Families of curve congruences on Lorentzian surfaces and pencils of quadratic forms

Ana Claudia Nabarro* and Farid Tari

Abstract

We define and study in this paper families of conjugate and reflected curve congruences associated to a self-adjoint operator $\mathbb{A}$ on a smooth and oriented surface M endowed with a Lorentzian metric g. These families trace parts of the pencil joining the equations of the $\mathbb{A}$-asymptotic and the $\mathbb{A}$-principal curves, and the pencil joining the $\mathbb{A}$-characteristic and the $\mathbb{A}$-principal curves respectively. The binary differential equations (BDEs) of these curves can be viewed as points in the projective plane. Using the polar lines of various BDEs with respect to the conic of degenerate quadratic forms, we obtain geometric results on the above pencils and their relation with the metric g, on the type of solutions of a given BDE and of its $\mathbb{A}$-conjugate equation, and on BDEs with orthogonal roots.

1 Introduction

Let M be a smooth and orientable surface in the Euclidean space $\mathbb{R}^3$ and $N : M \to S^2$ its Gauss map. The shape operator $S_p = -(dN)_p : T_pM \to T_pM$, $p \in M$, is a self-adjoint operator and determines three pairs of foliations on M in the following way. As T_pM inherits the Euclidean scalar product ".", it has a basis of orthonormal vectors given by the eigenvectors of S_p. The directions of these vectors are called the principal directions and their integral curves are the lines of principal curvature. Two tangent directions $u, v \in T_pM$ are conjugate if $S_p(u).v = 0$. A direction $u \in T_pM$ is asymptotic if it is self-conjugate, i.e., if $S_p(u).u = 0$. There are two asymptotic directions at hyperbolic points (these are points where the Gaussian curvature $K = det(S_p)$ is negative) and their integral curves are called the asymptotic curves. At an elliptic point (where

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$K > 0$) there is a unique pair of conjugate directions for which the included angle is extremal ([6]). These directions are called the characteristic directions and their integral curves the characteristic curves. The three pairs asymptotic, characteristic and principal foliations are given, in a local chart, by (quadratic) Binary Differential Equations (BDEs). These are equations in the form

$$(\omega): \qquad a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0, \tag{1}$$

where a, b, c are smooth functions in some open set U of $\mathbb{R}^2$. The set where $b^2 - ac = 0$ is called the *discriminant* of the BDE. In this paper, if (ω) denotes a given BDE (1) then ω denotes the left hand side of the same equation.

In [7] (resp. [3, 5]) is constructed a natural 1-parameter family of BDEs, called *conjugate curve congruence* (resp. *reflected conjugate curve congruence*) and denoted by $(\mathcal{C}_\alpha)$ (resp. $(\mathcal{R}_\alpha)$), which links the equation of the asymptotic (resp. characteristic) curves and that of the lines of principal curvature of M. (There is no natural family that links the equations of the asymptotic and characteristic curves; see Remark 6.4(3) in [5].) The family $(\mathcal{C}_\alpha)$ (resp. $(\mathcal{R}_\alpha)$) is obtained by seeking directions in the tangent plane T_pM that make an oriented angle α with their conjugate directions (resp. with the reflection of their conjugate directions with respect to a principal direction).

The families $(\mathcal{C}_\alpha)$ and $(\mathcal{R}_\alpha)$ have another interesting geometric interpretation ([5]). A BDE (1) can be viewed as a quadratic form and represented at each point in U by the point $(a(x,y) : 2b(x,y) : c(x,y))$ in the projective plane. Let Γ denote the conic of degenerate quadratic forms. To a point in the projective plane is associated a unique polar line with respect to Γ, and vice-versa. A triple of points is called a self-polar triangle if the polar line of any point of the triple contains the remaining two points. It turns out that, at non-parabolic and non-umbilic points on M, the triple asymptotic, characteristic and principal curves BDEs form a self-polar triangle. Also, the family of BDEs $(\mathcal{C}_\alpha)$ (resp. $(\mathcal{R}_\alpha)$) as α varies in $[-\pi/2, \pi/2]$ parametrises the *whole* pencil joining the principal and asymptotic (resp. characteristic) BDEs.

Now, the concepts of asymptotic, characteristic and principal foliations on a surface $M \subset \mathbb{R}^3$ are derived from the fact that S_p is a self-adjoint operator with respect to the Riemannian metric on M. This means that one can associate the same concepts to a self-adjoint operator on a smooth surface M endowed with a metric g which is not necessary Riemannian ([10, 15]).

In this paper, we suppose that M is endowed with a Lorentzian metric g and consider a self-adjoint operator $\mathbb{A}$ on (M, g), that is, a smooth map $TM \to TM$ with the property that its restriction $\mathbb{A}_p : T_pM \to T_pM$ is a linear map satisfying $g(\mathbb{A}_p(u), v) = g(u, \mathbb{A}_p(v))$ for any $p \in M$ and any $u, v \in T_pM$. Our aim is to define, as in [3, 5, 7], families of BDEs that link the BDE of $\mathbb{A}$-asymptotic (resp. $\mathbb{A}$-characteristic) curves and that of the lines of $\mathbb{A}$-principal curvature. For this, we use the concept of $\mathbb{A}$-conjugacy and say that two directions $u, v \in T_pM$ are $\mathbb{A}$-conjugate if $g(\mathbb{A}_p(u), v) = 0$, and the concept of oriented hyperbolic angle between two non-lightlike directions.

We prove in Theorem 3.2 (resp. Theorem 4.2) that there are at most two directions in T_pM that make an oriented angle α with their conjugate directions (resp. with the reflection of their conjugate directions with respect to an $\mathbb{A}$-principal direction). This yield two families of BDEs that we call *Lorentzian conjugate curve congruence* (resp. *Lorentzian reflected conjugate curve congruence*) and denote by $(\mathcal{L}C^i_\alpha)$ (resp. $(\mathcal{L}R^i_\alpha)$), $i = 1, 2$.

The families $(\mathcal{L}C^1_\alpha)$ and $(\mathcal{L}C^2_\alpha)$ (we have also similar results for the families $(\mathcal{L}R^1_\alpha)$ and $(\mathcal{L}R^2_\alpha)$) are best understood by considering pencils of forms in the projective plane (§5). At each point on M, the families $(\mathcal{L}C^1_\alpha)$ and $(\mathcal{L}C^2_\alpha)$, $\alpha \in \mathbb{R}$, parametrise two disjoint open intervals of the pencil joining the $\mathbb{A}$-asymptotic BDE and the lines of $\mathbb{A}$-principal curves BDE. The union of the closure of these intervals is the full pencil. The boundary BDEs $\mathcal{L}C^1_{-\infty} = \mathcal{L}C^2_{-\infty}$ and $\mathcal{L}C^1_{+\infty} = \mathcal{L}C^2_{+\infty}$ have the property that one of the solution curves of one BDE is a lightlike foliation and one of the solution curves of the other is the other lightlike foliation of M. The $\mathbb{A}$-asymptotic BDE is a member of $(\mathcal{L}C^1_\alpha)$ and the lines of $\mathbb{A}$-principal curves BDE a member of $(\mathcal{L}C^2_\alpha)$. Therefore, the BDEs $(\mathcal{L}C^1_{\pm\infty})$ form an obstruction for linking the $\mathbb{A}$-asymptotic and $\mathbb{A}$-principal curves BDEs via the families of conjugate curves congruence. This phenomenon is explained in more details in Theorem 5.2. We show in §5 (Theorem 5.3) that the various BDEs considered here are completely determined by the lightlike curves BDE and its $\mathbb{A}$-conjugate BDE. We also obtain results on BDEs with orthogonal roots (Theorem 5.5) and on the $\mathbb{A}$-conjugate BDE of a given BDE (Theorem 5.7).

Our results apply to, for example, the foliations determined by the shape operator of a timelike surface in the Minkowski $\mathbb{R}^3_1$; the shape operator along a normal vector of a timelike surface in the Minkowski $\mathbb{R}^4_1$; the shape operator derived from the de Sitter Gauss map ([8]) of a timelike surface in the de Sitter space $S^3_1 \subset \mathbb{R}^4_1$. The approach of considering general self-adjoint operators deals with all these cases (and others that we do not list here) in a unified way. It is also worth pointing out that this approach lead in [15] to a new definition of lines of principal curvature on a smooth surface in $\mathbb{R}^4$.

2 Preliminaries

Let M be a smooth and orientable surface endowed with a Lorentzian metric g. We say that a vector $v \in T_pM$ is *spacelike* if $g(v, v) > 0$, *lightlike* if $g(v, v) = 0$ and *timelike* if $g(v, v) < 0$. The norm of a vector $v \in T_pM$ is defined by $\|v\| = \sqrt{|g(v, v)|}$.

Let $\mathbf{r} : U \to M$ be a local parametrisation of M, where U is an open subset of $\mathbb{R}^2$. The first fundamental form of M at a point p is the quadratic form $\mathrm{I}_p : T_pM \to \mathbb{R}$ given by $\mathrm{I}_p(v) = g(v, v)$. If $p \in \mathbf{r}(U)$ and $v = a\mathbf{r}_x + b\mathbf{r}_y$, then $\mathrm{I}_p(v) = Ea^2 + 2Fab + Gb^2$, where

$$E = g(\mathbf{r}_x, \mathbf{r}_x), \quad F = g(\mathbf{r}_x, \mathbf{r}_y), \quad G = g(\mathbf{r}_y, \mathbf{r}_y),$$

are the coefficients of I_p with respect to the parametrisation $\mathbf{r}$.

As g is Lorentzian, $EG - F^2 < 0$ on M. This means that there are two linearly independent lightlike directions in T_pM at all points $p \in \mathbf{r}(U)$. These are determined by the BDE

$$(L): \quad Gdy^2 + 2Fdydx + Edx^2 = 0 \tag{2}$$

We call (L) the *lightlike* BDE and its solutions the *lightlike foliations*. We can take a local chart at any point on M in such a way that the coordinate curves are the lightlike curves (see Theorem 3.1 in [10]). In this chart, $E = G = 0$ and F is strictly positive or negative function. This parametrisation will prove very useful in subsequent sections.

Given a self-adjoint operator $\mathbb{A} : TM \to TM$, we denote by $\mathbb{A}_p$ the restriction of $\mathbb{A}$ to T_pM. If $v = a\mathbf{r}_x + b\mathbf{r}_y$, then $g(\mathbb{A}_p(v), v) = la^2 + 2mab + nb^2$, where

$$l = g(\mathbb{A}_p(\mathbf{r}_x), \mathbf{r}_x),\ m = g(\mathbb{A}_p(\mathbf{r}_x), \mathbf{r}_y) = g(\mathbb{A}_p(\mathbf{r}_y), \mathbf{r}_x),\ n = g(\mathbb{A}_p(\mathbf{r}_y), \mathbf{r}_y)$$

are referred to as the coefficients of $\mathbb{A}_p$. We still denote by $\mathbb{A}_p$ the matrix of the linear operator $\mathbb{A}_p$ with respect to the basis $\{\mathbf{r}_x, \mathbf{r}_y\}$. This matrix is given by

$$\mathbb{A}_p = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix}. \tag{3}$$

Because g is Lorentzian, the self-adjoint operator $\mathbb{A}_p$ does not always have real eigenvalues. When it does, we denote them by κ_i, $i = 1, 2$ and call them the $\mathbb{A}$-*principal curvatures*. The eigenvectors of $\mathbb{A}_p$ are called the $\mathbb{A}$-*principal directions* and the integral curves of their associated line fields are called *the lines of* $\mathbb{A}$-*principal curvature*. The equation of the lines of $\mathbb{A}$-principal curvature is analogous to that of surfaces in $\mathbb{R}^3$ and is given by

$$(P): \quad (Fn - Gm)dy^2 + (En - Gl)dydx + (Em - Fl)dx^2 = 0. \tag{4}$$

A direction $u \in T_pM$ is said to be $\mathbb{A}$-*asymptotic* if $g(\mathbb{A}_p(u), u) = 0$. It follows that the $\mathbb{A}$-asymptotic curves (whose tangent at all points are $\mathbb{A}$-asymptotic directions) are solutions of the BDE

$$(A): \quad ndy^2 + 2mdydx + ldx^2 = 0. \tag{5}$$

The $\mathbb{A}$-characteristic directions/curves are defined in [10] using the results in [5]. These are given by the BDE which form a self-polar triangle with the $\mathbb{A}$-principal and $\mathbb{A}$-asymptotic curves BDEs (see §5). It is simply the Jacobian of the asymptotic and principal curves BDEs and is given by

$$(C): \quad \begin{array}{r} (2m(mG - nF) - n(lG - nE))dy^2 + 2(m(lG + nE) - 2lnF)dydx + \\ (2m(mE - lF) - l(nE - lG))dx^2 = 0. \end{array} \tag{6}$$

Remark 2.1 *The $\mathbb{A}$-principal directions, when distinct, are orthogonal and one is spacelike while the other is timelike. The generic local configurations of the lines of $\mathbb{A}$-principal curves, the $\mathbb{A}$-asymptotic and $\mathbb{A}$-characteristic curves are obtained in* [10].

We denote the $\mathbb{A}$-conjugate direction of a direction u by $\bar{u}$ (so $g(\mathbb{A}_p(u), \bar{u}) = 0$). If we consider a parametrisation with $E = G = 0$, i.e., the coordinate curves are the lightlike foliations, then the conjugate of the lightlike direction $\mathbf{r}_x$ is $\bar{\mathbf{r}}_x = m\mathbf{r}_x - l\mathbf{r}_y$ and that of $\mathbf{r}_y$ is $\bar{\mathbf{r}}_y = n\mathbf{r}_x - m\mathbf{r}_y$. Thus, $\bar{\mathbf{r}}_x$ is lightlike at a point $p = \mathbf{r}(q)$ if and only if $l(q) = 0$ or $m(q) = 0$. The condition $l(q) = 0$ means that $\mathbf{r}_x$ is self-conjugate and p is on the discriminant of the BDE (4) which is labelled the Lightlike Principal Locus (LPL) in [9, 10]. If we call $H(p) = (lG - 2mF + nE)/(2(EG - F^2))$ the $\mathbb{A}$-*mean curvature* at p, then the condition $m(q) = 0$ is equivalent to $H(p) = 0$ when $\mathbf{r}$ is as above. When $H(p) = 0$, $\bar{\mathbf{r}}_x$ is parallel to the other lightlike direction. Also, observe that when $H(q) = 0$, the $\mathbb{A}$-characteristic directions at p are both lightlike and are along $\mathbf{r}_x$ and $\mathbf{r}_y$. We have thus the following result.

Proposition 2.2 *Let $\mathbb{A}$ be a self-adjoint operator on a smooth surface M with a Lorentzian metric g.*

(1) *A lightlike direction u at p is self-conjugate if and only if p is on the LPL and u is the unique $\mathbb{A}$-principal direction at p (which is also an asymptotic direction).*

(2) *The conjugate of a lightlike direction u at p is the other lightlike direction at p if and only if p is on the curve $H = 0$. At such points both lightlike directions are $\mathbb{A}$-characteristic directions.*

We have more results on $\mathbb{A}$-conjugacy in §5 (Theorem 5.7).

2.1 Angles in the Lorentzian plane

Consider the Minkowski plane $\mathbb{R}^2_1$, which is the vector space $\mathbb{R}^2$ with the pseudo scalar product $g(u, v) = u_1v_1 - u_2v_2$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. A timelike vector $u = (u_1, u_2)$ is said to be future pointing if $u_2 > 0$, otherwise it is said to be past-pointing.

The positive Lorentzian group of $\mathbb{R}^2_1$ (which acts transitively on $\mathbb{R}^2_1 \setminus 0$) is generated by the matrices

$$R(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}, \; \alpha \in \mathbb{R}.$$

The oriented hyperbolic angle $\alpha = \angle(u, v)$ between two non-lightlike vectors u, v in $\mathbb{R}^2_1$ is defined in [1, 2, 12, 13] as follows, and has similar properties to the Euclidean angle.

If u, v are both future or past pointing unit timelike vectors, then α is defined by the relation $R(\alpha).u = v$ and satisfies

$$\cosh(\alpha) = -g(u, v), \qquad \sinh(\alpha) = -g(u, Sv),$$

where $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Euclidean reflection with respect to the (lightlike) line $u_1 = u_2$.

If u, v are both unit timelike vectors but one is a future pointing and the other is a past pointing vector, then α is defined by the relation $R(\alpha).u = -v$ and satisfies

$$\cosh(\alpha) = g(u, v), \qquad \sinh(\alpha) = g(u, Sv).$$

If u, v are both unit spacelike vectors, Su and Sv are timelike and can point in any direction. Then $\angle(u, v) := -\angle(Su, Sv)$ if Su and Sv are pointing in the same direction and $\angle(u, v) := \angle(Su, Sv)$ otherwise.

If u and v are unit vectors of different type, and supposing that u is timelike, then $\angle(u, v) := -\angle(u, Sv)$.

For vectors u and v of arbitrary non zero lengths, we divide the above formulae by the product of the lengths of u and v. For example, if u, v are both future pointing timelike vectors

$$\cosh(\alpha) = -\frac{g(u, v)}{||u||||v||}, \qquad \sinh(\alpha) = -\frac{g(u, Sv)}{||u||||v||}.$$

The notion of hyperbolic angle can of course be defined in any Lorentzian plane, with pseudo scalar product g. If l_1 and l_2 are two lightlike independent vectors, then $e_1 = 2(l_1+l_2)/g(l_1+l_2, l_1+l_2)$ and $e_2 = 2(l_1-l_2)/g(l_1-l_2, l_1-l_2)$ form an orthonormal basis of the Lorentzian plane. We can suppose, without loss of generality, that e_1 is spacelike, so e_2 is timelike. Then the Lorentzian plane, with the basis $\{e_1, e_2\}$, can be identified with the Minkowski plane $\mathbb{R}^2_1$. We proceed as in the case $\mathbb{R}^2_1$ and the formulae for the angle are the same as above, where g now denotes the pseudo scalar product in the general Lorentzian plane.

Remarks 2.3 (1) The hyperbolic angle between a lightlike vector and any other vector is not well defined. If we fix u as a non lightlike vector and let v tend to a lightlike vector, then $\angle(u, v)$ tends to $\pm\infty$.

(2) For a Lorentzian surface M, we define the oriented angle between two tangent directions at p as the angle between any of their respective directional vectors u and v. The angle does not depend of the choice of the vectors as $\angle(u, v) = \angle(-u, v) = \angle(u, -v) = \angle(-u, -v)$.

3 The Lorentzian conjugate curve congruence

We follow the approach in [5, 7] for surfaces in the Euclidean space but consider instead oriented hyperbolic angles. We consider directions u in T_pM that make a fixed oriented (hyperbolic) angle α with their $\mathbb{A}$-conjugate directions $\bar{u}$. We need to consider two cases depending on whether u and $\bar{u}$ are of the same type (both spacelike or timelike) or are of different types (one is spacelike while the other is timelike).

Definition 3.1 *Let M be a smooth oriented Lorentzian surface and let PTM denote the projective tangent bundle to M. Given a self-adjoint operator $\mathbb{A}$ on M, define $\Theta_i : PTM \to \mathbb{R}$, $i = 1, 2$, by $\Theta_i(p, v) = \alpha = \angle(u, \bar{u})$, with $i = 1$ when u and its $\mathbb{A}$-conjugate direction $\overline{u}$ are of the same type, and $i = 2$ when they are of different types. The* $\mathbb{A}$-Lorentzian conjugate curve congruence*, for a fixed α, is defined to be $\Theta_i^{-1}(\alpha)$, $i = 1, 2$, and is denoted by $\mathcal{L}C^i_\alpha$. (Note that Θ_i, $i = 1, 2$, are not well defined at points corresponding to $\mathbb{A}$-asymptotic directions at $\mathbb{A}$-parabolic points.)*

We have the following result, where (P) and (A) are the BDEs (4) and (5) respectively.

Theorem 3.2 *The $\mathbb{A}$-Lorentzian conjugate curve congruences are given by BDEs and are as follows.*
If u and $\bar{u}$ are both spacelike or both timelike:

$$(\mathcal{L}C^1_\alpha) : \qquad \sinh(\alpha)P + \sqrt{F^2 - EG}\cosh(\alpha)A = 0.$$

If u is spacelike and $\bar{u}$ is timelike or vice-versa:

$$(\mathcal{L}C^2_\alpha) : \qquad \cosh(\alpha)P + \sqrt{F^2 - EG}\sinh(\alpha)A = 0.$$

Proof We take a local parametrisation $\mathbf{r} : U \to M$ of the surface and write, without loss of generality, $u = \mathbf{r}_x + \xi\mathbf{r}_y$ for a tangent direction $u \in T_pM$, and $\bar{u} = \mathbf{r}_x + \eta\mathbf{r}_y$ for its $\mathbb{A}$-conjugate direction, where ξ and η are real numbers.

We start with the case where $\mathbf{r}_x$ or $\mathbf{r}_y$ is not lightlike. We assume that $\mathbf{r}_x$ is spacelike (the case when it is timelike follows similarly) and choose the following orthonormal frame $\{e_1, e_2\} = \{\frac{\mathbf{r}_x}{\sqrt{E}}, \frac{\sqrt{E}}{\sqrt{F^2-EG}}(\mathbf{r}_y - \frac{F}{E}\mathbf{r}_x)\}$ in T_pM. The coordinates of u and $\bar{u}$, with respect to this frame, are

$$u = (\tfrac{E+\xi F}{\sqrt{E}}, \tfrac{\xi\sqrt{F^2-EG}}{\sqrt{E}}),$$
$$\bar{u} = (\tfrac{E+\eta F}{\sqrt{E}}, \tfrac{\eta\sqrt{F^2-EG}}{\sqrt{E}}).$$

We have $g(\mathbb{A}_p(u), \bar{u}) = 0$, therefore

$$l + (\xi + \eta)m + \xi\eta n = 0,$$

and we get $\eta = -(\xi m + l)/(\xi n + m)$.

If u and $\bar{u}$ are spacelike, then

$$\tanh(\alpha_1) = -\tfrac{\xi\sqrt{F^2-EG}}{E+\xi F},$$
$$\tanh(\alpha_2) = -\tfrac{\eta\sqrt{F^2-EG}}{E+\eta F} = \tfrac{(\xi m+l)\sqrt{F^2-EG}}{\xi(En-mF)+Em-lF},$$

with $\alpha_1 = \angle(u, e_1)$ and $\alpha_2 = \angle(\bar{u}, e_1)$. Substituting in

$$\tanh(\alpha_1 - \alpha_2) = \frac{\tanh(\alpha_1) - \tanh(\alpha_2)}{1 - \tanh(\alpha_1)\tanh(\alpha_2)}$$

the expressions for $\tanh(\alpha_1)$ and $\tanh(\alpha_2)$ as functions of ξ we get

$$\begin{array}{c}(\sinh(\alpha)(Fn-Gm)+n\cosh(\alpha)\sqrt{F^2-EG})\xi^2+\\ (\sinh(\alpha)(En-Gl)+2m\cosh(\alpha)\sqrt{F^2-EG})\xi+\\ (\sinh(\alpha)(Em-Fl)+l\cosh(\alpha)\sqrt{F^2-EG})=0.\end{array}$$

Setting $\xi=dy/dx$ and re-arranging the above equation yield the expression for $(\mathcal{L}C^1_\alpha)$ in the statement of the theorem.

If u and $\bar{u}$ are timelike, then

$$\begin{array}{rcl}\tanh(\alpha_1)&=&-\frac{E+\xi F}{\xi\sqrt{F^2-EG}},\\ \tanh(\alpha_2)&=&-\frac{E+\eta F}{\eta\sqrt{F^2-EG}}=\frac{\xi(En-mF)+Em-lF}{(\xi m+l)\sqrt{F^2-EG}}.\end{array}$$

Again, considering $\tanh(\alpha_1-\alpha_2)$ yields the equation $(\mathcal{L}C^1_\alpha)$ (which is the same as that for the case when u and $\bar{u}$ are spacelike).

If u is timelike and $\bar{u}$ is spacelike, then

$$\begin{array}{rcl}\tanh(\alpha_1)&=&-\frac{E+\xi F}{\xi\sqrt{F^2-EG}}\\ \tanh(\alpha_2)&=&-\frac{\eta\sqrt{F^2-EG}}{E+\eta F}=\frac{(\xi m+l)\sqrt{F^2-EG}}{\xi(En-mF)+Em-lF},\end{array}$$

and the expression for $(\mathcal{L}C^2_\alpha)$ follows in an analogous way.

When both $\mathbf{r}_x$ and $\mathbf{r}_y$ are lightlike, we consider the orthonormal frame $\{e_1,e_2\}=\{\frac{\mathbf{r}_x+\mathbf{r}_y}{\sqrt{2|F|}},\frac{\mathbf{r}_x-\mathbf{r}_y}{\sqrt{2|F|}}\}$ and suppose e_1 spacelike. We then proceed as above. $\square$

Remarks 3.3 (1) Theorem 3.2 states that there are at most two directions in T_pM that make a fixed oriented angle α with their $\mathbb{A}$-conjugate directions.

(2) For surfaces in the Euclidean space (and for self-adjoint operators on a Riemannian surface in general), the conjugate curve congruence is given by

$$(\mathcal{C}_\alpha):\ \cos(\alpha)P+\sqrt{EG-F^2}\sin(\alpha)A=0.$$

Thus, the family $(\mathcal{C}_\alpha)$ contains both the asymptotic BDE $(\mathcal{C}_0)$ and the principal BDE $(\mathcal{C}_{\pm\frac{\pi}{2}})$. However, in Theorem 3.2, $(\mathcal{L}C^1_\alpha)$ contains the BDE $(\mathcal{L}C^1_0)$ of the $\mathbb{A}$-asymptotic curves but not the BDE of the lines of $\mathbb{A}$-principal curves. Similarly, $(\mathcal{L}C^2_\alpha)$ contains the BDE $(\mathcal{L}C^2_0)$ of the lines of $\mathbb{A}$-principal curves but not the BDE of the $\mathbb{A}$-asymptotic curves.

We analyse now the limit of $(\mathcal{L}C^i_\alpha)$, $i=1,2$ as $\alpha\to\pm\infty$. If we divide both equations by $\cosh\alpha$, we get $\lim_{\alpha\to\pm\infty}\mathcal{L}C^1_\alpha=\lim_{\alpha\to\pm\infty}\mathcal{L}C^2_\alpha=\mathcal{L}C_{\pm\infty}$ which are given by the BDEs

$$(\mathcal{L}C_{\pm\infty}):\qquad \pm P+\sqrt{F^2-EG}A=0.$$

The above equations have a property that becomes more apparent when considering a special local parametrisation where the coordinate curves are lightlike, that is, when $E = G = 0$. Then, $P = F(ndy^2 - ldx^2)$ and we have (up to a factor of F),

$$\begin{array}{l}(\mathcal{L}C_{+\infty}):\ dy(mdx + ndy) = 0,\\(\mathcal{L}C_{-\infty}):\ dx(ldx + mdy) = 0\end{array}$$

if $F > 0$, otherwise the equations are interchanged. The BDEs $(\mathcal{L}C_{\pm\infty})$ determine two directions at each point, one is lightlike and the $\mathbb{A}$-conjugate of the second direction is the other lightlike direction. Thus, the angle between the directions determined by these BDEs and their $\mathbb{A}$-conjugate directions is infinite. (Theorem 5.2 sheds more light as to why $(\mathcal{L}C_{\pm\infty})$ are obstructions for linking the $\mathbb{A}$-asymptotic BDE with the lines of $\mathbb{A}$-principal curves BDE.)

4 The Lorentzian reflected congruence

$\mathbb{A}$-Conjugation $\bar{u} = C(u)$ gives an involution on PT_pM. Suppose that there are two $\mathbb{A}$-principal directions at p. (See Remark 4.3(2) for the case when there are no $\mathbb{A}$-principal directions at p.) Then, there is another involution $u \mapsto R(u)$ on PT_pM which is simply reflection in (either of) the principal directions. We use, as in [5], $R \circ C$ to determine families of BDEs by asking that the angle between a direction u and the image of $R \circ C(u) = R(\bar{u})$ is constant.

Definition 4.1 *Let M be a smooth oriented Lorentzian surface. Given a self-adjoint operator $\mathbb{A}$ on M, define $\Phi_i : PTM \to \mathbb{R}$, $i = 1, 2$, by $\Phi_i(p, v) = \alpha = \angle(u, R(\bar{u}))$, with $i = 1$ when u and $R(\bar{u})$ are of the same type, and $i = 2$ when they are of different types. The* $\mathbb{A}$-Lorentzian reflected curve congruence*, for a fixed α, is defined to be $\Phi_i^{-1}(\alpha)$, $i = 1, 2$, and is denoted by $\mathcal{L}R^i_\alpha$. (Note that Φ_i, $i = 1, 2$, are not well defined at points (p, u) with $\delta(p) \leq 0$, where δ is the discriminant of the lines of $\mathbb{A}$-principal BDE.)*

An alternative way of defining α at (p, u) is as the sum of the angles between u and a principal direction e and $\bar{u}$ and e. This does not depend on the choice of the principal direction. We have the following result, where (P) and (C) are the BDEs (4) and (6) respectively.

Theorem 4.2 *The $\mathbb{A}$-Lorentzian reflected curve congruences are given by BDEs and are as follows.*
If u and $\bar{u}$ are both spacelike or both timelike:

$$(\mathcal{L}R^1_\alpha): \qquad \cosh(\alpha)C + \frac{2Fm - Gl - En}{\sqrt{F^2 - EG}} \sinh(\alpha)P = 0.$$

If u is spacelike and $\bar{u}$ is timelike or vice-versa:

$$(\mathcal{L}R^2_\alpha): \qquad \sinh(\alpha)C + \frac{2Fm - Gl - En}{\sqrt{F^2 - EG}}\cosh(\alpha)P = 0.$$

Proof We follow the setting in the proof of Theorem 3.2. Suppose that there are two $\mathbb{A}$-principal directions in $\mathbf{r}(U)$. These are the solutions of the BDE (4) and are given by $u_1 = M_1\mathbf{r}_x + M_2\mathbf{r}_y$ and $u_2 = N_1\mathbf{r}_x + N_2\mathbf{r}_y$, with

$$\begin{array}{l} M_1 = N_1 = 2(Fn - mG), \\ M_2 = -(En - lG) + \sqrt{(En - lG)^2 - 4(Fn - mG)(Em - Fl)}, \\ N_2 = -(En - lG) - \sqrt{(En - lG)^2 - 4(Fn - mG)(Em - Fl)}. \end{array}$$

We use the orthonormal system $\{e_1, e_2\}$, $e_i = u_i/\sqrt{\mid g(u_i, u_i) \mid}$, $i = 1, 2$, and assume that e_1 is spacelike (see Remark 2.1). We write $e_1 = M_{11}\mathbf{r}_x + M_{21}\mathbf{r}_y$, where $M_{i1} = M_i/\sqrt{\mid g(u_1, u_1) \mid}$, $i = 1, 2$, and $e_2 = N_{12}\mathbf{r}_x + N_{22}\mathbf{r}_y$, where $N_{i2} = N_i/\sqrt{\mid g(u_2, u_2) \mid}$, $i = 1, 2$.

The coordinates of $u = \mathbf{r}_x + \xi\mathbf{r}_y$ in the coordinates system $\{e_1, e_2\}$ are given by

$$\frac{1}{M_{11}N_{22} - M_{21}N_{12}}(N_{22} - N_{12}\xi, -M_{21} + M_{11}\xi)$$

and those of $\bar{u} = \mathbf{r}_x + \eta\mathbf{r}_y$ (where $\eta = -(\xi m + l)/(\xi n + m)$, see the proof of Theorem 3.2) are given by

$$\frac{1}{M_{11}N_{22} - M_{21}N_{12}}(N_{22} - N_{12}\eta, -M_{21} + M_{11}\eta).$$

We use the alternative way of defining $\alpha = \angle(u, R(\bar{u}))$ as $\alpha = \angle(u, e_1) + \angle(\bar{u}, e_1)$. We observe that $\bar{u}$ and $R(\bar{u})$ are always of the same type.

If u and $\bar{u}$ are spacelike, then

$$\begin{array}{l} \tanh(\alpha_1) = \frac{M_{21} - M_{11}\xi}{N_{22} - N_{12}\xi}, \\ \tanh(\alpha_2) = \frac{M_{21} - M_{11}\eta}{N_{22} - N_{12}\eta} = \frac{(mM_{11} + nM_{21})\xi + mM_{21} + lM_{11}}{(nN_{22} + mN_{12})\xi + mN_{22} + lN_{12}}, \end{array}$$

with $\alpha_1 = \angle(u, e_1)$ and $\alpha_2 = \angle(\bar{u}, e_1)$. We obtain, as in the proof of Theorem 3.2, the expression for $\tanh(\alpha_1 + \alpha_2)$ in terms of ξ. Setting $\xi = dy/dx$ yields the expression for $(\mathcal{L}R^1_\alpha)$. A key observation here is that the terms involving square roots factor out and we end up with the equation of $\mathcal{L}R^1_\alpha$ as in the statement of the theorem times a function in (x, y).

If u and $\bar{u}$ are timelike, then

$$\begin{array}{l} \tanh(\alpha_1) = \frac{N_{22} - N_{12}\xi}{M_{21} - M_{11}\xi}, \\ \tanh(\alpha_2) = \frac{N_{22} - N_{12}\eta}{M_{21} - M_{11}\eta} = \frac{(nN_{22} + mN_{12})\xi + mN_{22} + lN_{12}}{(mM_{11} + nM_{21})\xi + mM_{21} + lM_{11}}, \end{array}$$

and we obtain the same expression for $(\mathcal{L}R^1_\alpha)$.

If u is timelike and $\bar{u}$ is spacelike, then

$$\begin{array}{l} \tanh(\alpha_1) = \frac{N_{22}-N_{12}\xi}{M_{21}-M_{11}\xi}, \\ \tanh(\alpha_2) = \frac{M_{21}-M_{11}\eta}{N_{22}-N_{12}\eta} = \frac{(mM_{11}+nM_{21})\xi+mM_{21}+lM_{11}}{(nN_{22}+mN_{12})\xi+mN_{22}+lN_{12}}. \end{array}$$

Similar calculations yield the expression for $(\mathcal{L}R^2_\alpha)$. □

Remarks 4.3 (1) Assuming that there are two $\mathbb{A}$-principal directions at p, Theorem 4.2 states that there are at most two directions in T_pM that make a fixed oriented angle α with the reflection of their $\mathbb{A}$-conjugate directions with respect to a $\mathbb{A}$-principal direction.

(2) When there are no $\mathbb{A}$-principal directions at p, we define the $\mathbb{A}$-Lorentzian reflected curve congruence, for a fixed α, as the solutions of the BDEs $(\mathcal{L}R^i_\alpha), i = 1, 2$, in Theorem 4.2. The solutions do not have the geometric characterisation in Definition 4.1 in terms of angles but have the same characterisation in terms of pencils of quadratic forms (section 5) as for the case when there are two $\mathbb{A}$-principal directions at p.

(3) For surfaces in the Euclidean 3-space (and for self-adjoint operators on a Riemannian surface in general), the reflected curve congruence $(\mathcal{R}_\alpha)$ is given by

$$(\mathcal{R}_\alpha) : \cos(\alpha)C + \frac{2Fm - Gl - En}{\sqrt{EG - F^2}} \sin(\alpha)P = 0.$$

Thus, the family $(\mathcal{R}_\alpha)$ contains both the characteristic curves BDE $(\mathcal{R}_0)$ and the principal BDE $(\mathcal{R}_{\pm\frac{\pi}{2}})$. However, in Theorem 3.2, $(\mathcal{L}R^1_\alpha)$ contains the BDE $(\mathcal{L}R^1_0)$ of the $\mathbb{A}$-characteristic curves but not the BDE of the lines of $\mathbb{A}$-principal curves. Similarly, $(\mathcal{L}R^2_\alpha)$ contains the BDE $(\mathcal{L}R^2_0)$ of the lines of $\mathbb{A}$-principal curves but not the BDE of the $\mathbb{A}$-characteristic curves.

We analyse the limit of $(\mathcal{L}R^i_\alpha)$, $i = 1, 2$ as $\alpha \to \pm\infty$. If we divide both equations by $\cosh\alpha$, we get $\lim_{\alpha\to\pm\infty} \mathcal{L}R^1_\alpha = \lim_{\alpha\to\pm\infty} \mathcal{L}R^2_\alpha = \mathcal{L}R_{\pm\infty}$ which are given by

$$(\mathcal{L}R_{\pm\infty}) : \qquad C \pm \frac{2Fm - Gl - En}{\sqrt{F^2 - EG}} P = 0.$$

We consider a special local parametrisation where the coordinate curves are lightlike, so $E = G = 0$, and suppose that $F > 0$ (if $F < 0$, the equations below are interchanged). Then, $P = F(ndy^2 - ldx^2)$, $C = -2F(mndy^2 + 2lndxdy + mldx^2)$, and we get (up to some factors)

$$\begin{array}{l} (\mathcal{L}R_{-\infty}) : \; dy(ldx + mdy) = 0, \\ (\mathcal{L}R_{+\infty}) : \; dx(mdx + ndy) = 0. \end{array}$$

The BDEs ($\mathcal{L}R_{\pm\infty}$) determine two directions at each point, one is lightlike and the second is its $\mathbb{A}$-conjugate direction. Thus, the angle between the directions determined by these BDE and their reflection with respect to a principal direction is infinite. (See Theorem 5.2 as to why $\mathcal{L}R_{\pm\infty}$ are obstructions for linking the $\mathbb{A}$-characteristic BDE with the lines of $\mathbb{A}$-principal curves BDE.)

5 Pencils of quadratic forms

The relation between the triple BDEs (asymptotic, characteristic and principal) on a smooth oriented surface in the Euclidean 3-space is explained in [5] using pencils of forms. As we commented elsewhere in this paper, the results in [5] are also valid for self-adjoint operators on a Riemannian surface. We investigate here the case when the metric on M is Lorentzian.

We shall not distinguish between a BDE (1) and its non-zero multiples, so at each point $(x, y) \in U$ we can view the BDE as a quadratic form $a\beta^2 + 2b\beta\gamma + c\gamma^2 = 0$ ($\beta = dy$ and $\gamma = dx$) and represent it by the point $Q = (a : 2b : c)$ in the projective plane $\mathbb{R}P^2$. In $\mathbb{R}P^2$ there is a conic $\Gamma = \{Q : b^2 - ac = 0\}$ of singular quadratic forms. These can be put in the form $(a_1\beta + b_1\gamma)^2$.

The *polar line* $\widehat{Q}$ of a point Q (with respect to the conic Γ) is the line that contains all points O such that Q and O are harmonic conjugate points with respect to the intersection points R_1 and R_2 of the conic Γ and a variable line through Q. Geometrically, if the polar line $\widehat{Q}$ meets Γ, then the tangents to Γ at the points of intersection meet at Q. A point $(a_1 : b_1 : c_1)$ is in the polar line of a point $Q = (a : 2b : c)$, if and only if $bb_1 - ac_1 - a_1c = 0$. Three points in the projective plane are said to form a *self-polar triangle* if the polar of any vertex of the triangle is the line through the remaining two points.

The next series of remarks are well known results and elementary, but very useful. They relate some of the invariants of pairs of binary forms to the geometry of the conic Γ of singular forms.

Remarks 5.1 (1) Let Q be a binary quadratic form, with distinct roots, determining a point in the plane $\mathbb{R}P^2$. Then $\widehat{Q}$ consists of the line through the two forms which are the squares of the factors of Q, that is, the tangents to the conic at these two points pass through Q. We refer to this intersection point as the polar form of the pencil. Conversely given any pencil meeting the conic Γ, the corresponding polar form is the binary form whose factors are the repeated factors at the two singular members of the pencil.

(2) The polar form of a pencil is given by the Jacobian of any two of the forms in the pencil, that is, the 2×2 determinant of the matrix of partial derivatives of the forms with respect to β and γ. The Jacobian is non-zero provided we have a genuine pencil, and is a square if and only if the forms have a factor in common.

(3) Three forms determine a self-polar triangle with respect to the conic Γ if and only if each is the Jacobian of the other two. Thus, given a self-adjoint operator $\mathbb{A}$ on M, the triangle (P, A, C) is self-polar by construction of $C = Jac(A, P)$.

(4) If the vertices of a quadrangle lie on Γ then the diagonal triangle (the triangle whose vertices are intersections of the lines joining distinct pairs of distinct points) is self-polar.

The expressions we use here simplify considerably and the geometry becomes more apparent when we take, as we shall do in the rest of this paper, a local parametrisation of M where the coordinate curves are lightlike curves (i.e., $E = G = 0$). We shall also assume, without loss of generality, that $F > 0$. Then, we can represent the BDEs of interest by the following points in the projective plane

$$\begin{array}{rll} \text{Lightlike BDE (the metric)} & (L): & (0:1:0) \\ \mathbb{A}\text{-principal BDE} & (P): & (n:0:-l) \\ \mathbb{A}\text{-asymptotic BDE} & (A): & (n:2m:l) \\ \mathbb{A}\text{-characteristic BDE} & (C): & (mn:2ln:ml) \end{array}$$

We will identify a BDE (ω) at $p \in M$ with its representative ω in the projective plane. We have the following where the pencil passing through two points Q_1 and Q_2 is denoted by (Q_1, Q_2).

Theorem 5.2 (1) *The lightlike BDE L belongs to the polar line $\widehat{P}$ of the lines of $\mathbb{A}$-principal curvature.*

(2) *BDEs whose one solution curves is a lightlike foliation of M form two pencils $\widehat{L}_1$ and $\widehat{L}_2$. These are the two tangent lines to Γ through the lightlike BDE L (Figure 1).*

(3) *The pencils $\widehat{L}_1$ and $\widehat{L}_2$ and the conic Γ partition the projective plane into four connected regions where the type of the solutions, when they exist, of the BDEs in each region is constant. The type of the solutions is as shown in Figure 1.*

(4) *The pencils $\widehat{L}_1$ and $\widehat{L}_2$ intersect the pencil (A, P) at $\mathcal{L}C_{+\infty}$ and $\mathcal{L}C_{-\infty}$, respectively. The points $\mathcal{L}C_{+\infty}$ and $\mathcal{L}C_{-\infty}$ separate A and P and are thus an obstruction for linking the $\mathbb{A}$-asymptotic curves with the lines of $\mathbb{A}$-principal curvature via the families of conjugate curve congruences (Figure 1).*

(5) *The pencils $\widehat{L}_1$ and $\widehat{L}_2$ intersect the pencil (C, P) at $\mathcal{L}R_{-\infty}$ and $\mathcal{L}R_{+\infty}$, respectively. The points $\mathcal{L}R_{-\infty}$ and $\mathcal{L}R_{+\infty}$ separate C and P and are thus an obstruction for linking the $\mathbb{A}$-characteristic curves with the lines of $\mathbb{A}$-principal curvature via the families of reflected curve congruences (Figure 1).*

Proof (1) A point $(a : b : c)$ is on the polar line of $P = (n : 0 : -l)$ if and only if $al - nc = 0$. It is clear that $L = (0 : 1 : 0)$ is on this polar line.

(2) One of the foliations determined by the BDE $Q = (a : 2b : c)$ is lightlike if and only if $a = 0$ or $c = 0$. When $c = 0$, we get the pencil $\widehat{L}_1$ which is parametrised by $(a : 2b : 0)$ and is the polar line of $L_1 = (0 : 0 : 1)$. When $a = 0$, we get the pencil $\widehat{L}_2$ which is parametrised by $(0 : 2b : c)$ and is the polar line of $L_2 = (1 : 0 : 0)$. The two pencils intersect when $a = c = 0$, i.e., at the point L. It is not difficult to show that $\widehat{L}_1$ (resp. $\widehat{L}_2$) is tangent to Γ at L_1 (resp. L_2).

(3) The solutions of the BDE $Q = (a : 2b : c)$ are given by $v_i = a\mathbf{r}_x + (-b + (-1)^i\sqrt{b^2 - ac})\mathbf{r}_y$, $i = 1, 2$, so $g(v_1, v_1)g(v_2, v_2) = 4F^2a^2(ac)$. If $ac > 0$, both solutions are of the same type (either spacelike or timelike), otherwise one is timelike and the other is spacelike. Of course Q has no solutions if it is inside the conic Γ which lies in the region $ac > 0$.

(4) Clearly, the points given by the families $(\mathcal{LC}^1_\alpha)$ and $(\mathcal{LC}^2_\alpha)$ are on the pencil (A, P). We have $\mathcal{LC}_{-\infty} = (0 : m : l)$ and $\mathcal{LC}_{+\infty} = (n : m : 0)$, so $\mathcal{LC}_{-\infty}$ is on the pencil $a = 0$ (i.e., $\widehat{L}_2$) and $\mathcal{LC}_{+\infty}$ is on the pencil $c = 0$ (i.e., $\widehat{L}_1$). In fact, $\mathcal{LC}_{-\infty} = A - P$ and $\mathcal{LC}_{+\infty} = A + P$. Therefore, $\mathcal{LC}_{-\infty}$ and $\mathcal{LC}_{+\infty}$ separate A and P and form an obstruction for linking these two BDEs by the families of conjugate curve congruences.

(5) The points given by the families $(\mathcal{LR}^1_\alpha)$ and $(\mathcal{LR}^2_\alpha)$ are on the pencil (C, P). We have $\mathcal{LR}_{-\infty} = (m : l : 0)$ and $\mathcal{LR}_{+\infty} = (0 : n : m)$, so $\mathcal{LR}_{-\infty}$ is on the pencil $c = 0$ (i.e., $\widehat{L}_1$) and $\mathcal{LR}_{+\infty}$ is on the pencil $a = 0$ (i.e., $\widehat{L}_2$). In fact, $\mathcal{LR}_{+\infty} = C + 2mP$ and $\mathcal{LR}_{-\infty} = C - 2mP$. Therefore, $\mathcal{LR}_{+\infty}$ and $\mathcal{LR}_{-\infty}$ separate C and P. Consequently $\mathcal{LR}_{+\infty}$ and $\mathcal{LR}_{-\infty}$ form an obstruction for linking C and P by the families of reflected curve congruences. □

We consider the BDE $(\overline{L})$ whose solutions at each point $p \in M$ are the $\mathbb{A}$-conjugate directions of the two lightlike directions in T_pM. Then,

$$(\overline{L}) : \ (mdx + ndy)(ldx + mdy) = 0.$$

A short calculation shows that $\overline{L}$ is on the polar line $\widehat{P}$.

We denote by $w_1 = dy$, $w_2 = dx$, $\overline{w}_1 = ldx + mdy$, and $\overline{w}_2 = mdx + ndy$ (so $w_1 = 0$ and $w_2 = 0$ determine the lightlike directions and $\overline{w}_1 = 0$ and $\overline{w}_2 = 0$ their respective $\mathbb{A}$-conjugate directions). Observe that $L_i = w_i^2$, $i = 1, 2$. In $\mathbb{R}P^2$, the tangent lines to the conic Γ through the points L and $\overline{L}$ determine the four points w_1^2, w_2^2, $\overline{w}_1{}^2$, $\overline{w}_2{}^2$ on Γ. These four points determine in turn the following polar lines (see Figure 2):

$\widehat{L}$ the line through w_1^2 and w_2^2,

$\widehat{\overline{L}}$ the line through $\overline{w}_1^2$ and $\overline{w}_2^2$,

$\widehat{L}_1$ the line tangent to Γ at $w_1{}^2$,

$\widehat{\overline{L}}_1$ the line tangent to Γ at $\overline{w}_1^2$,

$\widehat{L}_2$ the line tangent to Γ at $w_2{}^2$,

$\widehat{\overline{L}}_2$ the line tangent to Γ at $\overline{w}_2^2$,

$\widehat{\mathcal{LC}}_{+\infty}$ the line through w_1^2 and $\overline{w}_2^2$,

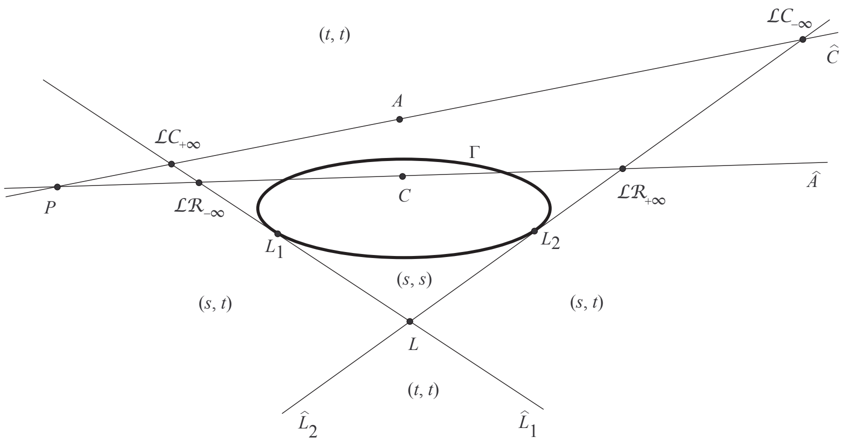

Figure 1: The lightlike pencils $\widehat{L}_1$ and $\widehat{L}_2$ obstructing the linking of the pairs (A, P) and (C, P) by the families of curve congruences. The figure also shows the type of the two solutions of a BDE (s for spacelike, t for timelike) in each region of the projective plane delimited by $\widehat{L}_1$, $\widehat{L}_2$ and the conic Γ.

$\widehat{\mathcal{LC}}_{-\infty}$ the line through $\overline{w}_1^2$ and w_2^2,
$\widehat{\mathcal{LR}}_{+\infty}$ the line through w_2^2 and $\overline{w}_2^2$,
$\widehat{\mathcal{LR}}_{-\infty}$ the line through w_1^2 and $\overline{w}_1^2$.

The following result follows from the fact that $\widehat{L} \cap \widehat{\overline{L}} = P$, $\widehat{\mathcal{LR}}_{\infty} \cap \widehat{\mathcal{LR}}_{-\infty} = A$ and $\widehat{\mathcal{LC}}_{\infty} \cap \widehat{\mathcal{LC}}_{-\infty} = C$.

Theorem 5.3 *The BDEs (A), (C), (P), $(\mathcal{LC}_{\pm\infty})$ and $(\mathcal{LR}_{\pm\infty})$ are completely determined by the BDEs (L) and $(\overline{L})$ (Figure 2).*

Remark 5.4 Note that (L) represents the metric g and $(\overline{L})$ determines the shape operator $\mathbb{A}$. There are other pairs that determine the remaining BDEs listed in Theorem 5.3. For example, (L) and (A), and (P) and $(\mathcal{LR}_{+\infty})$ will do. However, not all pairs determine the remaining BDEs. For instance, (P) and (C) do not determine the metric (L).

We observed in Remark 2.1 that the BDE (P) of the lines of $\mathbb{A}$-principal curves have orthogonal solutions (when they exist). In fact, P is the unique point on the polar line of A (or C) with this property, and in particular, A determines both P and C (see [15] when the metric g is Riemannian and [10] when it is Lorentzian). We have the following general result.

Theorem 5.5 (1) *The BDEs with orthogonal solutions are those represented by points on the polar line $\widehat{L}$ and which are located outside the conic Γ.*

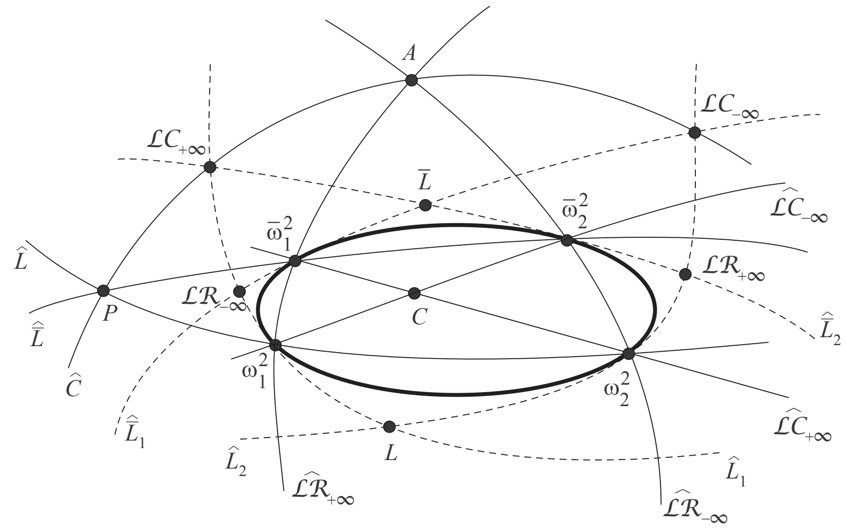

Figure 2: Quadrangle on Γ determining the polar lines.

(2) *If Q represents a BDE that determines two directions of the same type (both spacelike or timelike) or a BDE that has no solutions, then there is a unique BDE on $\widehat{Q}$ that has orthogonal solutions.*
(3) *If Q represents a BDE that determines two directions of different types, then there is no BDE on $\widehat{Q}$ that has orthogonal solutions.*

Proof (1) Let $(Q) : ady^2 + 2bdxdy + cdx^2 = 0$ be the BDE we are seeking. Suppose, without lost of generality, that $a \neq 0$ so the solutions of Q can be written in the form $\mathbf{r}_x + s_i\mathbf{r}_y$, $i = 1, 2$. These solutions are orthogonal if and only if

$$g(\mathbf{r}_x + s_1\mathbf{r}_y, \mathbf{r}_x + s_2\mathbf{r}_y) = (s_1 + s_2)F = 0$$

(we are taking a parametrisation with $E = G = 0$). We have $s_1 + s_2 = -2b/a$, so the condition for the solutions to be orthogonal is $b = 0$. This is equivalent to the condition for $Q = (a : 2b : c)$ to be on the polar line of $L = (0 : 1 : 0)$. For Q to have two solutions, it has to be outside the cone Γ.

(2) and (3) If $Q_1 = (a_1 : b_1 : c_1) \in \widehat{Q}$ represent a BDE with orthogonal solutions then $Q_1 = \widehat{Q} \cap \widehat{L}$. This implies that $b_1 = 0$. The results follow by observing that $sign(a_1c_1) = -sign(ac)$ and that $sign(ac)$ determines the regions in $\mathbb{R}P^2$ where the BDEs has solutions of the same or different types (Theorem 5.2(3)). □

Remarks 5.6 We still denote by (L) the BDE (2) on a Riemannian surface M. Then, given a self-adjoint operator on M, the statement in Theorem 5.5(1) is also true in this case, that is, $\widehat{L}$ coincides with the set of BDEs with orthogonal solutions. All points of $\widehat{L}$ lie outside Γ, so the polar line of any BDE contains a unique BDE with orthogonal solutions.

We return now to $\mathbb{A}$-conjugacy. Given a BDE (Q), we denote by $(\overline{Q})$ a BDE whose solutions are the $\mathbb{A}$-conjugate directions of the solutions of (Q). We call $(\overline{Q})$ the $\mathbb{A}$-conjugate of (Q) and seek to determine the position of $\overline{Q}$ in the projective plane given the position of Q.

Theorem 5.7 (1) *Let $\widehat{\overline{L}}_i$, $i = 1, 2$ be the pencils through $\overline{L}$ which are tangent to Γ. The pencil $\widehat{\overline{L}}_1$ (resp. $\widehat{\overline{L}}_2$) is the locus of the $\mathbb{A}$-conjugates of BDEs on $\widehat{L}_1$ (resp. $\widehat{L}_2$) and vice versa.*

(2) *The pencil $\widehat{\overline{L}}_1$ (resp. $\widehat{\overline{L}}_2$) intersects $\widehat{L}_1$ (resp. $\widehat{L}_2$) at $\mathcal{L}R_{-\infty}$ (resp. $\mathcal{L}R_{+\infty}$). The pencil $\widehat{\overline{L}}_2$ (resp. $\widehat{\overline{L}}_1$) intersects $\widehat{L}_1$ (resp. $\widehat{L}_2$) at $\mathcal{L}C_{+\infty}$ (resp. $\mathcal{L}C_{-\infty}$). Consequently, $\overline{\mathcal{L}C}_{+\infty} = \mathcal{L}C_{-\infty}$ and the points $\mathcal{L}R_{+\infty}$ and $\mathcal{L}R_{-\infty}$ are fixed under $\mathbb{A}$-conjugation.*

(3) *The pencils $\widehat{L}_i$, $\widehat{\overline{L}}_i$, $i = 1, 2$, and the conic Γ partition the projective plane into eleven connected regions as shown in Figure 3 (BDEs in region R11 have no solutions). Table 1 shows the position of $\overline{Q}$ given that of Q (and vice-versa) and Table 2 gives the type of the solutions of Q and $\overline{Q}$ in each region (s is for spacelike and t for timelike).*

Table 1: Position of $\overline{Q}$ given that of Q (and vice-versa) in Figure 3.

	Region					
Q	$R1$	$R2$	$R3$	$R4$	$R7$	$R8$
$\overline{Q}$	$R1$	$R6$	$R3$	$R5$	$R9$	$R10$

Table 2: The type of the solutions of Q and $\overline{Q}$ in each region of Figure 3.

	Region									
	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$	$R7$	$R8$	$R9$	$R10$
Q	(s,t)	(s,t)	(t,t)	(s,t)	(t,t)	(t,t)	(s,s)	(t,t)	(t,t)	(t,t)
$\overline{Q}$	(s,t)	(t,t)	(t,t)	(t,t)	(s,t)	(s,t)	(t,t)	(t,t)	(s,s)	(t,t)

Proof (1) This follows from the fact that the polar lines of $\overline{w}_1^2$ and $\overline{w}_2^2$ are, respectively, $\widehat{\overline{L}}_1$ and $\widehat{\overline{L}}_2$.

(2) This follows from the fact that points on the pencils $\widehat{\overline{L}}_1$, $\widehat{\overline{L}}_2$, $\widehat{L}_1$ and $\widehat{L}_2$ can be represent, respectively, in the form $\overline{w}_1\omega_1$, $\overline{w}_2\omega_2$, $w_1\omega_3$ and $w_2\omega_4$ where ω_i, $i = 1, \ldots, 4$ are any 1-forms and $w_i^2 = L_i$, $i = 1, 2$.

(3) The type of the solutions changes when crossing one of the lines $\widehat{L}_i$, $\widehat{\overline{L}}_i$, $i = 1, 2$ or Γ. The result follows by determining the type of the solutions of one BDE in each region (and similarly for determining the position of $\overline{Q}$ given that of Q). □

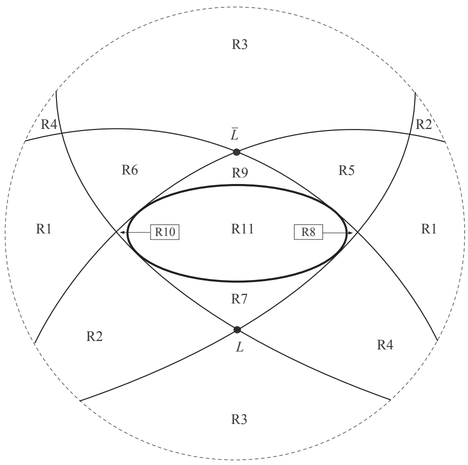

Figure 3: Partition of the projective plane $\mathbb{R}P^2 = S^2/\{p, -p\}$: the figure shows one hemisphere of the sphere S^2, the dotted boundary is the equator.

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ICMC-USP, Dept. de Matemática, Av. do Trabalhador São-Carlense, 400 Centro, Caixa Postal 668, CEP 13560-970, São Carlos (SP), Brazil
E-mail: anaclana@icmc.usp.br

Department of Mathematical Sciences, Durham University, Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
E-mail: farid.tari@durham.ac.uk