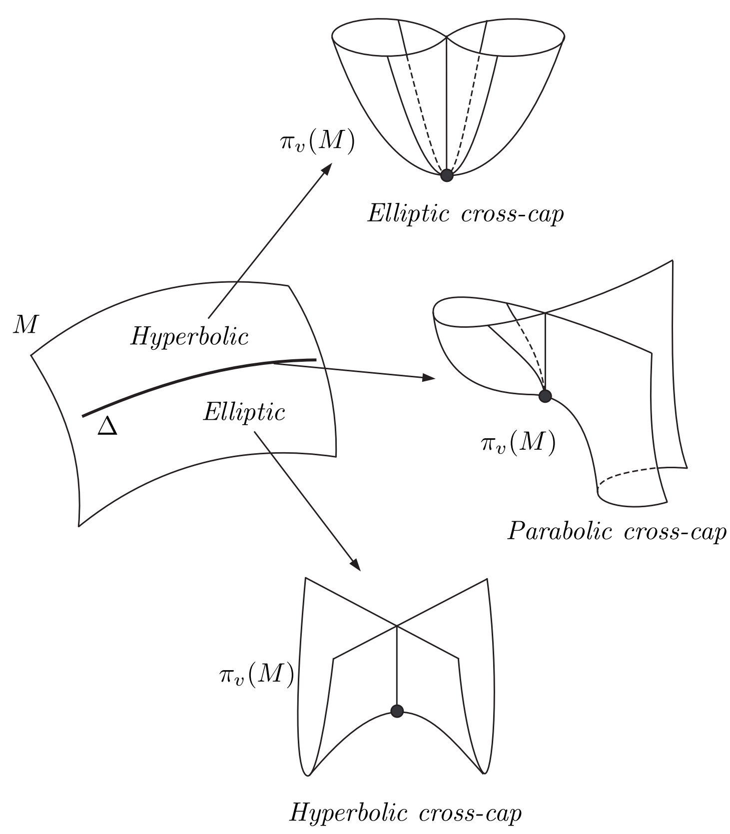

Figure 2: The three types of cross-caps of $\pi_v(M)$ and their parabolic sets.

Proof We follow the same notation as in previous section. Suppose that p is a hyperbolic point, so we can take $(Q_1, Q_2) = (x^2, y^2)$. The asymptotic directions are $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. We consider a tangent vector $v = (\alpha, \beta, 0, 0)$ which is not an asymptotic direction, that is, $\alpha\beta \neq 0$. The 2-jet of the v-PPS is given by

$$j^2 P_v(x, y) = -16\alpha\beta xy,$$

which has an A_1^--singularity. It is clear that its tangent directions coincide with the asymptotic directions of M at the origin.

Analogously, if p is an elliptic point, we take $(Q_1, Q_2) = (xy, x^2 - y^2)$ and $v = (\alpha, \beta, 0, 0)$, with $\alpha^2 + \beta^2 = 1$. (Recall that there are no asymptotic directions at an elliptic point.) The 2-jet of the v-PPS is given by

$$j^2 P_v(x, y) = -4(x^2 + y^2)$$

and this has A_1^+-singularity, that is the v-PPS is locally an isolated point.

Consider now the case where $p \in \Delta$. We take $(Q_1, Q_2) = (x^2, xy)$ and let $v = (\alpha, \beta, 0, 0)$ be a tangent vector with $\alpha \neq 0$ ($\alpha = 0$ would give the unique asymptotic direction). We have

$$j^2 P_v(x, y) = -4\alpha^2 x^2,$$

so the v-PPS has an A_k-singularity, with $k \geq 2$. Note that we have one tangent direction to the v-PPS which is exactly the unique asymptotic direction at the origin. A computation shows that the coefficient of y^3 in $j^3 P_v(x, y)$ is $12c_{33}\alpha^2$. Hence, for a generic point on the Δ-curve the v-PPS has an A_2-singularity. The points where $c_{33} = 0$ correspond to points of tangency of Δ and the B_2-curve (see [4]) and at these points the v-PPS has generically an A_3-singularity. □

Suppose now that v is an asymptotic direction, so the image $\pi_v(M)$ has a singularity worse than a cross-cap.

Theorem 4.2 *Suppose that v is an asymptotic direction at p.*

(i) *Let p be a hyperbolic point. Then the singularities of the v-PPS at p distinguish between the singularities of π_v. The correspondence between the singularities of the v-PPS and those of π_v are as follows:*

Singularities of the v-PPS	D_4	D_5	D_6	E_7	J_{10}	$X_{1,0}$
Singularities of π_v	B_1	B_2	B_3	S_2	S_3	C_3

(ii) *The v-PPS has a D_5-singularity at generic points on Δ. It has a J_{10}-singularity at the point of tangency of the B_2-curve with Δ.*

(iii) *Projecting along a tangent direction at an inflection point yields a v-PPS with a singularity of type $X_{1,0}$ or worse. The singularity is generically of type $X_{1,0}$ except for a finite number of directions (2, 4 or 6) in the tangent plane. Along these directions the singularity of the v-PPS is generically of type $X_{1,1}$.*

Proof (i) We take, as above, $(Q_1, Q_2) = (x^2, y^2)$ and v to be one of the asymptotic directions, for instance, $v = (1, 0, 0, 0)$. The 3-jet of the v-PPS is given, after scaling, by

$$j^3 P_v(x, y) = 3d_{30}x^3 - d_{32}xy^2.$$

If both coefficients d_{30} and d_{32} are not zero, the projection π_v has a singularity of type $S_1 = B_1$ (see [4]) and P_v has a D_4-singularity.

If $d_{30} = 0$ and $d_{32} \neq 0$, the point belongs to the B_2-curve and the v-PPS has a D_k-singularity, with $k \geq 5$. In fact, the coefficient of x^4 in $j^4 P_v(x, y)$ is a scalar multiple of $d_{31}^2 - 4d_{40}$. If this coefficient is not zero, P_v has a D_5-singularity and the projection has a B_2-singularity. Otherwise, the singularities of P_v and of the projection become generically of type D_6 and B_3 respectively.

If $d_{30} \neq 0$ and $d_{32} = 0$, the point belongs to the S_2-curve. Then the coefficient of y^4 in $j^4 P_v(x, y)$ is zero and the coefficient of xy^3 is a scalar multiple of $c_{32}d_{31} - d_{43}$. If this coefficient is not zero, P_v has an E_7-singularity. A computation shows that this condition corresponds exactly to the condition for the projection to have an S_2-singularity. If $c_{32}d_{31} - d_{43} = 0$, π_v has generically an S_3-singularity and P_v a singularity of type J_{10}.

If $d_{30} = d_{32} = 0$, the projection has a C_3-singularity and P_v has generically an $X_{1,0}$-singularity.

(ii) Suppose that $p \in \Delta$ and take $(Q_1, Q_2) = (x^2, y^2)$ and $v = (0, 1, 0, 0)$. Then the 3-jet of the v-PPS is given by

$$j^3 P_v(x, y) = 4(c_{32}x^3 + 3c_{33}x^2y).$$

At generic points on Δ, $c_{33} \neq 0$ and the v-PPS has a D_k-singularity. The coefficient of y^4 in $j^4 P_v(x, y)$ is $-9c_{33}^2$, hence P_v has a D_5-singularity. If $c_{33} = 0$ the point in consideration is a point of tangency of Δ and the B_2-curve. In this case, P_v has generically a J_{10}-singularity.

(iii) We consider finally the case when p is an inflection point. We take $(Q_1, Q_2) = (x^2 \pm y^2, 0)$ and $v = (\alpha, \beta, 0, 0)$, with $\alpha^2 + \beta^2 = 1$. In this case, the 4-jet of the v-PPS is in the form

$$j^4 P_v(x, y) = C_0(\alpha, \beta)x^4 + C_1(\alpha, \beta)x^3y + C_2(\alpha, \beta)x^2y^2 + C_3(\alpha, \beta)xy^3 + C_4(\alpha, \beta)y^4,$$

where $C_i(\alpha, \beta)$ are quadratic polynomials in α, β whose coefficients are polynomials in $d_{30}, \ldots, d_{33}$. The discriminant of $j^4 P_v(x, y)$ is given by $D_0(\alpha, \beta)D_1(\alpha, \beta)$, where $D_i(\alpha, \beta)$ are cubic polynomials in α, β whose coefficients are again polynomials in $d_{30}, \ldots, d_{33}$. It follows that for generic coefficients $d_{30}, \ldots, d_{33}$, there are 2, 4 or 6 directions in the (α, β)-plane where $j^4 P_v(x, y)$ has multiple roots. Away from these directions, $P_v(x, y)$ has a singularity of type $X_{1,0}$, while for such directions, it has an $X_{1,1}$-singularity. □

We analyse now in detail the subsets of the set of cubics $d_{30}x^3 + \ldots + d_{33}y^3$ where we have 2, 4 or 6 special directions.

At an inflection point of imaginary type ($(Q_1, Q_2) = (x^2 + y^2, 0)$) we can write the cubic $d_{30}x^3 + \ldots + d_{33}y^3$, by a rotation of the coordinate axes and rescaling, in the form $\Re\{z^3 + \gamma z^2 \overline{z}\}$, where $z = x + iy$ and $\gamma \in \mathbb{C}$. There are exceptional curves in the γ-plane that separate the regions corresponding to cubics where we have 2, 4 or 6 directions of projections for which the v-PPS has an $X_{1,1}$-singularity. These are the hypocycloids $\gamma = 2e^{2i\theta} + e^{-4i\theta}$ and $\gamma = -3(e^{-4i\theta} + 2e^{2i\theta})$, the circle $|\gamma| = 3$ and the line segments $\arg \gamma = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ (Figure 3, left). On these curves, there is a double direction where the singularity is worse than $X_{1,1}$. (If v is not one of the exceptional directions, then the v-PPS has an $X_{1,0}$-singularity.)

In [4] is given a partition of the γ-plane into regions corresponding to cubics where there are a certain number of directions v in T_pM yielding singularities of type B_2 or S_2 of π_v (see Figure 3, left). It is not hard to show that such directions coincide with the directions where the v-PPS has an $X_{1,1}$-singularity.

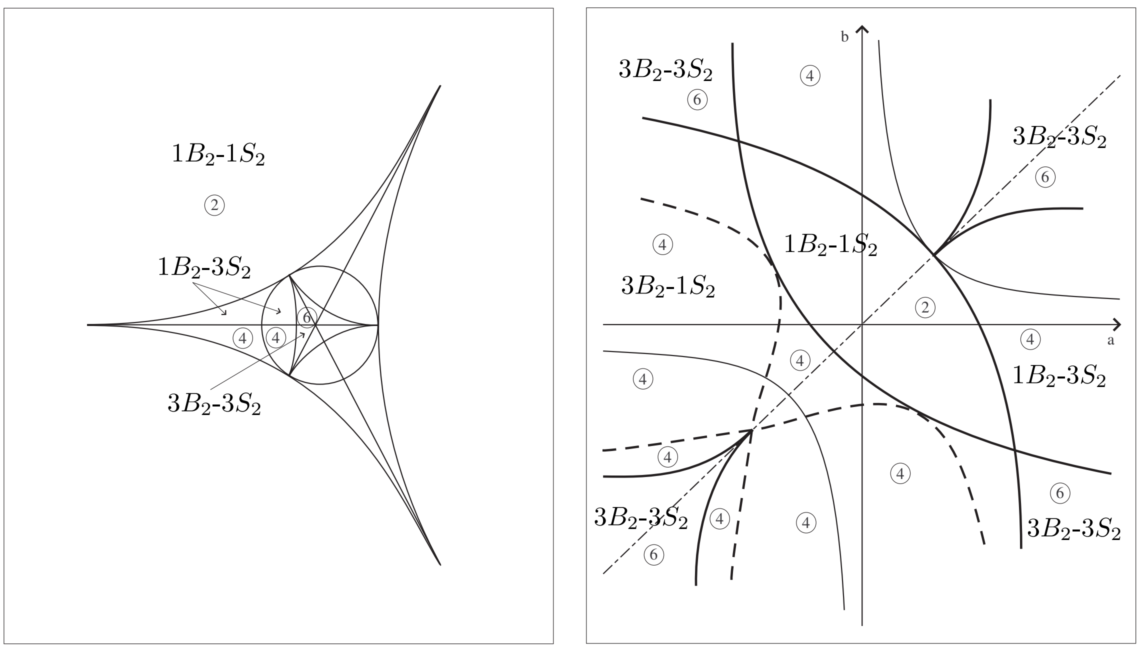

Figure 3: Partition of the γ-plane left and of the (a, b)-plane right. The encircled numbers indicate the number of directions yielding $X_{1,1}$-singularities of the v-PPS.

Analogously, at an inflection point of real type ($(Q_1, Q_2) = (x^2 - y^2, 0) \sim_{\mathcal{G}} (xy, 0)$), we take as in [4] the cubic $d_{30}x^3 + \ldots + d_{33}y^3$ in the form $x^3 + ax^2y + bxy^2 + y^3$, with $a, b \in \mathbb{R}$. There is a curve in the (a, b)-plane that separates the regions where there are 2, 4 or 6 special directions of projections. This is given by $(ab - 81)^2 - 4(b^2 + 9a)(a^2 + 9b) = 0$

(thick continuous in Figure 3 right). There are also other exceptional curves in Figure 3, right: the diagonal $a - b = 0$ (thin dashed), the hyperbola $ab - 9 = 0$ (thin continuous), and the curve $729 + 8a^3 + 54ab + a^2b^2 + 8b^3 = 0$ (thick dashed). These curves altogether give the set where the cubics $D_0(\alpha, \beta)$ and $D_1(\alpha, \beta)$ in the proof of Theorem 4.2 either have a multiple root or have a root in common. Again, on the exceptional curves there is a double direction for which the singularity of the v-PPS is more degenerated than $X_{1,1}$. (If v is not an exceptional direction, then the v-PPS has an $X_{1,0}$-singularity.)

Figure 3 right also indicates the number of B_2 and S_2 singularities of the projections as given in [4].

5 Bifurcations in the v-PPS

We describe in this section how the v-PPS changes as the direction of projection varies near the initial one. As pointed out in the proof of Proposition 3.2, when p is a hyperbolic point and v_0 is not a tangent direction, the family P_v with v varying in S^3 near v_0 is a $\mathcal{K}$-versal unfolding of the singularity of P_{v_0}. Therefore the deformations of P_{v_0} are modelled by those of a $\mathcal{K}$-versal deformation of a plane curve singularity.

If p is on the curve Δ but is not an inflection point and v_0 is not a tangent direction, then the structure of the v-PPS is stable (Proposition 3.4).

When p is an inflection point and v_0 is not a tangent direction, P_v is no longer a versal unfolding of the Morse singularity of P_{v_0}. Here we have sections of a cone as v varies in S^3 near v_0. The parabolic set on $\pi_v(M)$ is the discriminant of the equation of the asymptotic directions on $\pi_v(M)$ (see for example [7]). The equation is in the form $a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0$ and when p is an inflection point, all the coefficients of the equation vanish at p ([7, 11]). The discriminant of the equation is the determinant of the symmetric matrix formed by its coefficients. In fact we have a family of symmetric matrices parametrised by (x, y). The singularities of the discriminant are studied in [3] by considering the action of a group $\mathcal{H}$ on the set of families of symmetric matrices $\mathbb{R}^n, 0 \to S(n, \mathbb{R})$. The group $\mathcal{H}$ consists of smooth changes of parameters in the source and parametrised conjugation in the target. It turns out that the family P_v induces an $\mathcal{H}$-versal deformation of the singularity of the symmetric matrix of the asymptotic directions of $\pi_{v_0}(M)$ at p. Therefore the discriminant (and hence the v-PPS) undergoes the transitions given by sections of a cone ([3]).

When v_0 is a tangent direction, P_v is never a $\mathcal{K}$-versal deformation of P_{v_0}. (For example in Theorem 4.2, the singularities are of $\mathcal{K}$-codimension greater than 3, so they cannot be versally unfolded by P_v.) We shall describe below how the v-PPS changes as v varies in T_pM near $v_0 \in T_pM$.

If v_0 is not an asymptotic direction and $p \notin \Delta$, then P_v is $\mathcal{K}$-equivalent to P_{v_0}, i.e. we have a trivial local deformation so the v-PPS has one of the singularities in

Theorem 4.1(i). If $p \in \Delta$, then the changes in the v-PPS are sections of a Whitney umbrella, see Figure 4 and Theorem 4.1(ii).

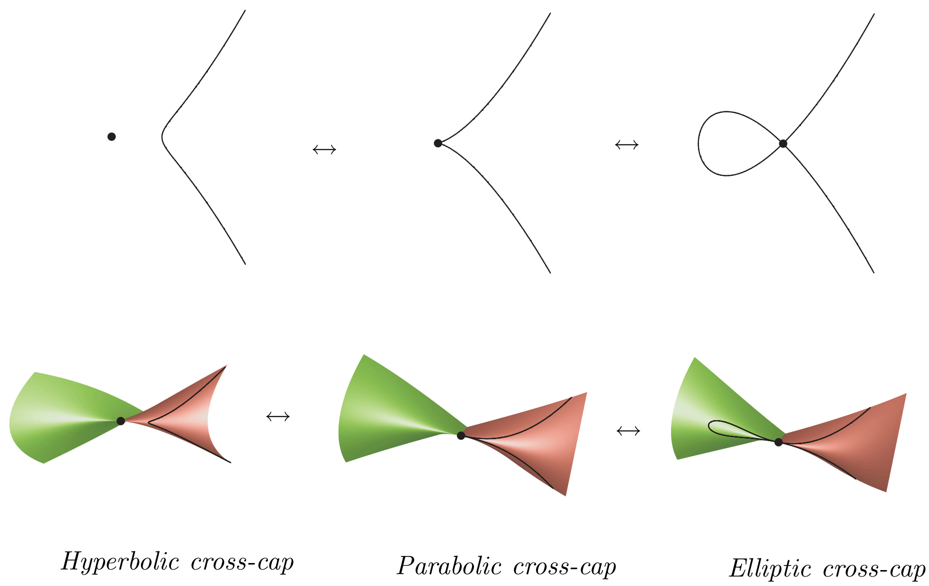

Figure 4: A change from a hyperbolic to an elliptic cross-cap at a parabolic cross-cap of the surface $\pi_v(M)$ (bottom) and the corresponding transitions on the v-PPS (top).

Suppose now that v_0 is an asymptotic direction and p is not an inflection point. We take the surface in Monge form $(x, y, f^1(x, y), f^2(x, y))$ at the origin. We suppose that $v_0 = (0, 1, 0, 0)$ and take $v = (u, 1, 0, 0)$. We compute the relevant jets of P_v in Lemma 2.2 and deduce the bifurcations in this set as u varies near zero. These are as shown in Figure 5 when p is not a parabolic point and in Figure 6 for the case when p is a parabolic point but not an inflection point.

At an inflection point, v can vary in $S^1 \subset T_pM$. There are several cases to consider depending on the position of the cubic $d_{30}x^3 + \ldots + d_{33}y^3$ in the γ-plane or the (a, b)-plane (see proof of Theorem 4.2 and Figure 3). Figure 7 shows an example of bifurcations at each type of inflections.

For an inflection of real type we take $\gamma = i$ in Figure 3 left, so the point γ is in the region $1B_2 - 3S_2$ with 4 exceptional directions for which the singularity of the v-PPS is of type $X_{1,1}$. Figure 7 (left) shows the bifurcations in the v-PPS at one of these exceptional directions.

For an inflection of real type we take $a = 6, b = 0$ in Figure 3 right, so the point (a, b) is in the region $1B_2 - 3S_2$ with 4 exceptional directions for which the singularity of the v-PPS is of type $X_{1,1}$. Figure 7 (right) shows the bifurcations in the v-PPS at one of these exceptional directions. In both examples, and for genericity reasons, the 4-jet of the parametrisation of the surface must contain some degree 4 terms.

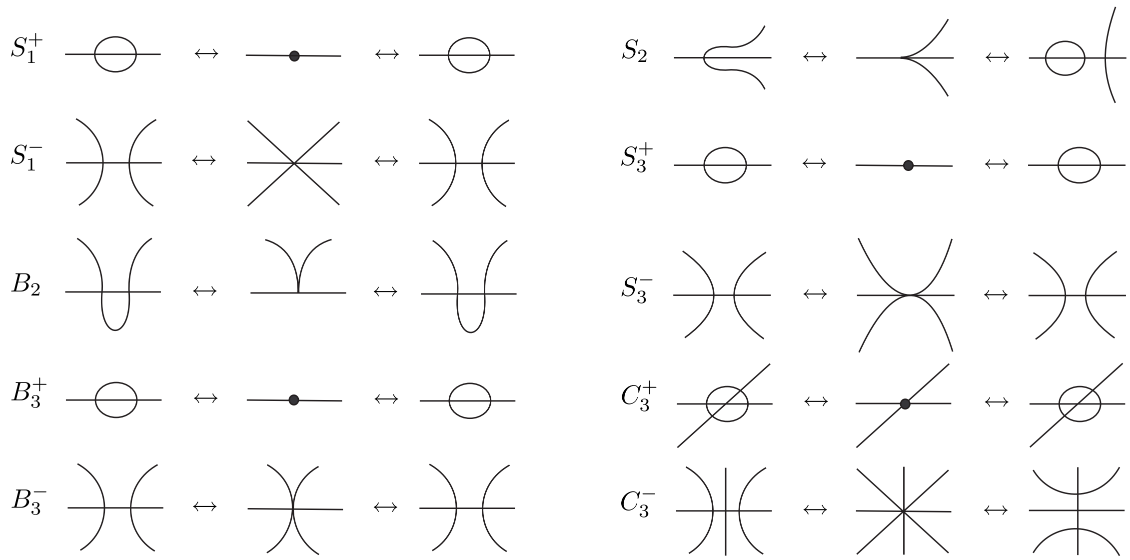

Figure 5: Bifurcations of the v-PPS at generic singularities of the projection π_v away from parabolic points.

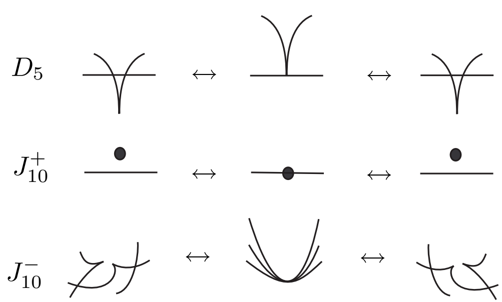

Figure 6: Bifurcations of the v-PPS at a non-inflection parabolic point. The singularities are those of the v_0-PPS (see Theorem 4.2).

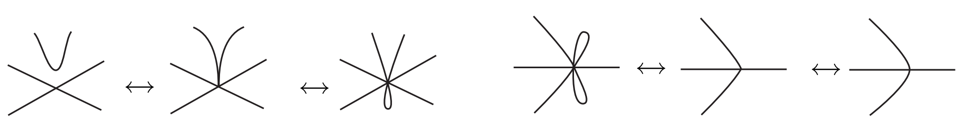

Figure 7: Examples of bifurcations of the v-PPS at an inflection point of imaginary type (left) and of real type (right).

The pictures in Figures 4 are computer generated and those in Figures 5, 6 and 7 are drawings from computer generated pictures.

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