Families of surfaces and conjugate curve congruences

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Abstract

Given a smooth and oriented surface M in the Euclidean space $\mathbb{R}^3$, the conjugate curve congruence $\mathcal{C}_\alpha$ is a family of pairs of foliations on M that links the lines of curvature and the asymptotic curves of M. This family is first introduced in [21] and is studied in [8, 13]. When the surface $M = M_0$ is deformed in a 1-parameter family of surfaces M_t, we obtain a 2-parameter family of conjugate curve congruence $\mathcal{C}_{\alpha,t}$. We study in this paper the generic local singularities in $\mathcal{C}_{\alpha_0,0}$ and the way they bifurcate in the family $\mathcal{C}_{\alpha,t}$, with (α, t) close to $(\alpha_0, 0)$.

1 Introduction

Changes in the geometry of a smooth surface in the Euclidean space $\mathbb{R}^3$ as the surface is deformed in 1-parameter families of surfaces are investigated in several papers; see for example [2, 10, 11, 12, 37]. In [12] is given a catalogue of the generic changes in the local and multi-local geometry of the surface that are governed by its contact with planes. This includes, for instances, the listing of the generic bifurcations in the parabolic curve as the surface is deformed. The changes are obtained by studying the singularities in the family of height functions on the deformed surface. A similar study is carried out in [10] for listing the generic changes in the local geometry of the surface that are governed by its contact with spheres. These are obtained by studying the singularities in the family of distance squared functions on the deformed surface. In [37] is given a catalogue of the changes in the flecnodal curve, which is the locus

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of geodesic inflections of the asymptotic curves, as the surface is deformed in generic 1-parameter families of surfaces.

We study in this paper the local changes in the configurations of a certain 1-parameter family of pairs of foliations on the surface, when the surface is deformed in a 1-parameter family of surfaces. There are three classical pairs of foliations defined on an oriented surface M in the Euclidean space $\mathbb{R}^3$. These are the lines of curvature defined away from umbilic points, the asymptotic curves defined in the hyperbolic region and the characteristic curves defined in the elliptic region of M. The three pairs of foliations are given, in a local chart $\mathbf{r} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, by binary differential equations (BDEs), also know as quadratic differential equations. These are equations in the form

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0. \tag{1}$$

where a, b, c are smooth functions in $(x,y) \in U$, U being an open set in $\mathbb{R}^2$.

The above three pairs of foliations were studied as separate objects for a long time. However, in [21] is constructed a natural 1-parameter family of BDEs, called *conjugate curve congruence* and denoted by $\mathcal{C}_\alpha$, which links the equation of the asymptotic curves and that of the lines of curvature of M. This link is explained in [13] as follows. A BDE can be viewed as a quadratic form and represented at each point in the plane by a point in the projective plane. If Γ denotes the set of degenerate quadratic forms, then the asymptotic, characteristic and principal BDEs represent a self-polar triangle with respect to Γ ([13]). In particular, any two of them determine the third one. (This idea is generalised in [32] for two dimensional surfaces in $\mathbb{R}^n$.) The construction in [13] also allows to generate other natural families of BDEs on the surface.

The local configurations of the pair of foliation determined by $\mathcal{C}_\alpha$ on a fixed surface M for a fixed value of α and the way they bifurcate when α varies are given in [8, 9]. When the surface is deformed in a 1-parameter family of surfaces M_t, we obtain a 2-parameter family of BDEs $\mathcal{C}_{\alpha,t}$. The aim of this paper is to give a catalogue of the generic local bifurcations in $\mathcal{C}_{\alpha,t}$. We recall that the local codimension 2 singularities of BDEs and their generic bifurcations are studied in [31, 35]. It is worth observing here that the family $\mathcal{C}_{\alpha,t}$ is special, so there is no guaranty that it will be a generic family at a codimension 2 singularity of one of its members. Indeed, we discover two interesting phenomena in this paper. In one case, the family $\mathcal{C}_{\alpha,t}$ is generic at a given codimension 2 singularity but this singularity is not generic in 1-parameter families of surfaces (Remark 3.3). In the other case, the singularity occurs in generic 1-parameter families of surfaces but the resulting 2-parameter family of BDEs $\mathcal{C}_{\alpha,t}$ is not generic (Theorem 4.2).

There is another family of BDEs, called the *reflected conjugate curve congruence* $\mathcal{R}_\alpha$, which links the equation of lines of curvature and that of the characteristic curves ([13]). The local bifurcations in the family $\mathcal{R}_{\alpha,t}$ will be dealt with elsewhere.

The paper is organised as follows. §2 is devoted to some preliminary results and notions that are used in the paper. In §3 we study the bifurcations in $\mathcal{C}_{\alpha,t}$ at parabolic points and in §4 those that occur at umbilic points. We deal with the bifurcations away from parabolic and umbilic points in §5. To make the paper self-contained, we give in the appendix (§6) a brief summary of the results on codimension ≤ 1 singularities in BDEs. A reader who is not familiar with work on implicit differential equations could have a look first at the appendix.

2 Preliminaries

In all the paper, M denotes a smooth oriented surface in $\mathbb{R}^3$. Given a local parametrisation $\mathbf{r} : U \to \mathbb{R}^3$ of the surface, the coefficients of the first fundamental form $\mathrm{I}_p : T_pM \times T_pM \to \mathbb{R}$, with $\mathrm{I}_p(u, v) = u.v$, are given by

$$E = \mathbf{r}_x.\mathbf{r}_x, \;\; F = \mathbf{r}_x.\mathbf{r}_y, \;\; G = \mathbf{r}_y.\mathbf{r}_y,$$

where "." denotes the scalar product in $\mathbb{R}^3$ and subscripts (in all the paper) denote partial derivatives. If S^2 denotes the unit sphere in $\mathbb{R}^3$, then the Gauss map is defined by $N : \mathbf{r}(U) \to S^2$, where $N(p) = (\mathbf{r}_x \times \mathbf{r}_y/||\mathbf{r}_x \times \mathbf{r}_y||)(p)$, is a unit normal vector to the surface. The differential of the Gauss map at p is an automorphism of T_pM. The shape operator S_p, is the self-adjoint map $S_p = -d_pN : T_pM \to T_pM$. The determinant $K(p)$ of S_p is the Gaussian curvature of the surface at p. Points where $K > / = / < 0$ are called elliptic/parabolic/hyperbolic points respectively.

The second fundamental form $\mathrm{II}_p : T_pM \times T_pM \to \mathbb{R}$, is given by $\mathrm{II}_p(u, v) = S_p(u).v$. Its coefficients, with respect to the parametrisation $\mathbf{r}$, are given by

$$l = S(\mathbf{r}_x).\mathbf{r}_x = N.\mathbf{r}_{xx}, \;\; m = S(\mathbf{r}_x).\mathbf{r}_y = N.\mathbf{r}_{xy}, \;\; n = S(\mathbf{r}_y).\mathbf{r}_y = N.\mathbf{r}_{yy}.$$

The eigenvectors of the shape operator are called the *principal directions*, and their associated eigenvalues are the *principal curvature*. Umbilic points are those points where the principal curvature coincide. Two directions $u, v \in T_pM$ are conjugate if $\mathrm{II}_p(u, v) = 0$. A direction $u \in T_pM$ is *asymptotic* if it is self-conjugate, that is $\mathrm{II}_p(u, u) = 0$.

The principal and asymptotic directions define two pairs of foliations on M. These are the lines of curvature and the asymptotic curves. A line of curvature is a curve whose tangent line at each point is along a principal direction. The lines of curvature form an orthogonal net away from umbilic points. Their configurations at umbilics are as in Figure 13 top (see [30, 7]). The study of the behaviour of the lines of curvature in a neighbourhood of a closed orbit and of their structural stability is also carried out in [30].

An asymptotic curve of M is a curve whose tangent line at each point is along an asymptotic direction. The asymptotic curves are defined in the hyperbolic region of the surface. They form a family of cusps at generic parabolic points. Their configurations at a cusp of Gauss (see §3 for definition) are given for example in [3, 4, 26]; for a more general approach for studying the singularities of their equation at such points see [18]. Global properties of this pair of foliations including the study of the close orbits are given in [22].

The above two pairs of foliations are given, in a local chart, by BDEs and were previously considered as separate objects. However in [21] (see also [8, 13]) is constructed a natural family of BDEs linking the two pairs. Consider the projective space PT_pM of all tangent directions through a point p of M which is neither an umbilic nor a parabolic point. Conjugation gives an involution on PT_pM, $v \mapsto \overline{v} = C(v)$. The involution C is used in [21, 8, 13] to determine a family of BDEs by asking that the angle between a direction v and the image of v under C is constant.

Definition 2.1 ([21]) *Let PTM denote the projective tangent bundle to M, and define $\Theta : PTM \to [-\pi/2, \pi/2]$ by $\Theta(p, v) = \alpha$ where α denotes the oriented angle between a direction v and the corresponding conjugate direction $\overline{v} = C(v)$. Note that Θ is not well defined at points corresponding to asymptotic directions at parabolic points. The* conjugate curve congruence *$(\mathcal{C}_\alpha)$, for a fixed α, is defined to be $\Theta^{-1}(\alpha)$.*

Proposition 2.2 ([21]) *The conjugate curve congruence $\mathcal{C}_\alpha$ of a parametrised surface is given by the BDE*

$$\begin{gathered}(\sin\alpha(Gm - Fn) - n\cos\alpha\sqrt{EG - F^2})dy^2 + \\ (\sin\alpha(Gl - En) - 2m\cos\alpha\sqrt{EG - F^2})dydx + \\ (\sin\alpha(Fl - Em) - l\cos\alpha\sqrt{EG - F^2})dx^2 = 0.\end{gathered}$$

Remark 2.3 1. BDEs (1) determine a pair of transverse foliations away from the discriminant $\Delta = \{\,(x, y)\,|\,\delta(x, y) = (b^2 - ac)(x, y) = 0\,\}$.

2. Observe that $\mathcal{C}_0$ is the BDE of the asymptotic curves and $\mathcal{C}_{\pm\pi/2}$ is that of the lines of curvature.

3. The discriminant Δ_α of $\mathcal{C}_\alpha$ satisfies $\Delta_\alpha = \Delta_{-\alpha}$ and foliates the elliptic region of the surface as α varies in $[-\pi/2, \pi/2]$.

The family $\mathcal{C}_\alpha$ is studied in [8, 9, 13]. The bifurcations of the local singularities in the members of the family $\mathcal{C}_\alpha$ when α varies in $[-\pi/2, \pi/2]$ are given in [9].

In this paper, the surface $M = M_0$ is deformed in a 1-parameter family of surface M_t. So we obtain a 2-parameter family of BDEs $\mathcal{C}_{\alpha,t}$. We study the local bifurcations in the members of this family by following the work in [31, 35]. We denote a BDE (1) by $w = (a, b, c)$. As our study is local, we consider germs of BDEs at a given point, which we can assume to be the origin. So a, b, c are considered as germs of functions at the origin. We require the following notions.

We adopt the notion of fibre topological equivalence for families of BDEs. Two germs of families of BDEs $\tilde{\omega}$ and $\tilde{\tau}$, depending smoothly on the parameters u and

v respectively, are said to be *locally fibre topologically equivalent* if, for any of their representatives, there exist neighbourhoods U and W of 0 in respectively the phase space (x, y) and the parameter space u, and a family of homeomorphisms k depending on $u \in W$, all defined on U such that k^u is a topological equivalence between $\tilde{\omega}^u$ and $\tilde{\tau}^{\psi(u)}$, where ψ is a homeomorphism defined on W. (In all the paper, superscripts denote the function/map with the superscript variable fixed. We exclude from our study the semi-local/global phenomena in $\tilde{\omega}^0$ but these can of course appear in $\tilde{\omega}^u$, for $u \neq 0$.)

We associate to a germ of an r-parameter family of BDEs $\tilde{\omega} = (\tilde{a}, \tilde{b}, \tilde{c})$ the jet-extension map

$$\begin{array}{rccc} \Phi: & \mathbb{R}^2 \times \mathbb{R}^r, (0,0) & \to & J^k(2,3) \\ & ((x,y),u) & \mapsto & j^k(\tilde{a}, \tilde{b}, \tilde{c})^u|_{(x,y)} \end{array}$$

where $J^k(2,3)$ denotes the vector space of polynomial maps of degree k from $\mathbb{R}^2$ to $\mathbb{R}^3$, and $j^k(\tilde{a}, \tilde{b}, \tilde{c})^u|_{(x,y)}$ is the k-jet of $(\tilde{a}, \tilde{b}, \tilde{c})$ at (x, y) with u fixed. This is simply the Taylor expansion of order k of $(\tilde{a}, \tilde{b}, \tilde{c})^u$ at (x, y). A singularity in the family is of *codimension* m if the conditions that define it yield a semi-algebraic set V of codimension $m + 2$ in $J^k(2, 3)$, for any k greater than some k_0. The set V is supposed to be invariant under the natural action of the $k + 1$-jets of diffeomorphisms in (x, y) and multiplication by non-zero functions in (x, y).

The family $\tilde{\omega}$ is said to be *generic* if the map Φ is transverse to V in $J^k(2, 3)$. Observe that a necessary condition for genericity is that $r \geq m$. It follows from Thom's Transversality Theorem that the set of generic families is residual in the set of smooth map germs $\mathbb{R}^2 \times \mathbb{R}^r, 0 \to \mathbb{R}^3, 0$.

Our interest in the paper is mainly when the codimension $m = 2$ (there is however one case in §4 where the codimension is higher). The *bifurcation set* of a generic 2-parameter family consists of the set of parameters u where the associated BDE has a singularity of codimension ≥ 1. (See §6 for a list of the generic local codimension ≤ 1 singularities in BDEs.) The components of the bifurcation set corresponding to local codimension 1 singularities are as follows.

($MT1$) *Morse Type 1 singularity*: this is the set of parameters u for which the discriminant of $\tilde{w}^u$ has a Morse singularity and the coefficients of $\tilde{w}^u$ do not vanish simultaneously at the singularity.

(FSN) *Folded saddle-node bifurcations*: this is the set of parameters u for which the lifted field ξ^u associated to $\tilde{w}^u$ has a saddle-node singularity.

(FNF) *Folded node-focus change*: this is the set of parameters u for which the lifted field ξ^u associated to $\tilde{w}^u$ has equal eigenvalues at the singular point.

($MT2$) *Morse Type 2 singularity*: this is the set of parameters u for which the discriminant of $\tilde{w}^u$ has a Morse singularity and the coefficients of $\tilde{w}^u$ all vanish at the singularity.

We also need to consider some semi-local singularities that emerge when the local codimension 2 singularity is deformed. The stratum of intersest here is the following.

> (FSC) *Folded saddle connection*: this is the set of parameters u for which the lifted field ξ^u has a saddle connection.

We obtain a stratification $\mathcal{S}$ of a neighbourhood U of the origin in the parameter space $\mathbb{R}^2$, given by the origin, the above strata and the complement of the union of these sets. In [31, 35] is given a strategy for obtaining models of generic families of BDEs at a codimension 2 singularities. It consists in showing that two generic families have homeomorphic bifurcation sets and that the configurations in each stratum of $\mathcal{S}$ is constant.

For generic surfaces, the family $\mathcal{C}_\alpha$ is a generic family at the codimension ≤ 1 singularities of its members ([8, 9]). We consider now 1-parameter families of surfaces M_t, with $M_0 = M$, and t close to zero. We take $\alpha_0 \in [-\pi/2, \pi/2]$ and deal with the bifurcations in the family $\mathcal{C}_{\alpha,t}$ as (α, t) varies near $(\alpha_0, 0)$. In all the cases but one, the singularities in the members of $\mathcal{C}_{\alpha,t}$ are of codimension ≤ 2, so we need to show that the family $\mathcal{C}_{\alpha,t}$ is generic and apply the results in [31, 35]. The remaining case however is not covered in [31] as the singularity is of codimension > 2. So we follow the strategy in [31] to deal with it.

In all the paper one foliation is drawn in blue and the other in red. The discriminant is drawn in thick black and the singularities are represented by thick dots. The figures in this paper are also checked using a computer programme written by A. Montesinos ([28]).

We shall use the following notation in the paper. Given a germ of a family of BDEs $\tilde{\omega} = (\tilde{a}, \tilde{b}, \tilde{c})$, we write

$$\begin{aligned}
\tilde{a} &= a_0(\alpha,t) + a_1(\alpha,t)x + a_2(\alpha,t)y + a_{20}(\alpha,t)x^2 + a_{21}(\alpha,t)xy + a_{22}(\alpha,t)y^2 + \ldots,\\
\tilde{b} &= b_0(\alpha,t) + b_1(\alpha,t)x + b_2(\alpha,t)y + b_{20}(\alpha,t)x^2 + b_{21}(\alpha,t)xy + b_{22}(\alpha,t)y^2 + \ldots,\\
\tilde{c} &= c_0(\alpha,t) + c_1(\alpha,t)x + c_2(\alpha,t)y + c_{20}(\alpha,t)x^2 + c_{21}(\alpha,t)xy + c_{22}(\alpha,t)y^2 + \ldots.
\end{aligned}$$

We take the family of surfaces M_t, depending smoothly on the parameter t, in Monge form $z = h(x, y, t)$ in a neighbourhood of the origin in $\mathbb{R}^2$ and with t close to zero. We assume that the tangent space T_0M_0 is the (x, y)-plane and write

$$h(x,y,t) = h_0(t) + h_1(t)x + h_2(t)y + \frac{1}{2}\sum_{i=0}^{2} \tbinom{i}{2}\, h_{2i}(t)x^{2-i}y^i + \frac{1}{6}\sum_{i=0}^{3} \tbinom{i}{3}\, h_{3i}(t)x^{3-i}y^i + \ldots,$$

where the coefficients h_{ij} are germs, at the origin, of smooth functions. To simplify the notation, we write $h_{ij}(0) = h_{ij}$, $a_{ij}(0,0) = a_{ij}$, $b_{ij}(0,0) = b_{ij}$ and $c_{ij}(0) = c_{ij}$. It follows from the the assumptions above, that $h_0(0) = h_1(0) = h_2(0) = 0$.

We split our study into three cases, depending on whether the singularity $p \in M_0$ is a parabolic point, an umbilic point or neither of these two. We refer to [1, 5] for the singularity theory concepts used here.

3 Bifurcations at parabolic points

The parabolic set is the discriminant of the asymptotic BDE $\mathcal{C}_0$ and the zeros of this BDE are the cusps of Gauss (see §6 for definition of a zero of a BDE). A cusp of Gauss is a point on the parabolic set where the Gauss map is equivalent, by changes of coordinates in the source and target, to the cusp map $(x, xy + y^3)$. Other geometric characterisations of the cusp of Gauss are given in [3], see also [38] for some new invariants associated to these points.

The cusp of Gauss can also be picked up by the singularities of the height functions on $M = M_0$. Recall that the family of height functions on M is given by

$$\begin{array}{rcl} H : M \times S^2 & \to & \mathbb{R} \\ (p, u) & \mapsto & p.u \end{array}$$

For a fixed direction $u \in S^2$ and a point $p \in M$, the height function H^u can be viewed as a germ of a function $\mathbb{R}^2, 0 \to \mathbb{R}$. One can make any changes of coordinates in the source and this leads to the action of the right group $\mathcal{R}$ on the set of germs of functions $\mathbb{R}^2, 0 \to \mathbb{R}$. When u is along the normal direction of M at p, then the height function H^u has generically a singularity of type A_2 if p is a parabolic point and of type A_3 if it is a cusp of Gauss (A_k-singularities are those that are $\mathcal{R}$-equivalent to $\pm y^2 \pm x^{k+1}$, [1]).

Using the Monge form setting for $t = 0$ and the notation in §2, if the origin is a parabolic point, then we can take without loss of generality $j^2 h = h_{22} y^2$. The origin is then a cusp of Gauss if and only if $h_{30} = 0$ and $h_{31}^2 - 4h_{22}h_{40} \neq 0$. In generic 1-parameter families of surfaces, we expect the following local singularities to occur at isolated points on the parabolic set (see [12]):

(i) An A_4-singularity on a smooth parabolic set. This occurs when $h_{30} = h_{31}^2 - 4h_{22}h_{40} = 0$ and $h_{22}(4h_{50}h_{22}^2 + h_{32}h_{31}^2 - 2h_{41}h_{31}h_{22}) \neq 0$.

(ii) A non-versal A_3-singularity, which occurs when the parabolic set is singular (generically a Morse singularity). That is, when $h_{30} = h_{31} = 0$ and $h_{22}h_{40} \neq 0$.

(iii) A flat umbilic point, which occurs when the height function has a $D_4^{\pm}$ singularity (modeled by $x^3 \pm xy^2$). So we have $h_{20} = h_{21} = h_{22} = 0$ and $j^3 h^0$ is not a degenerate cubic form.

In (i) the lifted field ξ has a saddle-node singularity (see §6 for definitions). In (ii) the asymptotic curves BDE has a Morse Type 1 singularity and in (iii) it has a Morse Type 2 singularity. So as far as BDEs are concerned the phenomena (i), (ii) and (iii) are all of codimension 1. The case (i) is dealt with in [9] were it is shown there that the family $\mathcal{C}_\alpha$ is in general a generic family. Therefore the family $\mathcal{C}_{\alpha,t}$ is also generic and is trivial along the parameter t. The bifurcations in $\mathcal{C}_\alpha$ as α varies near $\alpha_0 = 0$ are given in Figure 10.

We deal now with cases (ii) and (iii). We start with the case (ii).

Theorem 3.1 *The family $\mathcal{C}_\alpha$ is not generic at a non-versal A_3-singularity of the height function. A generic 1-parameter family of surfaces M_t induces a generic family $\mathcal{C}_{\alpha,t}$ of the Morse Type 1 singularity of the asymptotic BDE $\mathcal{C}_{0,0}$. The bifurcations in $\mathcal{C}_{\alpha,t}$, for α fixed, are as in Figure 12.*

Proof According to [9], a family of BDEs is generic at a Morse Type 1 singularity if and only if the associated family of discriminant functions is an $\mathcal{R}$-versal deformation of the Morse singularity of the discriminant. The condition for a family of functions $g(x, y, s)$ depending smoothly on a parameter s to be an $\mathcal{R}$-versal deformation of a Morse singularity at $p_0 = (x_0, y_0, 0)$ is $g_s(p_0) \neq 0$.

At a non-versal A_3, and using the Monge form setting, we can take $p_0 = (0, 0, 0)$ and $j^2h = h_{22}y^2 + h_{32}xy^2 + h_{33}y^3 + \text{h.o.t.}$. The 2-jet of the discriminant (up to a scalar multiple) is then given by

$$j^2\delta(x, y, \alpha) = -24h_{22}h_{40}x^2 - 12h_{22}h_{41}xy + 4(h_{32}^2 - h_{22}h_{42})y^2 + h_{22}^2\alpha^2.$$

Therefore $\delta_\alpha(p_0) = 0$, so $\mathcal{C}_\alpha$ is not generic. (A geometric explanation for this is the following. As $\Delta_\alpha = \Delta_{-\alpha}$, we do not get the full Morse transitions by varying α near zero.) For generic 1-parameter families of surfaces M_t, the parabolic set undergoes Morse transitions [12]. That is, the family $\delta(x, y, \alpha, t)$ is a versal deformation of the Morse singularity. So the induced family $\mathcal{C}_{\alpha,t}$ is generic.

The bifurcation set in the (α, t)-space consists of the Morse Type 1 stratum (MT1). This is given by the set of parameters (α, t) such that $\delta = \delta_x = \delta_y = 0$. By the implicit functions theorem, the MT1-stratum is a smooth curve parametrised by α, say $t = k(\alpha)$, for some germ of a smooth function k. One can show that $k'(0) = 0$ as $\delta_\alpha(0) = 0$. In general $k''(0) \neq 0$ as $\delta_{\alpha\alpha}(0) \neq 0$. Therefore the MT1-stratum has an ordinary tangency at the origin with the α-axis. Recall that the Morse Type 1 singularity of $\mathcal{C}_{0,0}$ is of codimension 1. The MT1 singularity of $\mathcal{C}_{\alpha,t}$, for $\alpha \neq 0$, is no longer a parabolic point. The bifurcations for any two fixed values of α are equivalent and are as in Figure 12. There are several cases depending on the type of singularity of the parabolic set (A_1^+ or A_1^-) and on the type of the zeros that appear in the bifurcations (folded saddles or folded foci). □

We consider now the bifurcations at a flat umbilic (case (iii) above). If we write the surface $M = M_0$ in Monge form $z = h(x, y, 0)$, then at a flat umbilic $h(x, y, 0) = C(x, y) + \text{h.o.t}$, where $C(x, y)$ is a cubic in (x, y). There are two generic types of flat umbilic points, the elliptic umbilic (D_4^-) where C has three real roots and the hyperbolic umbilic (D_4^+) where it has one real root. We can make changes of coordinates in the source and set $C(x,y)=x^2y\pm y^3$. We consider the family $\mathcal{C}_\alpha$ with $[0,\pi/2]$ (the case $[-\pi/2,0]$ is identical). The coefficients of the second fundamental form all vanish at the origin, so the coefficients of $\mathcal{C}_\alpha$, for all $\alpha\in[0,\pi/2]$, also vanish at the origin. If (a,b,c) denotes the coefficients of $\mathcal{C}_\alpha$, then

$$\begin{array}{rcl} j^1a &=& 2\sin(\alpha)\,x\mp 6\cos(\alpha)\,y,\\ j^1b &=& -4\cos(\alpha)\,x+(2\mp 6)\sin(\alpha)\,y,\\ j^1c &=& -2\sin(\alpha)\,x-2\cos(\alpha)\,y. \end{array}$$

At an elliptic umbilic the 2-jet of the discriminant of $\mathcal{C}_\alpha$ (up to a scalar multiple) is given by

$$j^2\delta^\alpha(x,y)=x^2+(4-\cos^2(\alpha))y^2.$$

It is clear that $\delta^\alpha(x,y)$ has a Morse A_1^+ singularity at the origin for all $\alpha\in[0,\pi/2]$. So $\mathcal{C}_\alpha$ has a Morse Type 2 singularity (see §6). To determine the configuration of $\mathcal{C}_\alpha$ we need to determine the number and type of the zeros of ξ^α on the exceptional fibre. We have

$$\begin{array}{rcl} \phi(p) &=& 6\cos(\alpha)p^3+10\sin(\alpha)p^2-6\cos(\alpha)p-2\sin(\alpha),\\ \alpha_1(p) &=& 6\cos(\alpha)p^2+6\sin(\alpha)p-2\cos(\alpha). \end{array}$$

When $\alpha\neq\pi/2$, the cubic ϕ has three distinct roots and $-\phi'(p)\alpha_1(p)$ is negative at these roots. So ξ has 3 saddle singularities for all values of α, with $\alpha\neq\pi/2$ (see §6). Therefore $\mathcal{C}_\alpha$ has a singularity of type MT2 A_1^+(3S) at the origin (Figure 13(3S), top). When $\alpha=\pi/2$ we need to consider the chart $q=dx/dy$ instead of $p=dy/dx$ (see §6), and one can show that $\mathcal{C}_{\pi/2}$ has also a singularity of type MT2 A_1^+(3S) at the origin.

At a hyperbolic umbilic the 2-jet of the discriminant of $\mathcal{C}_\alpha$ (up to a scalar multiple) is given by

$$j^2\delta^\alpha(x,y)=x^2-(4\cos^2(\alpha)-1)y^2.$$

The discriminant has a Morse singularity of type A_1^- if $\alpha<\pi/3$ and of type A_1^+ if $\alpha>\pi/3$. We have

$$\begin{array}{rcl} \phi(p) &=& -6\cos(\alpha)p^3-2\sin(\alpha)p^2-6\cos(\alpha)p-2\sin(\alpha),\\ \alpha_1(p) &=& -2\cos(\alpha)(p^2+1). \end{array}$$

Following the same arguments as above, one can show that $\mathcal{C}_\alpha$ has a singularity of type MT2 A_1^-(1S) at the origin (Figure 14(1S), top) when $\alpha<\pi/3$, and a singularity of type MT2 A_1^+(1S) (Figure 13(1S), top) when $\alpha>\pi/3$.

When $\alpha=\pi/3$, the 3-jet of the discriminant is given by

$$j^2\delta^{\frac{\pi}{3}}=x^2+3h_{41}x^3+\frac{9}{2}(h_{42}-2h_{40})x^2y+\frac{9}{2}(h_{43}-h_{41})xy^2+\frac{3}{2}(2h_{44}-h_{42})y^3.$$

This has an A_2-singularity (a cusp) at the origin provided $2h_{44}-h_{42}\neq 0$. Consider the discriminant surface

$$D=\{(x,y,\alpha)|(x,y)\in(\mathbb{R}^2,0),\,\alpha\in[0,\pi/2]\}$$

obtained by putting together the discriminants of the family $\mathcal{C}_\alpha$. The surface D has a Whitney umbrella singularity at $(0,0,\pi/3)$ and its intersection with the planes $\alpha =$constant yields generic sections of the Whitney umbrella.

We need to identify the type of the BDE $\mathcal{C}_{\pi/3}$. As its discriminant is a cusp, the topological type of the equation is determined by the singularities of the cubic ϕ ([31], Theorem 3.4). We have $\phi(p) = -3(p^2+1)(p+\frac{\sqrt{3}}{3})$, so it has one real root. It follows then by Theorem 3.4 in [31] that $\mathcal{C}_{\pi/3}$ is topologically equivalent to

$$ydy^2 + (x+y)dxdy + y^2dx^2 = 0,$$

see Figure 1 ①. A generic 2-parameter family of this equation is given by

$$ydy^2 + (x+y)dxdy + (y^2+uy+v)dx^2 = 0,$$

(Theorem 4.4 in [31]) and the bifurcations in the family are given in Figure 1.

We summarise below the above calculations and consider the local bifurcations in $\mathcal{C}_{\alpha,t}$ with α varying near a fixed value in $[0,\pi/2]$.

Theorem 3.2 (1) *At a flat umbilic of elliptic type, the members of the family $\mathcal{C}_\alpha$ have a Morse Type 2 A_1^+(3S) singularity. The family $\mathcal{C}_\alpha$ is not a generic family of such singularities. A generic 1-parameter family of surfaces M_t induces a generic family $\mathcal{C}_{\alpha,t}$. The bifurcations in $\mathcal{C}_{\alpha,t}$, for α fixed, are as in Figure 13(3S).*

(2) *At a flat umbilic of hyperbolic type, the members of the family $\mathcal{C}_\alpha$ have a Morse Type 2 A_1^-(1S) singularity if $\alpha<\pi/3$ and of type A_1^+(1S) if $\alpha>\pi/3$. The family $\mathcal{C}_\alpha$ is not a generic family. A generic 1-parameter family of surfaces M_t induces a generic family $\mathcal{C}_{\alpha,t}$. The bifurcations in $\mathcal{C}_{\alpha,t}$, for α fixed, are as in Figure 14(1S) for $\alpha<\pi/3$ and Figure 13(1S) for $\alpha>\pi/3$.*

The BDE $\mathcal{C}_{\pi/3}$ has a Cusp Type 2 singularity. A generic 1-parameter family of surfaces M_t induces a generic family $\mathcal{C}_{\alpha,t}$. The bifurcations in $\mathcal{C}_{\alpha,t}$, for α near $\pi/3$, are as in Figure 1.

Proof A family of BDEs with coefficients (a,b,c) depending on a parameter s and with a Morse Type 2 singularity at the origin is generic if and only if

$$\begin{vmatrix} a_x & a_y & a_s \\ b_x & b_y & b_s \\ c_x & c_y & c_s \end{vmatrix} \neq 0,$$

where the partial derivatives are evaluated at the origin ([15]). This is precisely the condition for the jet extension map to be transverse to the $MT2$-variety in $J^1(2,3)$. At a flat umbilic, $a_\alpha = b_\alpha = c_\alpha = 0$ at the origin for all $\alpha \in [0,\pi/2]$. So the family $\mathcal{C}_\alpha$ is not generic at the MT2 singularities. For generic 1-parameter families of surfaces (which yield a family of height functions that unfolds versally the $D_4^\pm$ singularity),

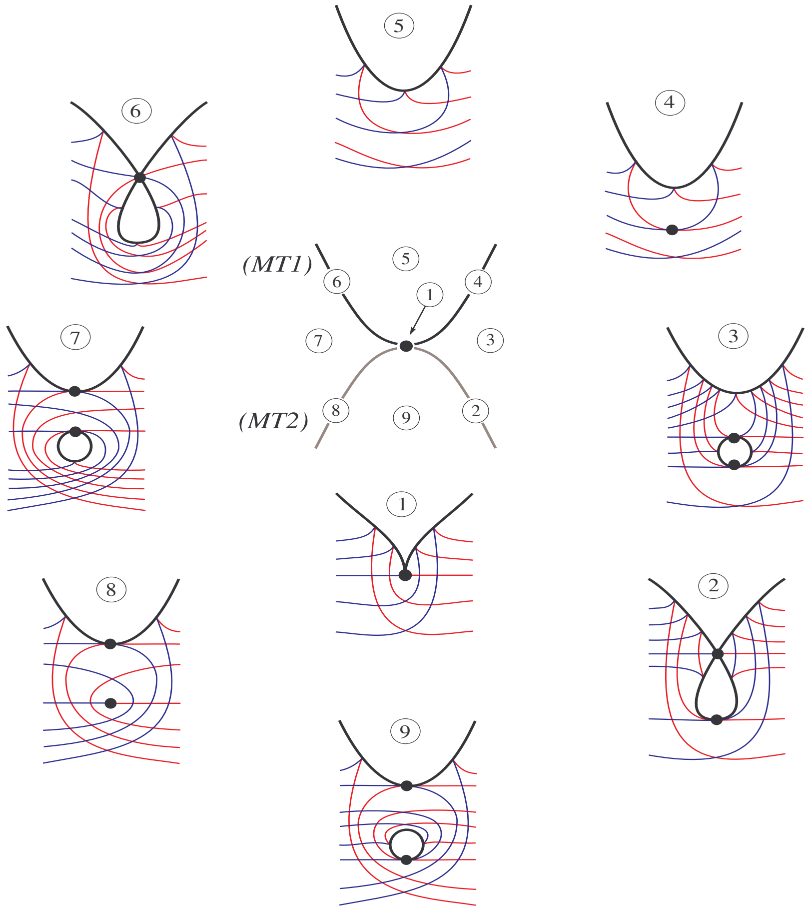


Figure 1: Bifurcations in $\mathcal{C}_{\alpha,t}$ at a hyperbolic flat umbilic with $\alpha_0 = \pi/3$.

the induced family $\mathcal{C}_{\alpha,t}$ becomes generic as the determinant above is non zero when differentiating with respect to t.

The Morse Type 2-stratum of the bifurcation set in the (α, t)-space is given by the set of parameters (α, t) such that $a = b = c = 0$ for some (x, y). This is just the curve $t = 0$. We can now use the results in [15] to draw the bifurcations in $\mathcal{C}_{\alpha,t}$ as (α, t) varies near $(\alpha_0, 0)$. We get a trivial family along the α-parameter, with α close to the initial value α_0. At an elliptic umbilic the bifurcations are as in Figure 13(3S) and at a hyperbolic umbilic (with $\alpha \neq \pi/3$) they are as in Figure 14(1S) for $\alpha < \pi/3$ and Figure 13(1S) for $\alpha > \pi/3$.

When $\alpha = \pi/3$ we need to show that a generic family of surfaces induces a generic family of equations $\mathcal{C}_{\alpha,t}$. We shall suppose that the family of surfaces M_t is given in Monge form $z = h(x, y, t)$, with $j^3h^0(x, y) = x^2y - y^3$. Let $\omega^{\pi/3} = (a, b, c)$ denote the coefficients of $\mathcal{C}_{\pi/3,0}$. We need to consider the jet extension map Φ in §2 and its transversality with $V \subset J^k(2,3)$ at $j^k\omega^{\pi/3}$, where V is the variety of the Cusp Type 2 singularities. If we identify in $J^k(2,3)$ polynomials with their coefficients, the Cusp Type 2 variety is given by

$$V = \{a_0 = b_0 = c_0 = 0, g = (2b_1b_2 - a_1c_2 - a_2c_1)^2 - 4(b_2^2 - a_2c_2)(b_1^2 - a_1c_1) = 0\}.$$

We can work in $J^2(2,3)$. The variety V is smooth at $j^2\omega^{\pi/3}$ and its tangent space at $j^2\omega^{\pi/3}$ is given by the intersection of the kernels of the 1-forms α_i, $i = 1,2,3,4$ with

$$\alpha_1 = \mathrm{d}a_0,\ \alpha_2 = \mathrm{d}b_0,\ \alpha_3 = \mathrm{d}c_0,\ \alpha_4 = dg,$$

The map Φ is transverse to V at $j^2\omega^{\pi/3}$ if and only if there is no non-zero vector $Z = \lambda_1\Phi_x + \lambda_2\Phi_y + \lambda_3\Phi_u + \lambda_4\Phi_v$ that belongs to the intersection of the kernels of the 1-forms α_i, $i = 1,2,3,4$. This gives a linear system $\alpha_i(Z) = 0$, $i = 1,2,3,4$ in $(\lambda_1, ..., \lambda_4)$, and the transversality condition is equivalent to the matrix of this linear system having a non zero determinant. A calculation shows that the determinant is non zero if and only if

$$(3h_{txx} - h_{tyy})(0,0,0) \neq 0.$$

Now a simple calculation shows that the above condition is exactly that for the (big) family of height functions to be a versal unfolding of the D_4^--singularity of h_0. It is satisfied for generic families of surfaces M_t. Hence, for generic families of surfaces, the induced family of equations $\mathcal{C}_{\alpha,t}$ is generic. It follows then from Theorem 4.3 in [31] that the bifurcations in $\mathcal{C}_{\alpha,t}$ when (α, t) varies near $(\pi/3, 0)$ are as Figure 1. □

Remark 3.3 Bifurcations at an A_5-singularity. An A_5-singularity of the height occurs generically at isolated points on M only when the surface is deformed in 2-parameter families of surfaces. In general, the parabolic set of M is smooth at this singularity. Also, one can show that at an A_5-singularity, the lifted field ξ has a degenerate elementary singularity (see Table 1 in §5) so the BDE of the asymptotic curves has a codimension 2 singularity. One can apply the results in [35] to show that for an open and dense set of 1-parameter families of surfaces M_t, the induced family $\mathcal{C}_{\alpha,t}$ is generic. This is an interesting phenomena as it shows that $\mathcal{C}_{\alpha,t}$ is generic at an A_5-singularity, but this singularity does not occur in generic 1-parameter families of surfaces.

When the parabolic set has a cusp singularity ($\mathcal{C}_0$ has a Cusp Type 1 singularity; Table 1 in §5)) or when it has a Morse singularity of type A_1^- with the unique asymptotic direction there tangent to one branch of the discriminant ($\mathcal{C}_0$ has a Non transverse Morse singularity; Table 1 in §5), then one can show that the family $\mathcal{C}_{\alpha,t}$ is not generic at such singularities.

4 Bifurcations at umbilics

The family of distance squared functions on M is given by

$$\begin{array}{rcl} d^2 : M \times \mathbb{R}^3 & \to & \mathbb{R} \\ (p, a) & \mapsto & ||p - a||^2 \end{array}$$

The function $d(-, a)$, with a fixed, has a degenerate singularity (worse than Morse) if and only if a is the centre of curvature at p, that is, a is a point on the focal set. At umbilic points the singularity of $d(-, a)$ is generically of type D_4.

On a generic surface, the only member of $\mathcal{C}_\alpha$ which is singular umbilic points is $\mathcal{C}_{\pm\pi/2}$ (equation of the lines of curvature). We shall deal with $\mathcal{C}_{\pi/2}$ as $\mathcal{C}_{-\pi/2-s} = \mathcal{C}_{\pi/2+s}$. We take the surface in Monge form $z = h(x, y, 0)$ and suppose that the origin is an umbilic point. Then we can write

$$j^3 h(x, y, 0) = \frac{\kappa}{2}(x^2 + y^2) + C(x, y),$$

where the cubic $C(x, y)$ is given by

$$C(x, y) = \Re(z^3 + \beta z^2 \bar{z}),$$

with $z = x + iy$ and $\beta = u + iv$. The configuration of the lines of curvature is determined by the values of β (see for example [29]). There are two exceptional curves in the β-plane, the circle $|\beta| = 3$ and the hypercycloid $\beta = 3(2e^{i\theta} - e^{-2i\theta})$ (Figure 2). Away from these curves, the configuration of the lines of curvature are stable and are as in Figure 13 top. These configuration were first drawn by Darboux and a rigorous proof is given in [30] (see also [7]) where they are labelled D_1 for Figure 13 top left, D_2 for Figure 13 top centre and D_3 for Figure 13 top right. They are also called, respectively, Lemon, Monstar and Star.

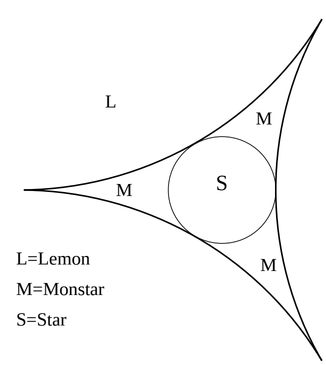


Figure 2: Partition of the β-plane.

We need to make an important observation here. Generic umbilics (i.e., with β away from the exceptional curves) are stable on immersed surfaces in $\mathbb{R}^3$. This means that if a surface M with a generic umbilic p_0 is deformed in a 1-parameter family of surfaces M_t with $M = M_0$, then the configurations of the lines of curvature of the surfaces M_t (for t near zero) are all topologically equivalent at their umbilic points that are the deformations of p_0. However, the equation $\mathcal{C}_{\pi/2}$ of the lines of curvature of M has a Morse Type 2 singularity at a generic umbilic, and this singularity is of codimension 1 in the set of all BDEs. In this case, the family $\mathcal{C}_\alpha$ is a generic family of the singularity of $\mathcal{C}_{\pi/2}$. Indeed, if we consider the Monge form setting above and denote by (a, b, c) the coefficients of $\mathcal{C}_\alpha$, then

$$\begin{vmatrix} a_x & a_y & a_\alpha \\ b_x & b_y & b_\alpha \\ c_x & c_y & c_\alpha \end{vmatrix} (0, 0, \frac{\pi}{2}) = -16\kappa(u^2 + v^2 - 9) \neq 0.$$

So by the result in [15], the family $\mathcal{C}_\alpha$ is a generic family and the bifurcations in $\mathcal{C}_\alpha$ as α varies near $\pi/2$ are then as in Figure 13.

On the special curves $\beta = 3(2e^{i\theta} - e^{-2i\theta})$ and $|\beta| = 3$, the following happens. When $\beta = 3(2e^{i\theta} - e^{-2i\theta})$, the discriminant of $\mathcal{C}_{\pi/2}$ has still a Morse singularity but the lifted field ξ has a saddle-node singularity on the exceptional fibre. This singularity is denoted by D_2^1 in [24] and by MT2 saddle-node in [31]. On $|\beta| = 3$, the discriminant has a degenerate singularity (of type A_3^+), the cubic $C(x, y)$ has two orthogonal roots, and the family of distance squared function is not a versal unfolding of the D_4-singularity. This singularity is denoted by $D_{2,3}^1$ in [24]. We expect the singularities D_2^1 and $D_{2,3}^1$ to occur at isolated points on generic 1-parameter families of surfaces. We start with the D_2^1 singularity.

Theorem 4.1 *Let M be a smooth surface with an umbilic point of type D_2^1 (i.e., $\beta = 3(2e^{i\theta} - e^{-2i\theta})$). Then a generic family M_t with $M = M_0$ induces a generic family $\mathcal{C}_{\alpha,t}$ of the singularity of $\mathcal{C}_{\pi/2}$. The bifurcations in $\mathcal{C}_{\alpha,t}$ are as in Figure 3.*

Proof As pointed out above, on $\beta = 3(2e^{i\theta} - e^{-2i\theta})$, the BDE $\mathcal{C}_{\pi/2}$ has generically an MT2 saddle-node singularity at the origin. (Generic here means excluding the cusp points of the hypercycloid and its points of tangency with the circle $|\beta| = 3$.) It follows by Theorem 1.1 in [31] that it is topologically equivalent to $ydy^2 + 2(x + y)dxdy - ydx^2 = 0$. Theorem 1.2 in [31] states that any generic deformation of this equation is topologically equivalent to $ydy^2 + 2(x + y)dxdy + (-y + ux + v)dx^2 = 0$.

We seek to show that a generic family of surfaces M_t induce a generic family $\mathcal{C}_{\alpha,t}$. We proceed as in the proof of Theorem 3.2. If we take the 2-jet of a general BDE as in §2, then MT2 saddle-node singularities determine a variety V in $J^k(2, 3)$ given by

$$V = \{a_0 = b_0 = c_0 = 0, \text{Resultant}(\phi, \phi', p) = 0\},$$

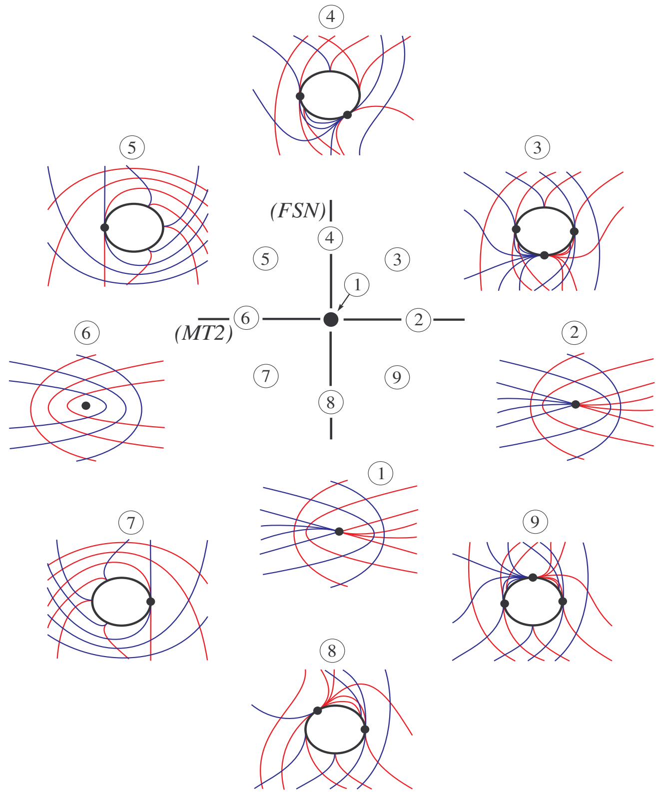


Figure 3: Bifurcations in $\mathcal{C}_{\alpha,t}$ at a D_2^1 umbilic.

where $\phi(p) = a_2p^3 + (2b_2 + a_1)p^2 + (2b_1 + c_2)p + c_1$ is the cubic determining the zeros of the lifted field on the exceptional fibre (see §6).

We shall suppose that the family of surfaces M_t is given in Monge form $z = h(x, y, t)$, with $h(x, y, 0) = \kappa/2(x^2+y^2)+\Re(z^3+\beta z^2\bar{z})+$h.o.t. Let ω denote the coefficients of $\mathcal{C}_{\pi/2}$. We need to consider the jet extension map Φ in §2 and its transversality with $V \subset J^k(2, 3)$ at $j^k\omega$. It is enought to work in $J^2(2, 3)$. The variety V is smooth at $j^2\omega$ and the map Φ is transverse to V at $j^2\omega$ if and only if there is no non-zero vector $\lambda_1\Phi_x + \lambda_2\Phi_y + \lambda_3\Phi_\alpha + \lambda_4\Phi_t$ that belongs to the intersection of the kernels of the 1-forms

$$\alpha_1 = \mathrm{d}a_0,\ \alpha_2 = \mathrm{d}b_0,\ \alpha_3 = \mathrm{d}c_0,\ \alpha_4 = dg,$$

where $g = \mathrm{Resultant}(\phi, \phi', p)$. This is a linear algebra calculation (carried out with Maple), and the required determinant is a polynomial of degree 9 in $\cos\theta$ and $\sin\theta$ with coefficients depending on the coefficients of the 4-jet of $h^t(x, y)$ and their derivatives with respect to t. We have transversality when this polynomial is not zero. This is the case, for generic families of surfaces, at all points on $\beta = 3(2e^{i\theta} - e^{-2i\theta})$ except for a finite number of them. (For example, the cusp points of the hypercycloid are exceptional points where transversality fails.) So for generic families of surfaces and at generic points on the hypercycloid, the family $\mathcal{C}_{\alpha,t}$ is generic. Therefore the bifurcations in $\mathcal{C}_{\alpha,t}$ are as described in [31] (see Figure 3). In Figure 3 the bifurcation set consists of two transverse curves. On one of them we have folded saddle-node bifurcations and on the other $MT2$-bifurcations. The BDEs on the $MT2$-stratum are the topological models of the configurations of the lines of curvature on the surfaces M_t. The bifurcations in $\mathcal{C}_{\alpha,t}$ along this stratum agree with the results in [24]. $\square$

We turn now to the case $D^1_{2,3}$ on $|\beta| = 3$. Here the family of distance squared functions is not a versal unfolding of the D_4-singularity (see for example [10]). For generic families of surfaces (generic here means that the resulting family of distance squared functions is versal) we have a birth/death of two umbilic points, one is a Star (3S) and the other a Monstar (2S+1N). The bifurcations in the lines of curvature as the surface is deformed are studied in [24]. In [24] the surface is taken in Monge form $z = h(x, y, 0)$ with

$$j^4h(x,y,0) = \frac{\kappa}{2}(x^2+y^2)+\frac{1}{6}(ax^3+3bxy^2+cy^3)+\frac{1}{24}(Ax^4+4Bx^3y+6Cx^2y^2+4Dxy^3+Ey^4).$$

The condition to have a genuine $D^1_{2,3}$ is $a = b \neq 0$ and $(A - 2\kappa^3 - C)b + cB \neq 0$ ([24]). The condition $a = b$ is equivalent to $|\beta| = 3$ when the cubic in g is taken in the form $\Re(z^3 + \beta z^2\bar{z})$. Now a simple calculation shows that $(A - 2\kappa^3 - C)b + cB \neq 0$ is precisely the condition for the discriminant of $\mathcal{C}_{\pi/2}$ to have a genuine A_3^+-singularity (i.e., is $\mathcal{R}$-equivalent, up to a sign, to $x^2 + y^4$).

We continue to take here the surface in Monge form as in the beginning of this section. Suppose that $|\beta| = 3$, so that $\beta = 3(\cos(\theta) + i\sin(\theta))$, for some $\theta \in [0, 2\pi]$. Then the 2-jets of the coefficients of $\mathcal{C}_{\pi/2}$ are given by

$$\begin{aligned}
j^2a &= 6\sin(\theta)x + 6(1-\cos(\theta))y - 3a_{41}x^2 + (\kappa^3 - 4a_{42})xy - 3a_{43}y^2,\\
j^2b &= -12(\cos(\theta)+1)x - 12\sin(\theta)y + (2a_{42} - 12a_{40} - 12a_{44} + \kappa^3)x^2\\
&\quad -6(a_{41} - a_{43})xy - (2a_{42} + \kappa^3)y^2,\\
j^2c &= -6\sin(\theta)x + 6(\cos(\theta)-1)y + 3a_{41}x^2 + (4a_{42} - \kappa^3)xy + 3a_{43}y^2.
\end{aligned}$$

If $\theta \neq 0$, then the 1-jet of $\mathcal{C}_{\pi/2}$ is equivalent (by smooth changes of coordinates in (x, y) and multiplication by non-zero functions) to $(x + y)dxdy$. If the singularity of the discriminant is of type A_3^+, then the 2-jet of $\mathcal{C}_{\pi/2}$ is equivalent to $y^2dy^2 + (x + y)dxdy + \lambda y^2dx^2$ (see for example [14]), with $\lambda < 0$. The result in [24] states that this 2-jet is in fact topologically sufficient and one can set $\lambda = -1$. Therefore $\mathcal{C}_{\pi/2}$ is topologically equivalent to

$$-y^2dy^2 + (x + y)dxdy + y^2dx^2 = 0$$

at a $D^1_{2,3}$-singularity. We consider now the bifurcations in $\mathcal{C}_{\alpha,t}$ when $\mathcal{C}_{\pi/2,0}$ has a $D^1_{2,3}$-singularity.

Theorem 4.2 *Let M be a smooth surface with an umbilic point of type $D^1_{2,3}$ (i.e., $|\beta| = 3$). Then any deformation of the surface induces a non-generic family $\mathcal{C}_{\alpha,t}$ of the singularity of $\mathcal{C}_{\pi/2}$. For generic families of surfaces, the family $\mathcal{C}_{\alpha,t}$ is topologically equivalent to*

$$(-y^2+u)dy^2 + (x+y)dxdy + (y^2+w)dx^2 = 0.$$

The bifurcations in $\mathcal{C}_{\alpha,t}$ are as in Figure 4.

Proof In [6], Bruce associated to a BDE (1) a symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ whose determinant is the discriminant of the BDE, and classified families of symmetric matrices up to an equivalence relation that preserves the singularities of the determinant. Let $S(n,\mathbb{K})$ denote the space of $n \times n$-symmetric matrices with coefficients in the field $\mathbb{K}$ of real or complex numbers. A family of symmetric matrices is a smooth map germ $\mathbb{K}^r, 0 \to S(n,\mathbb{K})$. Denote by $\mathcal{G}$ the group of smooth changes of parameters in the source and parametrised conjugation in the target. So if $(k, X) \in \mathcal{G}$ and $A \in S(n,\mathbb{K})$, then the action is given by $X^T(A \circ k)X$. A classification of all $\mathcal{G}$-simple symmetric matrices is given in [6].

If we represent the matrix A by (a, b, c), then the matrix of $\mathcal{C}_{\pi/2}$ at a $D^1_{2,3}$-singularity is $\mathcal{G}$-equivalent to $(-y^2, x, y^2)$. This germ has $\mathcal{G}$-codimension 3 ([6]). Therefore, any generic deformation of $\mathcal{C}_{\pi/2}$ needs to have at least three parameters. The family $\mathcal{C}_{\alpha,t}$ has only two, so cannot be generic.

In order to determine the bifurcations in $\mathcal{C}_{\alpha,t}$ we adopt the strategy outlined in [31, 35]. We first determine the bifurcation set of $\mathcal{C}_{\alpha,t}$. The bifurcation set determines a stratification of the (α, t)-plane. On each stratum of this stratification, we determine the configuration of the discriminant curve, the number, position and type of singularity of $\mathcal{C}_{\alpha,t}$ on this discriminant. We then draw the configuration of the integral curves of $\mathcal{C}_{\alpha,t}$ and show that these are topologically constant on each stratum.

The bifurcations related to the singularities of the discriminant are best studied using the symmetric matrices framework in [6]. A $\mathcal{G}$-versal unfolding of the singularity $(-y^2, x, y^2)$ is given by $\tilde{A} = (-y^2 + u_0, x, y^2 + v_1 y + v_0)$. This matrix has zero determinant if $x^2 + (y^2 - u_0)(y^2 + v_1 y + v_0) = 0$. This curve is singular if and only if

$$u_0(v_1^2 - 4v_0)((u_0 + v_0)^2 - u_0 v_1^2) = 0.$$

The above equation gives the bifurcation set of the family $\tilde{A}$. It has two components. One is given by the set of points (u_0, v_0, v_1) for which there exist (x, y) where all the coefficients of the matrix vanish. This is the cross-cap $(u_0 + v_0)^2 - u_0 v_1^2 = 0$ and is parametrised by $(y^2, -v_1 y - y^2, v_1)$. The other component consists of the two smooth surfaces $u_0 = 0$ and $v_1^2 - 4v_0 = 0$ and corresponds to matrices with coefficients not all zero, but with a singular determinant (see Figure 5)

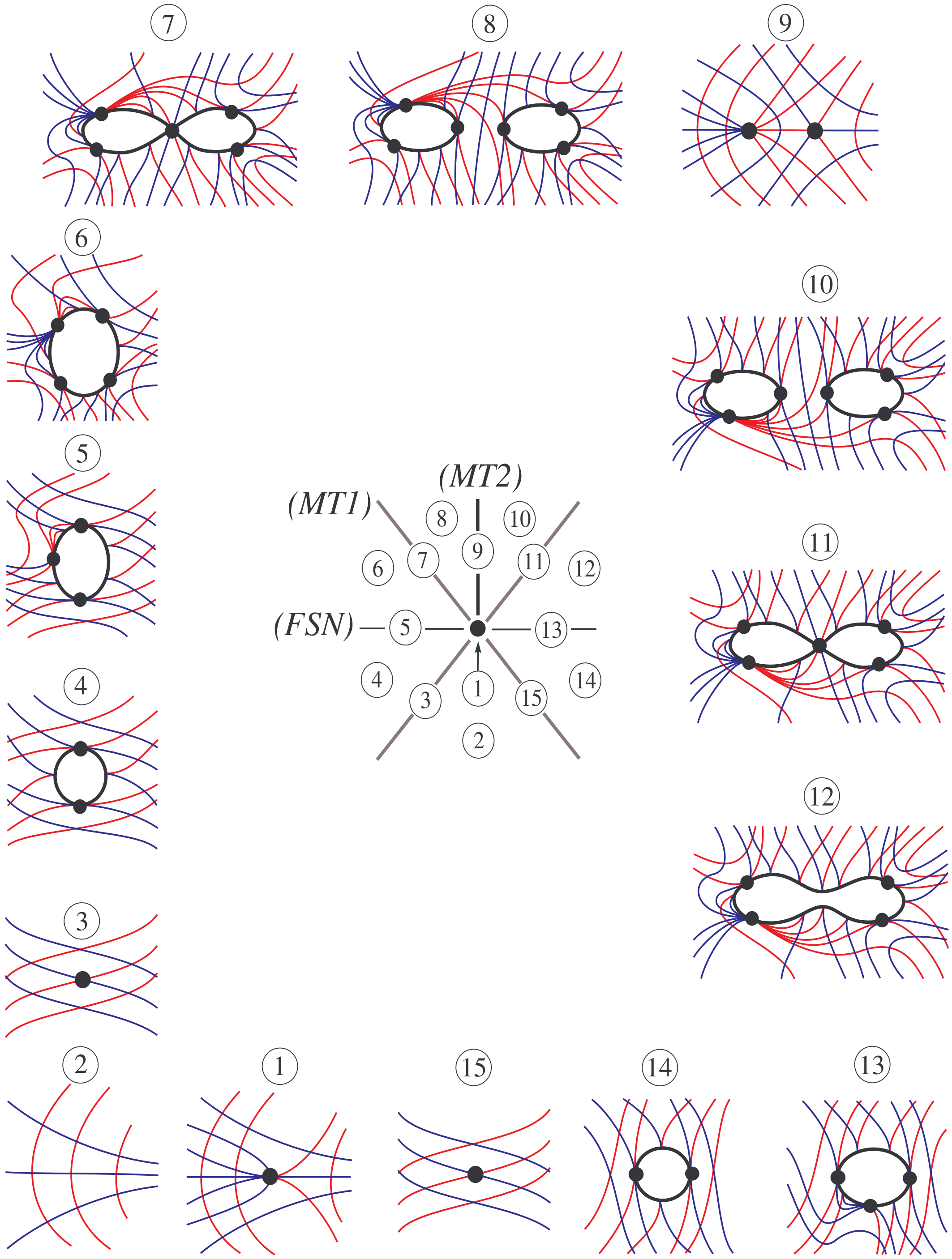


Figure 4: Bifurcations in $\mathcal{C}_{\alpha,t}$ at a $D^1_{2,3}$ umbilic.

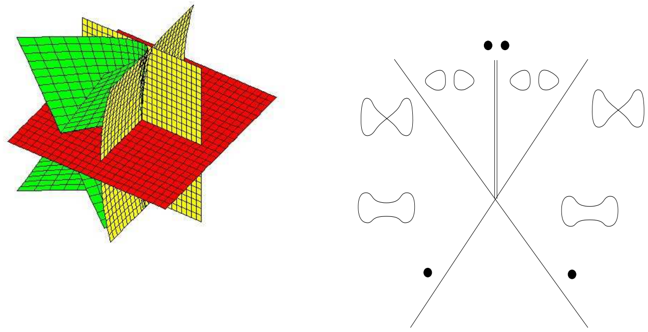


Figure 5: Bifurcation set of $(-y^2+u_0, x, y^2+v_1y+v_0)$ in green and yellow and the variety W in red, left. Bifurcations in the determinant curve on W, right.

We consider now the matrix $\omega = (a, b, c)$ of the family $\mathcal{C}_{\alpha,t}$. As $\tilde{A}$ is a versal deformation, it follows by definition (of a versal deformation) that there exist parametrised diffeomorphisms $k(x, y, \alpha, t)$, invertible matrices $X(x, y, \alpha, t)$, and a map $\psi(\alpha, t) = (u_0, v_0, v_1)$, such that

$$\omega(x, y, \alpha, t) = X(x, y, \alpha, t)^T \tilde{A}(k(x, y, \alpha, t), \psi(\alpha, t))X(x, y, \alpha, t).$$

This means that each member $\omega^{\alpha,t}$ is $\mathcal{G}$-equivalent to $\tilde{A}^{\psi(\alpha,t)}$. So the family $\mathcal{C}_{\alpha,t}$ can be represented by a 2-dimensional variety W (the image of ψ) in the (u_0, v_0, v_1)-space. The variety W is smooth. To show this it is enough to consider the 1-jet of ω at $(0, 0, \pi/2, 0)$. We have

$$\begin{array}{rcl} j^1a & = & -6x\sin(\theta) + 6(\cos(\theta) - 1)y + h_{txy}t + \kappa(\alpha - \frac{\pi}{2}), \\ j^1b & = & 12(\cos(\theta) + 1)x + 12y\sin(\theta) + 2(h_{txx} - h_{tyy})t, \\ j^1c & = & 6x\sin(\theta) + 6(1 - \cos(\theta))y - h_{txy}t + \kappa(\alpha - \frac{\pi}{2}), \end{array}$$

where the partial derivatives are evaluated at $(0, 0, \pi/2, 0)$. We have assumed above that $\theta \neq 0$, otherwise β is also on the hypercycloid in Figure 2. Suppose that $\theta \neq \pi$ (if $\theta = \pi$ we need to make different changes of variables but get the same result). Then we can change variables in the source and set $X = j^1b$, $Y = y$. We can eliminate X in the first and last entries by elements in the group $\mathcal{G}$. Then the 1-jet of the family ω is $\mathcal{G}$-equivalent to (A, B, C) with

$$\begin{array}{l} A = (h_{txy} - \frac{\cos(\theta)-1}{\sin(\theta)}(h_{txx} - h_{tyy}))t + \kappa(\alpha - \frac{\pi}{2}), \\ B = X, \\ C = (-h_{txy} + \frac{\cos(\theta)-1}{\sin(\theta)}(h_{txx} - h_{tyy}))t + \kappa(\alpha - \frac{\pi}{2}). \end{array}$$

If $\sin(\theta)h_{txy} - (\cos(\theta) - 1)(h_{txx} - h_{tyy}) \neq 0$, which is the case for generic families of surfaces, we can make a further change of variable so that $j^1\omega$ becomes $\mathcal{G}$-equivalent to a family of matrices (u, X, v) parametrised by (u, v). This shows that for generic families of surfaces, the map $\psi(\alpha, t)$ is indeed of rank 2 at $(\pi/2, 0)$.

Suppose now that the family of distance squared functions is a versal unfolding of the D_4 singularity. This is the case if $\theta \neq \pm 2\pi/3$ and $\sin(\theta)h_{txy} - (\cos(\theta) - 1)(h_{txx} - h_{tyy}) \neq 0$. (When $\theta = \pm 2\pi/3$, β is also on the hypercycloid in Figure 2, so we have a higher codimension singularity of $\mathcal{C}_{\pi/2}$.) Then we have a birth of two umbilics in the family. This means that the variety W must contain the double point curve of the cross-cap in the bifurcation set of $\tilde{A}$ (Figure 2) as this corresponds to the presence of two umbilic points (two singularities of type $(-y, x, y)$).

The components of the bifurcation set of the family $\mathcal{C}_{\alpha,t}$ that are associated with the bifurcations of the discriminant can be modelled by the intersection of W with the bifurcation set $\tilde{A}$. It follows from the discussion above that the $MT2$-stratum of the bifurcation set of $\mathcal{C}_{\alpha,t}$ consists of a segment of a curve ending at $(\pi/2, 0)$, and the $MT1$-stratum consists of two curves intersecting transversaly at $(\pi/2, 0)$ (Figures 4 and 2 left). It also follows that the changes in the discriminant curves as (α, t) varies near $(\pi/2, 0)$ are as in Figure 5 right (see also Figures 4).

In [14] we computed the multiplicities of BDEs with zero coefficients (this is the maximum number of zeros that can appear in a deformation of the equation). As $\mathcal{C}_{\pi/2}$ is equivalent to $(-x^2, x + y, x^2)$, its multiplicity is 6 (see Table 1 in [14]). At a birth of umbilics, one is a Star and the other is a Monstar, so as these bifurcates, there zeros appear on each component of the discriminant (so we have six in total).

We deal now with the folded saddle-node (FSN) stratum. Following [31, 35] we take, an affine chart $p = \frac{dy}{dx}$ (we also consider the chart $q = \frac{dx}{dy}$) and set $F(x, y, \alpha, t, p) = \tilde{a}(x, y, \alpha, t)p^2 + 2\tilde{b}(x, y, \alpha, t)p + \tilde{c}(x, y, \alpha, t)$. (We still denote by F the restriction of this function to (α, t) =constant.) A zero of the BDE $\tilde{w}^{\alpha,t}$, for (α, t) fixed, at a smooth point on the discriminant is given by $F = F_p = F_x + pF_y = 0$. Suppose, without loss of generality, that $F_y \neq 0$ at the point in consideration, say $q_0 = (x_0, y_0, p_0)$, so that the surface of the equation is given by $y = g(x, p)$ for some germ of a smooth function g at (x_0, p_0). So $F(x, g(x, p), p) = 0$ and $F_x + g_xF_y = F_yg_p + F_p = 0$. Therefore, the linear part of the projection of the lifted field $\xi^{\alpha,t}$ to the (x, p)-plane is given by

$$A = \begin{pmatrix} F_{xp} - \frac{F_x}{F_y}F_{yp} & F_{pp} - \frac{F_p}{F_y}F_{yp} \\ \\ -F_{xx} + \frac{F_x}{F_y}F_{xy} - p(F_{xy} - \frac{F_x}{F_y}F_{yy}) & -F_y - F_{xp} + \frac{F_p}{F_y}F_{xy} - p(F_{yp} - \frac{F_p}{F_y}F_{yy}) \end{pmatrix},$$

where the entries are evaluated at q_0. Then the (FSN) stratum is given by the set of parameters (α, t) for which there exists (x, y, p) such that

$$F = F_p = F_x + pF_y = \det(A) = 0$$

at (x, y, p, α, t). We can solve the above system (with Maple) inductively on the jet level and get the 1-jet of the (FSN) stratum. It is given by

$$(h_{xxx}(h_{xxt}-h_{yyt})+h_{yyy}h_{xyt})t-(h_{yyy}h_{xx})(\alpha-\frac{\pi}{2}),$$

where the partial derivatives are evaluated at $(0,0,0)$. So this stratum is generically a smooth curve. We also find that the tangent directions to the $MT1$-stratum are given by

$$(h_{xxx}(h_{xxt}-h_{yyt})+h_{yyy}h_{xyt})^2t^2-h_{xx}^2(h_{xxx}^2+h_{yyy}^2)(\alpha-\frac{\pi}{2})^2=0.$$

where the partial derivatives are evaluated at $(0,0,0)$. The $MT2$ stratum is a segment of the line $\alpha=\pi/2$. Therefore the (FSN) stratum is a smooth curve transverse to the branches of the $MT1$-stratum and to the limiting tangent direction of the $MT2$-stratum. (See the bifurcation set in Figure 4.)

One can also show, using the transversality of the pair of foliation of $\mathcal{C}_{\alpha,t}$ away from the discriminant, that there are no other codimension 1 local and semi-local strata. We can now use the results on the bifurcations of codimension 1 singularities of BDEs [9, 15, 24, 27, 34] to determine the position and type of the zeros on the discriminant in each stratum of the stratification determined by the bifurcation set. We can also use the results on the bifurcations of the codimension 1 singularities to draw the integral curves in each stratum. At $(\pi/2,0)$ and on the $MT2$-stratum the configurations are those of the lines of curvature and are given in [24]. On other strata, they are as in Figure 4. Given any two configurations on the same stratum, one can use the technique in [9, 34, 35] to construct a homeomorphism that takes one to the other.

The bifurcation set of the family $(-y^2+u)dy^2+(x+y)dxdy+(y^2+w)dx^2=0$ is homeomorphic to that of $\mathcal{C}_{\alpha,t}$, and following the discussion above, it is topologically equivalent to $\mathcal{C}_{\alpha,t}$. (The configurations in Figure 4 are checked on the model family using the Motesinos' programme [28].) $\square$

5 Bifurcations away from umbilics and parabolic points

Away from umbilics and parabolic points the coefficients of $\mathcal{C}_{\alpha_0,0}$ do not all vanish. So we expect $\mathcal{C}_{\alpha_0,0}$ to have the codimension 2 singularities listed in [35] at such points on M_0. It is shown in [35] that the singularities of codimension ≤ 2 of this type of equations (i.e., with coefficients not all vanishing at the singularity) are locally topologically equivalent to $dy^2+f_{(0,0)}(x,y)dx^2=0$, with $f_{(0,0)}$ as in Table 1, second column. Any generic 2-parameter family of such BDE is fibre topologically equivalent to $dy^2+f(x,y,u,v)dx^2=0$, with f as in Table 1, third column.

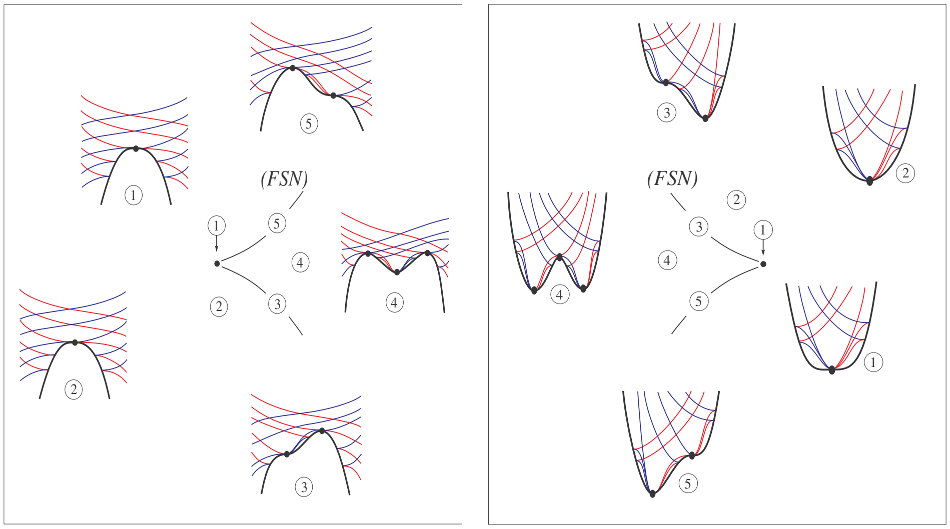


Figure 6: Bifurcations at a folded degenerate elementary singularity, $\epsilon = -1$ left and $\epsilon = +1$ right.

Table 1: Codimension 2 singularities of BDEs and their generic deformations, [35].

Name	Normal form	Generic family	Figure
Folded degenerate elementary singularity	$-y \pm x^4$	$-y \pm x^4 + ux^2 + vx$	Figure 6
Non-transverse Morse singularity	$xy + x^3$	$xy + x^3 + ux^2 + v$	Figure 7
Cusp type 1 singularity	$\pm x^2 + y^3$	$\pm x^2 + y^3 + uy + v$	Figure 8

Theorem 5.1 *All the local codimension 2 singularities in Table 1 can occur on $\mathcal{C}_{\alpha_0,0}$ at isolated points on M_0 that are neither parabolic nor umbilic. Generic 1-parameter families of surfaces M_t induce generic families $\mathcal{C}_{\alpha,t}$ of these singularities. The bifurcations in $\mathcal{C}_{\alpha,t}$ are thus as in Figures 6, 7, 8.*

Proof We take the family of surface M_t in the Monge form $(x, y, h(x, y, t))$ as in §2, and suppose that the origin is a singularity of $C_{\alpha_0,0}$. We rotate the coordinates axes so that $(1, 0)$ is tangent to a solution of $C_{\alpha_0,0}$ at the origin. Then

$$\sin\alpha_0 = \frac{h_{20}}{(h_{20}^2 + h_{21}^2)^{\frac{1}{2}}} \quad \text{and} \quad \cos\alpha_0 = -\frac{h_{21}}{(h_{20}^2 + h_{21}^2)^{\frac{1}{2}}}.$$

We shall assume that $\alpha_0 \neq 0, \frac{\pi}{2}$, equivalently, $h_{20}h_{21} \neq 0$, that is $(1, 0)$ is neither a principal nor an asymptotic direction. The origin is a point on the discriminant of

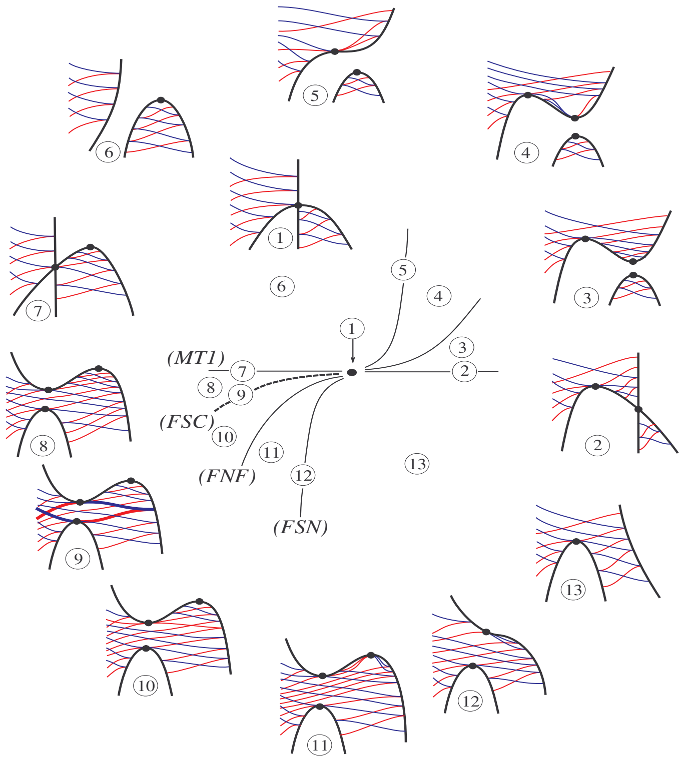


Figure 7: Bifurcations at a non-transverse Morse singularity.

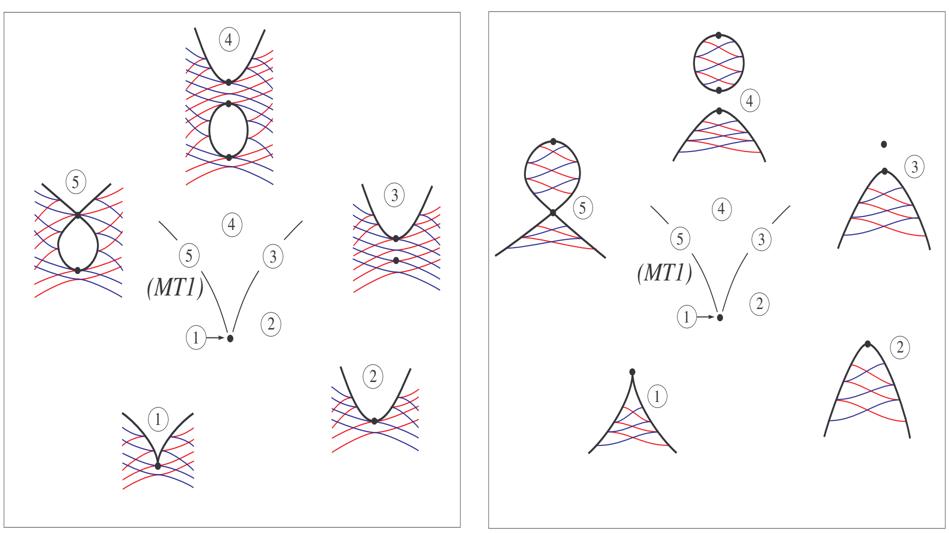


Figure 8: Bifurcations at a cusp type 1 singularity, $\epsilon = -1$ left and $\epsilon = +1$ right.

$C_{\alpha_0,0}$ if and only if

$$h_{20}^2 - h_{20}h_{22} + 2h_{21}^2 = 0,$$

and a singularity of $C_{\alpha_0,0}$ if furthermore

$$h_{30}h_{21} - h_{31}h_{20} = 0.$$

As the coefficients of $C_{\alpha_0,0}$ do not all vanish at the origin, we know from [14] that $\mathcal{C}_{\alpha,t}$ is equivalent, for t small enough, by smooth changes coordinates and multiplication by a non-zero function, to a family in the form

$$dy^2 + f(x, y, \alpha, t)dx^2 = 0. \tag{2}$$

In fact, we only need to reduce an appropriate k-jet of $\mathcal{C}_{\alpha,t}$ to the above form as the conditions for a given singularity in the members of the family (2) and that for the family to be generic depend only on some k-jet of f at the origin ([35]). The reduction is done inductively on the jet level following the algorithm in [17, 14]. Once the reduction done, we apply the results in [35] to obtain information about $\mathcal{C}_{\alpha,t}$

We use the above setting for $C_{\alpha_0,0}$ in all the cases below. The calculations are carried out using Maple. The genericity conditions are very lengthy for us to be able to reproduce them here. We start with the following case.

(i) *Folded degenerate elementary singularity*: This singularity occurs when the discriminant is smooth and the lifted field has a degenerate elementary singularity of multiplicity 3. It occurs at $q_0 = (0, 0, \alpha_0, 0)$ in (2) if and only if

$$f(q_0) = f_x(q_0) = f_{xx}(q_0) = f_{xxx}(q_0) = 0 \text{ and } f_y(q_0)f_{xxxx}(q_0) \neq 0.$$

Following similar calculations to those in the previous sections, the family (2) is generic if and only if

$$\begin{pmatrix} f_y & f_\alpha & f_t \\ f_{xy} & f_{\alpha x} & f_{tx} \\ f_{xxy} & f_{\alpha xx} & f_{txx} \end{pmatrix}$$

has maximal rank at q_0 (see [35]. The matrix in [35] is slightly different as the equation there is reduced to a pre-normal form at $(\alpha_0, 0)$. This observation is also valid for the two other cases below).

For $C_{\alpha_0,0}$ (and with the above setting), the origin is a folded degenerate elementary singularity if and only if

$(-h_{31}^2 + 4h_{21}^4)h_{20}^5 + 3h_{20}^4 h_{21}h_{32}h_{31} + (-4h_{41}h_{21} - 5h_{31}^2 - 2h_{32}^2 + 8h_{21}^4)h_{21}^2 h_{20}^3 + (4h_{40}h_{21} + 7h_{32}h_{31})h_{21}^3 h_{20}^2 + (-4h_{21}h_{41} + 4h_{21}^4 - 6h_{31}^2)h_{21}^4 h_{20} + 4h_{21}^6 h_{40} = 0$

and

$(24h_{21}^4 h_{31} - 3h_{31}^3)h_{20}^9 + (-36h_{21}^5 h_{32} + 13h_{31}^2 h_{21}h_{32})h_{20}^8 + (156h_{21}^6 h_{31} - 12h_{21}^3 h_{41}h_{31} + (-3h_{31}^2 h_{33} - 18h_{32}^2 h_{31} - 13h_{31}^3)h_{21}^2)h_{20}^7 + (-16h_{21}^7 h_{32} + (24h_{31}h_{42} + 16h_{41}h_{32})h_{21}^4 + (51h_{31}^2 h_{32} + 7h_{31}h_{32}h_{33} + 8h_{32}^3)h_{21}^3)h_{20}^6 + (232h_{21}^8 h_{31} - 16h_{21}^6 h_{51} + (-28h_{32}h_{42} - 76h_{41}h_{31})h_{21}^5 + (-13h_{31}^2 h_{33} - 58h_{32}^2 h_{31} -$

$4h_{32}^2h_{33}-12h_{31}^3)h_{21}^4)h_{20}^5+(76h_{21}^9h_{32}+16h_{21}^7h_{50}+(76h_{31}h_{42}+60h_{41}h_{32})h_{21}^6+(76h_{31}^2h_{32}+18h_{32}^3+15h_{31}h_{32}h_{33})h_{21}^5)h_{20}^4+(92h_{21}^{10}h_{31}-32h_{21}^8h_{51}+(-144h_{41}h_{31}-28h_{32}h_{42})h_{21}^7+(-14h_{31}^2h_{33}+4h_{31}^3-66h_{32}^2h_{31})h_{21}^6)h_{20}^3+(56h_{21}^{11}h_{32}+32h_{21}^9h_{50}+(44h_{41}h_{32}+52h_{31}h_{42})h_{21}^8+60h_{21}^7h_{32}h_{31}^2)h_{20}^2+(-16h_{21}^{10}h_{51}-80h_{21}^9h_{41}h_{31}-8h_{21}^{12}h_{31})h_{20}+16h_{21}^{11}h_{50}=0.$

We then reduce the 4-jet of $\mathcal{C}_{\alpha,t}$ at $(0,0,\alpha_0,0)$ to the form $dy^2+f(x,y,\alpha,t)dx^2$ and calculate the determinant of the above matrix. The expression of this determinant is very lengthy to reproduce here. However it does not vanish for generic families of surfaces. Therefore, for generic families of surfaces, the family $\mathcal{C}_{\alpha,t}$ is generic at a folded degenerate elementary singularity of $C_{\alpha_0,0}$. It follows then from [35] that the bifurcations in $\mathcal{C}_{\alpha,t}$ are as shown in Figure 6.

(ii) *Non-transverse Morse singularity*: occurs when the discriminant has a Morse singularity of type node (i.e., given by $x^2-y^2=0$ in some coordinates system) and the unique direction determined by the BDE at the origin has an ordinary tangency with one of the branches of the discriminant. It occurs at $q_0=(0,0,\alpha_0,0)$ in (2) if and only if

$$f(q_0)=f_x(q_0)=f_y(q_0)=f_{xx}(q_0)=0 \text{ and } f_{xy}(q_0)f_{xxx}(q_0)\neq 0.$$

The family (2) with a non-transverse Morse singularity at the origin is generic if and only if

$$\begin{pmatrix} 0 & 0 & f_\alpha & f_t \\ 0 & f_{xy} & f_{\alpha x} & f_{tx} \\ f_{xy} & f_{yy} & 0 & f_{ty} \\ f_{xxx} & f_{xxy} & f_{\alpha xx} & f_{txx} \end{pmatrix}$$

has maximal rank at q_0 ([35]).

We reduce the 3-jet of $\mathcal{C}_{\alpha,t}$ at q_0 to the form $dy^2+f(x,y,\alpha,t)dx^2$ as in the previous case. We can then read the conditions for a $\mathcal{C}_{\alpha_0,0}$ to have a non-transverse Morse singularity at the origin. Under these conditions, the determinant of the above matrix has maximal rank for generic families of surfaces. (Again the expression of this determinant is too lengthy to reproduce here.) Therefore the family $\mathcal{C}_{\alpha,t}$ is generic at a non-transverse Morse singularity of $C_{\alpha_0,0}$. It follows then from [35] that the bifurcations in $\mathcal{C}_{\alpha,t}$ are as shown in Figure 7.

(iii) *Cusp type 1 singularity*: occurs when the discriminant has a cusp singularity with a limiting tangent transverse to the unique direction determined by the BDE. It occurs at $q_0=(0,0,\alpha_0,0)$ in (2) if and only if

$$f(q_0)=f_x(q_0)=f_y(q_0)=(f_{xy}^2-f_{xx}f_{yy})(q_0)=0 \text{ and } f_{xx}(q_0)f_{yyy}(q_0)\neq 0.$$

The family (2) with a cusp singularity at the origin is generic if and only if

$$\begin{pmatrix} 0 & 0 & f_\alpha & f_t \\ f_{xx} & f_{xy} & f_{\alpha x} & f_{tx} \\ f_{xy} & f_{yy} & 0 & f_{ty} \\ m_1 & m_2 & * & * \end{pmatrix}$$

has maximal rank at q_0 ([35]), where $m_1 = (2f_{xy}f_{xxy} - f_{xx}f_{xyy} - f_{yy}f_{xxx})(q_0)$, $m_2 = (2f_{xy}f_{xyy} - f_{xx}f_{yyy} - f_{yy}f_{xxy})(q_0)$. Observe that $(f_{xy}^2 - f_{xx}f_{yy})(q_0) = 0$, so the determinant of the above matrix does not depend on the entries marked $*$.

We reduce the 3-jet of $\mathcal{C}_{\alpha,t}$ at q_0 to the form $dy^2 + f(x, y, \alpha, t)dx^2$ and compute the conditions for having a cusp singularity and for $\mathcal{C}_{\alpha,t}$ to be generic. We find that for generic families of surfaces $\mathcal{C}_{\alpha,t}$ is indeed generic (the determinant of the above matrix does not vanish). It follows then from [35] that the bifurcations in $\mathcal{C}_{\alpha,t}$ are as shown in Figure 8. □

6 Appendix

BDEs (1) determine a pair of transverse foliations away from the discriminant. The pair of foliations together with the discriminant are called the *configuration* of the solutions of the BDE. We analyse here the configurations of the BDEs at points on their discriminants and describe the stable and codimension ≤ 1 local phenomena in these equations. We shall consider the point of interest to be the origin and separate the study into two cases depending on whether all the coefficients vanish at the origin or not.

6.1 Not all the coefficients vanish at the origin

A BDE with coefficients not all vanishing at a given point can be considered as an implicit differential equation (IDE) $F(x, y, p) = 0$, $p = dy/dx$, with (x, y, p) in some open set in $\mathbb{R}^3$. Conversely, any IDE that satisfies $F = F_p = 0$ and $F_{pp} = 0$ at a given point, is locally smoothly equivalent to a BDE ([14]). (So in particular, one can deform a BDE in this situation in the set of all IDEs.) One approach for investigating BDEs with coefficients not all vanishing at a given point consists of lifting the bi-valued direction field defined in the plane to a single direction field ξ on the surface $N = F^{-1}(0) \subset \mathbb{R}^3$. If ξ does not vanish at the point in consideration then the BDE can be reduced locally (by smooth changes of coordinates in the plane and multiplication by non-zero functions) to $dy^2 - xdx^2 = 0$. The integral curves in this case is a family of cusps transverse to the discriminant. (See [18, 25] for details and references.)

If ξ has an elementary singularity (saddle/node/focus), then the corresponding point in the plane is called a *folded singularity* of the BDE. At folded singularities, the equation is locally smoothly equivalent to $dy^2 + (-y + \lambda x^2)dx^2 = 0$, with $\lambda \neq 0, \frac{1}{16}$, provided that ξ is linearzable at the singular point; see [18]. There are three topological models (see [18] for references): a folded saddle if $\lambda < 0$, a folded node if $0 < \lambda < \frac{1}{16}$ and a folded focus if $\frac{1}{16} < \lambda$; Figure 9.

The family of cusps and the folded singularities are the only locally structurally stable configurations of singular BDEs (and indeed of IDEs) .

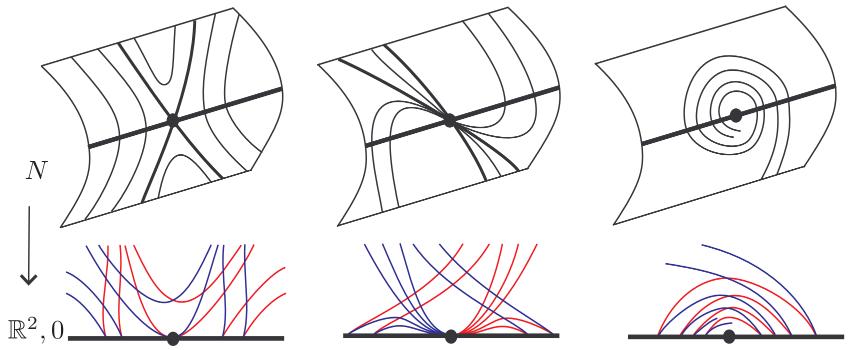


Figure 9: Folded saddle (left), node (centre) and focus (right).

The bifurcations in generic 1-parameter families have also been classified. One of these is the *folded saddle-node bifurcation* ($\lambda = 0$ above) and occurs when the discriminant is smooth and the lifted field ξ has a saddle-node singularity. Then the equation is locally smoothly equivalent to $dy^2 + (-y + x^3 + \mu x^4)dx^2 = 0$ ([19]). A generic 1-parameter family of BDEs with a folded saddle-node singularity at $t = 0$ is fibre topologically equivalent to $dy^2 + (-y + x^3 + tx)dx^2 = 0$ ([34], Figure 10).

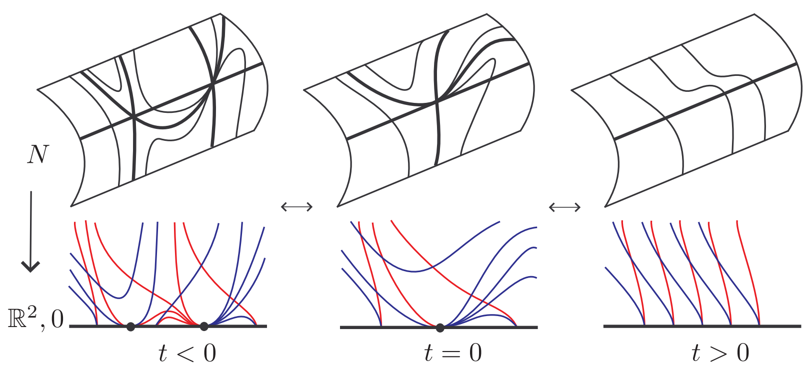


Figure 10: Bifurcations at a folded saddle-node.

When $\lambda = \frac{1}{16}$, it is shown in [20] that the IDE is still smoothly equivalent to $dy^2 + (-y + \frac{1}{16}x^2)dx^2 = 0$. We label this singularity a *folded node-focus change*. One can show that a generic 1-parameter family of BDEs with this singularity at $t = 0$ is fibre topologically equivalent $dy^2 + (-y + (\frac{1}{16} + t)x^2)dx^2 = 0$ ([34], Figure 11).

Bifurcations can also occur when the discriminant has a Morse singularity (i.e., the discriminant is given by $x^2 \pm y^2 = 0$ in some coordinates system). BDEs with discriminants as above are labelled *Morse Type 1* in [9]. Generic Morse Type 1 sin-

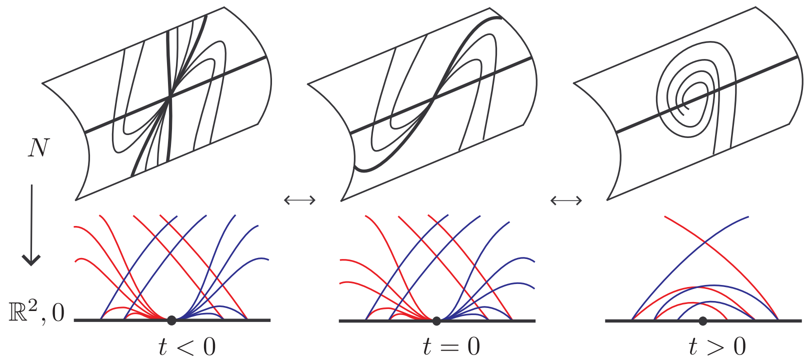


Figure 11: Change from a folded node to a folded focus.

gularities are locally topologically equivalent to $dy^2 + (\pm x^2 \pm y^2)dx^2 = 0$ ([9, 27]). A generic 1-parameter family of BDEs with a Morse Type 1 singularity at $t = 0$ is fibre topologically equivalent to $dy^2 + (\pm x^2 \pm y^2 + t)dx^2 = 0$ ([9], Figure 12). As t passes through 0, two folded saddles or foci singularities appear on one side of the transition and none on the other. The saddle or focus type are distinguished by the sign of x^2 in the normal form (+ for focus and − for saddle).

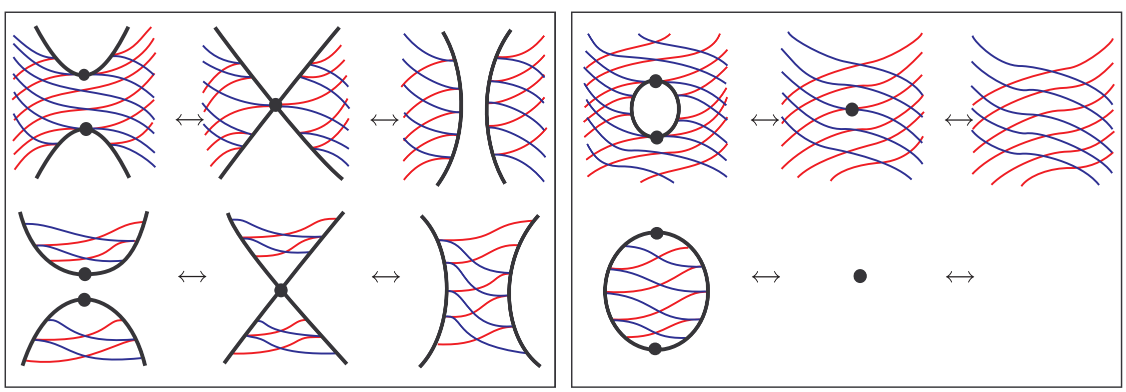


Figure 12: Bifurcations at Morse Type 1 singularities: A_1^- left and A_1^+ right.

In [35] are studied the local codimension 2 singularities and their bifurcations in generic families. The topological normal forms of these singularities and models of their generic families are given in Table 1. It is also shown in [35] that there are no Poincaré-Andronov (Hopf) bifurcations on the lifted field ξ of a BDE at a regular point on the criminant.

6.2 All the coefficients vanish at the origin

BDEs with coefficients all vanishing at a given point are of infinite codimension in the set of all IDEs. So one has to restrict the study of their deformations to the set of all BDEs. General BDEs with vanishing coefficients at a given point are studied for example in [7, 16, 23, 27, 33]. The bi-valued field in the plane is lifted to a single direction field ξ on a surface

$$N = \{(x, y, [\alpha : \beta]) \in \mathbb{R}^2, 0 \times \mathbb{R}P^1 : a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\}.$$

If we consider the affine chart $p = \beta/\alpha$ and set $F(x, y, p) = a(x, y)p^2 + 2b(x, y)p + c(x, y)$, then the lifted direction field is parallel to the vector field $\xi = F_p\partial/\partial x + pF_p\partial/\partial y - (F_x + pF_y)\partial/\partial p$. The whole exceptional fibre $(0, 0) \times \mathbb{R}P^1$ is an integral curve of ξ. The surface N is regular along the exceptional fibre if and only if the discriminant of the BDE has a Morse singularity ([16]). It turns out that when this is the case and when ξ has only elementary singularities on the exceptional fibre, the topological models of the integral curves of the BDE are completely determined by the singularity type of the discriminant (an isolated point or a crossing), the number (1 or 3) and the type (saddle or node) of the singularities of ξ (see for example [16]). If $j^1(a, b, c) = (a_1x + a_2y, b_1x + b_2y, c_1x + c_2y)$, then the singularities of ξ on the exceptional fibre are given by the roots of the cubic

$$\phi(p) = (F_x + pF_y)(0, 0, p) = a_2p^3 + (2b_2 + a_1)p^2 + (2b_1 + c_2)p + c_1.$$

The eigenvalues of the linear part of ξ at a singularity are $-\phi'(p)$ and $\alpha_1(p)$, where

$$\alpha_1(p) = 2(a_2p^2 + (b_2 + a_1)p + b_1).$$

So the cubic ϕ and the quadratic α_1 determine the number and the type of the singularities of ξ (see [16] for details).

When the discriminant has a Morse singularity we can set $j^1(a, b, c) = (y, b_1x + b_2y, \epsilon y)$, $\epsilon = \pm 1$ (or $j^1(a, b, c) = (x + a_2y, 0, y)$, $a_2 > \frac{1}{4}$), [16, 23]. In the (b_1, b_2)-plane, there are curves where singularities of codimension > 1 occur. These are:

(i) $b_1 = 0$, where the discriminant has a degenerate singularity (worse than Morse);
(ii) $2b_1 + \epsilon = 0$ or $b_1 = \frac{1}{2}(b_2^2 - \epsilon)$ where ϕ has a double root;
(iii) $\epsilon = 1$, $b_1 = \pm b_2 - 1$ where α_1 and ϕ have a common root.

The topological type of the BDE is constant in the complement of the above curves. These singularities are of codimension 1 and their bifurcations in generic families are studied in [15] (see also [27] for the case $\epsilon = -1$). The figures on both sides of the bifurcations are equivalent, so only one of them is shown in each case in Figures 13 and 14. (The 2S+1N case (b) in Figure 14 was missing in [15]. It was completed in [31].)

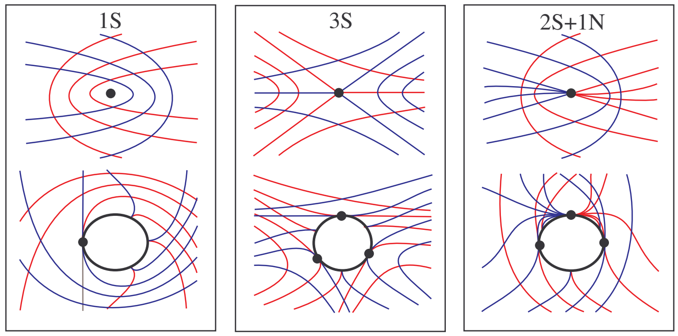
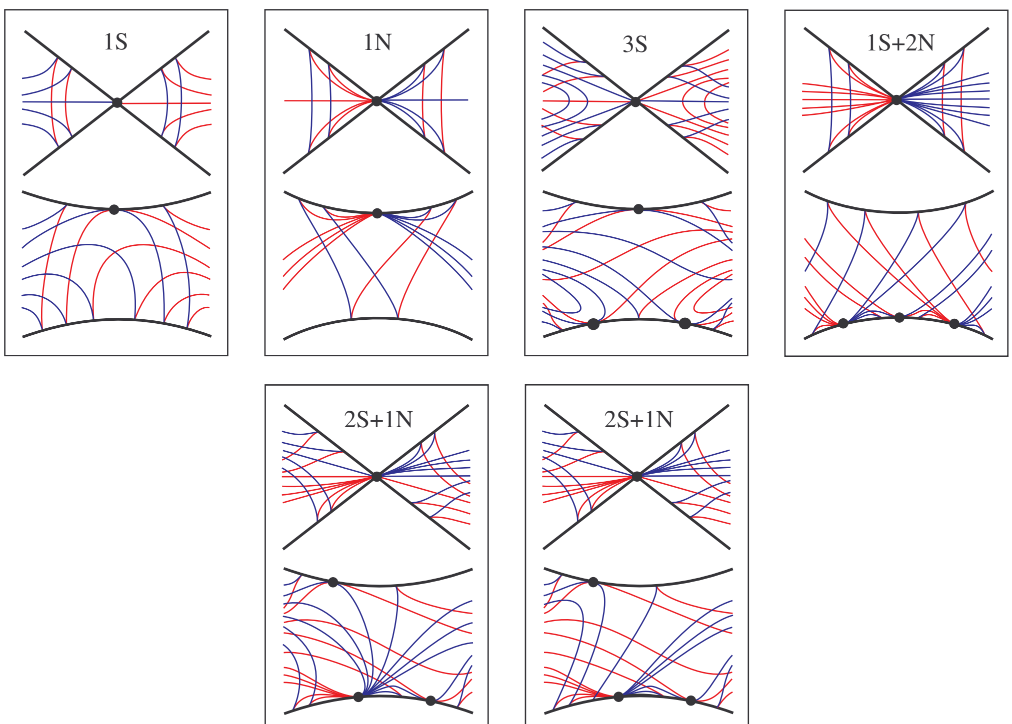


Figure 13: Bifurcations at an A_1^+ Morse Type 2 singularities.



case (a) case (b)

Figure 14: Bifurcations at an A_1^- Morse Type 2 singularities.

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