# Self-adjoint operators on surfaces with singular metrics

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**Abstract**

We define and study the asymptotic, characteristic and principal direction fields associated to a self-adjoint operator on a smooth surface $M$ endowed with a metric $g$ which is singular along a smooth curve on $M$.

## 1 Introduction

Let $M$ be a smooth and orientable surface in the Euclidean space $\mathbb{R}^3$ and $N : M \to S^2$ be its Gauss map. The shape operator $S_p = -(dN)_p : T_pM \to T_pM$, $p \in M$, is a self-adjoint operator and provides information about the local shape of $M$ in $\mathbb{R}^3$. It also determines on $M$ three pairs of foliations which are defined as follows. As $T_pM$ inherits the Euclidean scalar product ".", the shape operator $S_p$ has real eigenvalues at any point $p$ on $M$. These are called the principal curvatures and points where they coincide are labelled umbilic points. Umbilic points are also points where $S_p$ is a multiple of the identity map. The eigenvectors of $S_p$ are called the principal directions and their integral curves the lines of principal curvature (they are the solutions of a binary/quadratic differential equation). The lines of principal curvature form a pair

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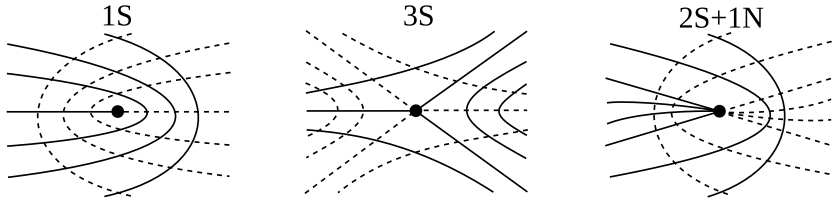


Figure 1: Generic configurations of the lines of principal curvature at umbilic points.

of orthogonal foliations on $M$ away from umbilic points. Their generic configurations at an umbilic point were first drawn by Darboux and a rigorous proof was given in [2, 13]. See Figure 1.

Two tangent directions $\boldsymbol{u}, \boldsymbol{v} \in T_pM$ are said to be conjugate if $S_p(\boldsymbol{u}).\boldsymbol{v} = 0$. A direction $\boldsymbol{u} \in T_pM$ is said to be asymptotic if it is self-conjugate, i.e., if $S_p(\boldsymbol{u}).\boldsymbol{u} = 0$. There are two asymptotic directions at each hyperbolic point (these are points where the Gaussian curvature $K = det(S_p)$ is negative) and their integral curves are called the asymptotic curves. At an elliptic point (where $K > 0$) there is a unique pair of conjugate directions for which the included angle is extremal ([10]). These directions are called the characteristic directions and their integral curves the characteristic curves. The pairs of foliations determined by the principal, asymptotic and characteristic directions are related (see [4] and §3.3).

A key observation here is that the concepts of principal, asymptotic and characteristic foliations on a surface $M$ in $\mathbb{R}^3$ are derived from the fact that $S_p$ is a self-adjoint operator with respect to the Euclidean metric on $M$. This means, and it is the aim of this paper, that one can associate these concepts to a self-adjoint operator on a two dimensional manifold endowed with a metric $g$ which could be of varying signature.

We consider in this paper a smooth two dimensional manifold (i.e., a surface) $M$ with a countable basis. We suppose that $M$ is endowed with a metric $g$, which is possibly singular on a smooth curve on $M$. Suppose given on $(M, g)$ a self-adjoint operator $A$, that is, a smooth map $TM \to TM$ with the property that its restriction $A_p : T_pM \to T_pM$ is a linear map satisfying $g(A_p(\boldsymbol{u}), \boldsymbol{v}) = g(\boldsymbol{u}, A_p(\boldsymbol{v}))$ at any $p \in M$ and for any $\boldsymbol{u}, \boldsymbol{v} \in T_pM$. When $A_p$ has real eigenvalues, we call them the $A$-principal curvatures and their associated eigenvectors the $A$-principal directions. The integral curves of the $A$-principal directions are labelled the lines of $A$-principal curvature.

When the metric $g$ is Riemannian, the lines of $A$-principal curvature have the same local behaviour as that of the lines of principal curvature of a surface in $\mathbb{R}^3$. (This follows from [21] and is true for the other two pairs of foliations in this paper.) For instance, their generic local configurations at spacelike umbilic points, which are points where $A_p$ is a multiple of the identity, are as in Figure 1. However, when the metric $g$ is Lorentzian, i.e., when $g$ has signature 1 (§3), $A_p$ does not always have real eigenvalues.

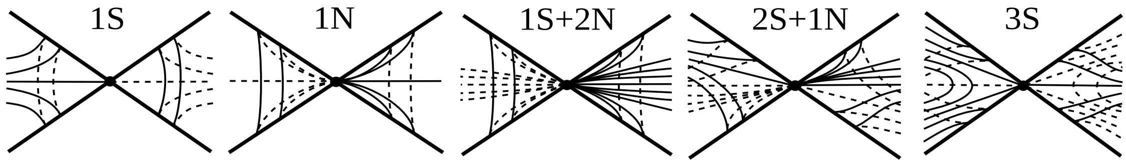


Figure 2: Generic topological configurations of the lines of $A$-principal curvature at timelike umbilic points.

The lines of $A$-principal curvature present interesting behaviour in this case (§3.1). For instance (Theorem 3.4), at timelike umbilic points, which are points where $A_p$ is a multiple of the identity map, the lines of $A$-principal curvature have generically one of the five configurations in Figure 2.

We can also define the concepts of $A$-asymptotic and $A$-characteristic directions on a Lorentzian surface (§3.2, §3.3). We study in §3 the local behaviour of these foliations when $g$ is Lorentzian and apply the results to surfaces in the de-Sitter space $S^3_1$ (§3.4). We show in §4 how to extend the $A$-foliations (asymptotic, characteristic, principal) to the singular locus of the metric. We apply the results to closed surfaces in the Minkowski space $\mathbb{R}^3_1$ (§4.4).

The pairs of foliations in this paper are the solution curves of certain binary differential equations (BDEs). We give a brief review on BDEs in §2. An important aspect of BDEs which is not included here is the global behaviour of their pairs of foliations (the study of their structural stability which includes the study of their behaviour near a periodic orbit). This study is initiated in the pioneering work of Sotomayor and Gutierrez [19] for the lines of principal curvature on a surface in $\mathbb{R}^3$. The global behaviour of the asymptotic and characteristic curves on a surface in $\mathbb{R}^3$ is studied in [11, 12]. (Similar work on special surfaces in $\mathbb{R}^3$ and on surfaces in $\mathbb{R}^4$ is also carried out by Garcia, Gutierrez, Mello, Sotomayor.)

It is worth pointing out that our approach of considering general self-adjoint operators lead in [21] to a new definition of lines of principal curvature on a smooth surface in $\mathbb{R}^4$. In [18] are defined lines of $\mu$-principal curvature. These are the lines of $S_\mu$-principal curvature, where $S_\mu$ is the shape operator along a smooth normal vector field $\mu$ on $M$. These pairs of foliations depend of course on the choice of $\mu$. The asymptotic curves are well defined on surfaces in $\mathbb{R}^4$. One can recover a self-adjoint operator $A$ on $M$ from the equation of the asymptotic curves. The lines of $A$-principal curvature are then called the lines of curvature of $M$. It turns out that there exists a unique normal vector field $\mu_0$ on $M$ such that the lines of $A$-principal curvature are the lines of $S_{\mu_0}$-principal curvature ([21]). It is shown in [1] that $\mu_0$ is in fact the mean curvature vector of $M$.

# 2 Preliminaries

We review in this section results on Binary Differential Equations that are needed in the paper and set some notation about surfaces endowed with a self-adjoint operator.

## 2.1 Binary Differential Equations (of order 2)

The pairs of foliations that we study in this paper are given by Binary Differential Equations (BDEs). These are equations of the form

$$a(u,v)dv^2 + 2b(u,v)dvdu + c(u,v)du^2 = 0, \tag{1}$$

where the coefficients $a, b, c$ are smooth functions on some open set $U \subset \mathbb{R}^2$ (here smooth means $C^\infty$). BDEs determine a pair of transverse foliations away from the discriminant, which is the set of points where the function $\delta = b^2 - ac$ vanishes. The pair of foliations together with the discriminant are called the *configuration* of the solutions of the BDE. In all the figures in this paper, we draw one foliation in continuous line and the other in dashed line. The discriminant is drawn in thick black.

We consider here topological equivalence among BDEs and say that two BDEs are topologically equivalent if there is a local homeomorphism in the plane taking the configuration of one equation to the configuration of the other. We describe below some singularities of BDEs that are of interest in this paper (see [20] for a survey article and references). We suppose the point of interest to be the origin. There are two cases to consider depending on whether all the coefficients of the BDE vanish or not at the origin.

One approach for investigating BDEs with coefficients not all vanishing at the origin consists of lifting the bi-valued direction field defined in the plane to a single direction field $\xi$ on the surface $N = F^{-1}(0) \subset \mathbb{R}^3$, where

$$F(u,v,p) = a(u,v)p^2 + 2b(u,v)p + c(u,v), \ p = \frac{dv}{du}$$

(we assume here, without loss of generality, that $a(0,0) \neq 0$, that is, the direction $du = 0$ is not locally a solution of the BDE). If $\xi$ is regular, then the BDE is locally smoothly equivalent to $dy^2 - xdx^2 = 0$ ([8]), i.e., it can be transformed to the model BDE by a smooth local change of coordinates in the plane and multiplication by a non-zero function. In this case, the solution curves of the equation form a family of cusps, with the cusps tracing the discriminant curve.

If $\xi$ has an elementary singularity (saddle/node/focus), then the corresponding point in the plane is called a *folded (non-degenerate elementary) singularity* of the BDE. At folded singularities, and when $\xi$ has non-resonant eigenvalues, the equation is locally smoothly equivalent to ([9])

$$dy^2 + (-y + \lambda x^2)dx^2 = 0, \ \lambda \neq 0, \frac{1}{16}.$$

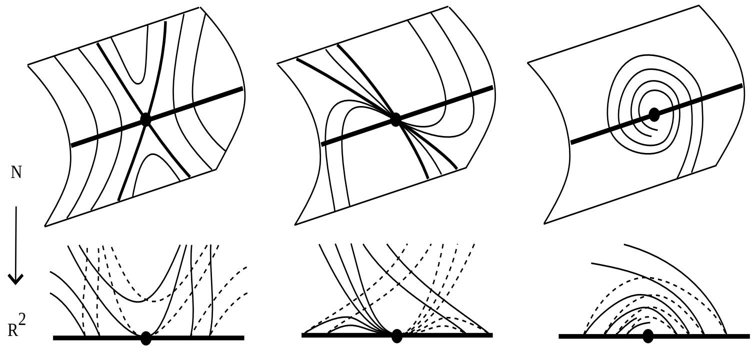


Figure 3: Folded singularities: saddle (left), node (centre), focus (right).

There are three stable topological models at folded singularities: a folded saddle, a folded node and a folded focus; Figure 3 (see [9] for references). These can be modelled by taking a fixed value of $\lambda$ in the above BDE, with $\lambda < 0$ for the folded saddle, $0 < \lambda < \frac{1}{16}$ for the folded node and $\frac{1}{16} < \lambda$ for the folded focus. The index of a folded saddle is defined to be $-\frac{1}{2}$ and that of a folded node or focus to be $+\frac{1}{2}$.

The folded singularities are completely determined by the 2-jet of $F$ at the origin. (The $k$-jet of a smooth map $f$ at a point $p$ is the Taylor polynomial of degree $k$ of $f$ at $p$ and is denoted by $j^k f(p)$, or simply by $j^k f$.) Following Lemma 2.1 in [5], if we write

$$j^2F = a_0p^2 + 2(b_1x + b_2y)p + (c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2),$$

then the origin is a folded singularity if and only if

$$c_1 = 0,\ c_2 \neq 0,\ \lambda = \frac{1}{4c_2^2}(4a_0c_3 - b_1^2 - b_1c_2) \neq 0, \frac{1}{16}.$$

The BDE is then topologically equivalent to $dy^2 + (-y + \lambda x^2)dx^2 = 0$, with $\lambda$ as above.

We also require in this paper models for non-stable singularities. Those of interest here occur when the discriminant function $\delta$ has a Morse singularity $A_1^{\pm}$. (We say that a function $f$ has a Morse singularity at the origin if it is $\mathcal{R}$-equivalent to $\pm(u^2 \pm v^2)$. That is, there exists a local diffeomorphism $h$ of the source such that $f \circ h^{-1}(u, v) = \pm(u^2 + v^2)$ and the singularity is labelled $A_1^+$, or $f \circ h^{-1}(u, v) = \pm(u^2 - v^2)$ and the singularity is labelled $A_1^-$. At an $A_1^+$-singularity, the set $f^{-1}(0)$ is locally a point and at an $A_1^-$-singularity it is a node, i.e., a union of two smooth curves intersecting transversally at the origin.) BDEs with discriminant having a Morse singularity are generically topologically equivalent to

$$dy^2 + (\pm x^2 \pm y^2)dx^2 = 0,$$

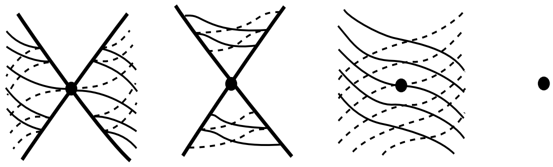

Figure 4: Topological configurations of Morse Type 1 BDEs in the following order from left to right: $A_1^-$ saddle type, $A_1^-$ focus type, $A_1^+$ saddle type, $A_1^+$ focus type.

and are labelled Morse Type 1 BDEs ([3]; see Figure 4). The Morse Type 1 singularities are distinguished by the type of the singularity of the discriminant, $A_1^+$ (isolated point) or $A_1^-$ (node), and by the type of the folded singularities that appear in a generic deformation (two folded saddles or foci), see [3].

When the coefficients of the BDE all vanish at the origin, the bi-valued field in the plane is lifted to a single direction field $\xi$ on a surface

$$N = \{(x, y, [\alpha : \beta]) \in \mathbb{R}^2, 0 \times \mathbb{R}P^1 : a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\}.$$

If we consider the affine chart $p = \beta/\alpha$ (we also need to consider the chart $q = \alpha/\beta$) and set $F(u, v, p) = a(u, v)p^2 + 2b(u, v)p + c(u, v)$, then the lifted direction field is parallel to the vector field $\xi = F_p\partial/\partial u + pF_p\partial/\partial v - (F_u + pF_v)\partial/\partial p$. The whole exceptional fibre $(0, 0) \times \mathbb{R}P^1$ is an integral curve of $\xi$. The surface $N$ is regular along the exceptional fibre if and only if the discriminant function $\delta$ of the BDE has a Morse singularity ([6]). If $j^1(a, b, c) = (a_1x + a_2y, b_1x + b_2y, c_1x + c_2y)$, then the singularities of $\xi$ on the exceptional fibre are given by the roots of the cubic

$$\phi(p) = (F_u + pF_v)(0, 0, p) = a_2p^3 + (2b_2 + a_1)p^2 + (2b_1 + c_2)p + c_1.$$

The eigenvalues of the linear part of $\xi$ at a singularity are $-\phi'(p)$ and $\alpha_1(p)$ (see [6] for details), where

$$\alpha_1(p) = 2(a_2p^2 + (b_2 + a_1)p + b_1).$$

These are non zero if $\phi$ has distinct roots and these are distinct from those of $\alpha_1$. If this is the case, the topological models of the BDE are completely determined by the singularity type of the discriminant ($A_1^+$ or $A_1^-$), the number (1 or 3) and the type (saddle or node) of the singularities of $\xi$ (see [6] for the analytic case and Remark 2 in [22] for the smooth case). We label such BDEs Morse Type 2 BDEs (MT2). There are three generic models of an MT2 BDE when the discriminant has a Morse singularity of type $A_1^+$. These are as in Figure 1 where the labels there indicate the number and type of the singularities of $\xi$ ($S$ for saddle and $N$ for node). There are five generic models of an MT2 BDE when the discriminant has a Morse singularity of type $A_1^-$. These are as in Figure 2.

## 2.2 Surfaces

Let $M$ be a smooth surface endowed with a metric $g$. We say that a vector $\boldsymbol{v} \in T_pM$ is *spacelike* if $g(\boldsymbol{v},\boldsymbol{v}) > 0$, *lightlike* if $g(\boldsymbol{v},\boldsymbol{v}) = 0$ and *timelike* if $g(\boldsymbol{v},\boldsymbol{v}) < 0$. The norm of a vector $\boldsymbol{v} \in T_pM$ is defined by $\|\boldsymbol{v}\| = \sqrt{|g(\boldsymbol{v},\boldsymbol{v})|}$.

Let $\boldsymbol{x} : U \to M$ be a local parametrisation of $M$, where $U$ is an open subset of $\mathbb{R}^2$. The first fundamental form of $M$ at a point $p$ is the quadratic form $\mathrm{I}_p : T_pM \to \mathbb{R}$ given by $\mathrm{I}_p(\boldsymbol{v}) = g(\boldsymbol{v},\boldsymbol{v})$. If $p \in \boldsymbol{x}(U)$ and $\boldsymbol{v} = a\boldsymbol{x}_u + b\boldsymbol{x}_v$, then $\mathrm{I}_p(\boldsymbol{u}) = Ea^2 + 2Fab + Gb^2$, where

$$E = g(\boldsymbol{x}_u,\boldsymbol{x}_u), \quad F = g(\boldsymbol{x}_u,\boldsymbol{x}_v), \quad G = g(\boldsymbol{x}_v,\boldsymbol{x}_v)$$

are the coefficients of $\mathrm{I}_p$ with respect to the parametrisation $\boldsymbol{x}$.

Given a self-adjoint operator $A : TM \to TM$, we denote by $A_p$ the restriction of $A$ to $T_pM$. If $\boldsymbol{v} = a\boldsymbol{x}_u + b\boldsymbol{x}_v$, then $\langle A_p(\boldsymbol{v}),\boldsymbol{v}\rangle = la^2 + 2mab + nb^2$, where

$$l = g(A_p(\boldsymbol{x}_u),\boldsymbol{x}_u),\ m = g(A_p(\boldsymbol{x}_u),\boldsymbol{x}_v) = g(A_p(\boldsymbol{x}_v),\boldsymbol{x}_u),\ n = g(A_p(\boldsymbol{x}_v),\boldsymbol{x}_v)$$

are referred to as the coefficients of $A_p$.

Let $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be the matrix of $A_p$ (which we will still denote by $A_p$) with respect to the basis $\{\boldsymbol{x}_u,\boldsymbol{x}_v\}$. Calculating $g(A_p(\boldsymbol{x}_u),\boldsymbol{x}_u)$, $g(A_p(\boldsymbol{x}_u),\boldsymbol{x}_v)$ and $g(A_p(\boldsymbol{x}_v),\boldsymbol{x}_v)$ we get the following systems of linear equations in $a, b$ and $c, d$

$$\begin{array}{rclcrcl} aE + bF &=& l & \qquad & cE + dF &=& m \\ aF + bG &=& m & \qquad & cF + dG &=& n. \end{array} \tag{2}$$

We require some genericity concept in this paper. (We shall impose some genericity conditions on $g$ when it is singular, see Proposition 4.1.) We denote by $\mathcal{S}$ the set of self adjoint operators on $(M, g)$, which is a subset of smooth maps $TM \to TM$. The set $\mathcal{S}$ is given the induced Whitney $C^\infty$-topology. We say that a self-adjoint operator $A$ is *generic* if it belongs to a residual subset of $\mathcal{S}$.

# 3 Pairs of foliations on Lorentzian surfaces

We suppose in this section that the metric $g$ is Lorentzian on $M$ (this includes the case of a region of a surface where the metric is Lorentzian). Given a local parametrisation $\boldsymbol{x}$ of $M$, we have $EG - F^2 \neq 0$, so we can solve the linear systems (2) to get

$$A_p = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} l & m \\ m & n \end{pmatrix}. \tag{3}$$

We call

$$K(p) = \det(A_p) = \frac{ln - m^2}{EG - F^2}$$

the $A$-*Gaussian curvature* of $M$ at $p$. The set of points where this curvature vanishes is labelled the $A$-*parabolic set.*

Because $g$ is Lorentzian, there are two lightlike directions in $T_pM$ at any point $p \in \boldsymbol{x}(U)$. These are the solutions of $\mathrm{I}_p(\boldsymbol{v}) = 0$ and yield a pair of smooth direction fields on $\boldsymbol{x}(U)$. One can show the following (the proof is identical to the Euclidean case; see for example [7] page 216).

**Theorem 3.1** *Let $(M, g)$ be a Lorentzian surface. At any point $p \in M$, there is a local parametrisation of a neighbourhood $V$ of $p$, such that for any $p' \in V$, the coordinate curves through $p'$ are tangent to the lightlike directions. Equivalently, there exist a local parametrisation $\boldsymbol{x} : U \to V \subset M$ with $E = G = 0$ in $U$.*

We consider now the three pairs of foliations associated to $A$.

## 3.1 Lines of $A$-principal curvature

The self-adjoint operator $A_p$ does not always have real eigenvalues. When it does, we denote them by $\kappa_i$, $i = 1, 2$ and call them the $A$-*principal curvatures.* Then, $K(p) = \kappa_1(p)\kappa_2(p)$. The eigenvectors of $A_p$ are called the $A$-*principal directions* and the integral curves of their associated line fields are called *the lines of $A$-principal curvature.* The equation of the lines of $A$-principal curvature is analogous to that of a surface in $\mathbb{R}^3$, and is given by

$$\begin{vmatrix} dv^2 & -dvdu & du^2 \\ E & F & G \\ l & m & n \end{vmatrix} = 0,$$

equivalently, by

$$(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)du^2 = 0. \tag{4}$$

The discriminant function of this equation is

$$\delta(u, v) = \big((Gl - En)^2 - 4(Gm - Fn)(Fl - Em)\big)(u, v).$$

We label the locus of points where $\delta(u, v) = 0$ the $A$-*Lightlike Principal Locus* ($LPL$). We have the following, which is a generalisation of a result in [15] for Lorentzian surfaces in the de Sitter space $S^3_1$.

**Proposition 3.2** (1) *For a generic self-adjoint operator $A$ on a Lorentzian surface $(M, g)$, the LPL is a curve on $M$. It can be characterised as the set of points on $M$ where the two $A$-principal directions coincide and become a lightlike direction.*

(2) *The LPL divides the surface into two regions. In one of them there are no principal directions and in the other there are two distinct principal directions at each point. In the latter case, the principal directions are orthogonal and one is spacelike while the other is timelike.*

We have an extra information about the $LPL$.

**Proposition 3.3** *For a generic self-adjoint operator $A$ on a Lorentzian surface $(M,g)$, the LPL is a smooth curve except possibly at isolated points where it has Morse singularities of type $A_1^{-1}$ (node). The singular points are where $A_p$ is a multiple of the identity. For this reason, we label them $A$-*timelike umbilic points*.*

*Proof.* We take a local parametrisation as in Theorem 3.1. Then, the equation of the lines of $A$-principal curvature becomes

$$ndv^2 - ldu^2 = 0.$$

The discriminant function of this equation is $\delta(u,v) = (ln)(u,v)$ and its zero set is the $LPL$. It is clear that this is a curve when $A$ is a generic self-adjoint operator. Generically, this curve is smooth unless $l(q) = n(q) = 0$ at some point $q \in U$. (The curve can also be singular if $l(q) = l_u(q) = l_v(q), n(q) \neq 0$ or $n(q) = n_u(q) = n_v(q), l(q) \neq 0$, but this is not generic as we have three equations with two unknowns.) The singularities of the $LPL$ are in general isolated points on this set. At such points the $LPL$ has generically a Morse singularity of type $A_1^-$.

The matrix of $A_p$, with $p = \boldsymbol{x}(q)$, with respect to the above parametrisation is

$$-\frac{1}{F^2}\begin{pmatrix} 0 & -F \\ -F & 0 \end{pmatrix}\begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

It is a multiple of the identity at $q$ if and only if $l(q) = n(q) = 0$. □

We seek the local topological configurations of the lines of $A$-principal curvature. Away from the $LPL$, we either have locally a pair of transverse foliations or no lines of $A$-principal curvature. We analyse the configurations at points on the $LPL$ of a generic $A$. (See §2.1 for terminology on BDEs.)

**Theorem 3.4** (1) *At regular points of the LPL, the lines of $A$-principal curvature form a family of cusps with the cusps tracing the LPL, except maybe at some isolated points on this curve. At such points, the equation of the lines of $A$-principal curvature has generically a folded singularity of type saddle, node or focus (Figure 3).*

(2) *At a timelike umbilic point, the BDE of the lines of $A$-principal curvature has generically a Morse Type 2 singularity of type $A_1^-$. All the five generic cases of such singularities can occur (Figure 2).*

*Proof.* We take a local parametrisation as in Theorem 3.1, so that the BDE of the lines of $A$-principal curvature is given by $ndv^2 - ldu^2 = 0$.

(1) At a regular point $p_0 = \boldsymbol{x}(q_0)$ of the $LPL$, we have either $l(q_0)$ or $n(q_0)$ non zero. Assume that $n(q_0) \neq 0$ and write the equation at $q_0$ in the form

$$dv^2 - \frac{l}{n}du^2 = 0.$$

We can now use the criteria for recognition of the singularities of a BDE (see §2.1). The equation is locally smoothly equivalent to $dv^2 - udu^2 = 0$ if and only if $l(q_0) = 0$ and $\frac{\partial l}{\partial u}(q_0) \neq 0$. In this case the configuration is a family of cusps. When $l(q_0) = \frac{\partial l}{\partial u}(q_0) = 0$, the BDE has a folded saddle/node/focus singularity if and only if

$$\frac{\partial l}{\partial v}(q_0) \neq 0 \text{ and } \lambda = \frac{n(q_0)\frac{\partial^2 l}{\partial u^2}(q_0)}{2\frac{\partial l}{\partial v}(q_0)} \neq 0, \frac{1}{16}.$$

It is clear that the three types can occur.

(2) If $p_0$ is a timelike umbilic point, $l(q_0) = n(q_0) = 0$. Then, the coefficients of the BDE vanish at $q_0$. We proceed following §2.1. As the $LPL$ has a Morse singularity of type $A_1^-$, the equation has generically a Morse Type 2 singularity of type $A_1^-$ (§2.1). We assume that $q_0$ is the origin. If we write $j^1 l = l_1 u + l_2 v$ and $j^1 n = n_1 u + n_2 v$, then the singularities of the lifted field $\xi$ are at the roots of the cubic $\phi = n_2 p^3 + n_1 p^2 - l_2 p - l_1$. We need $\phi$ to have simple roots. The eigenvalues of the linear part of $\xi$ at the roots of $\phi$ are $-\phi'(p)$ and $\alpha_1(p)$, where $\alpha_1(p) = 2p(n_2 p + n_1)$ (see §2.1). We also need $\alpha_1$ not to vanish at the roots of $\phi$. The above conditions are satisfied at timelike umbilic points of generic self-adjoint operators, so $\xi$ has either saddle or node singularities at the roots of $\phi$. It is clear that we can have all the five possible generic cases in Figure 2. □

### 3.2 $A$-Asymptotic curves

We shall say that a direction $\boldsymbol{v} \in T_pM$ is *$A$-asymptotic* if $g(A_p(\boldsymbol{v}), \boldsymbol{v}) = 0$. It follows that the $A$-asymptotic curves (whose tangent at all points are $A$-asymptotic directions) are solutions of the BDE

$$ndv^2 + 2mdudv + ldu^2 = 0. \tag{5}$$

The discriminant of equation (5) is the set of points where $m^2 - nl$ vanishes, which is the $A$-parabolic set. As $EG - F^2 < 0$, there are two distinct $A$-asymptotic directions in the region $K > 0$ and no $A$-asymptotic directions in the region $K < 0$ (the opposite happens if $g$ is Riemannian). On the $A$-parabolic set there is a unique double $A$-asymptotic direction. The $A$-parabolic set of a generic $A$, when not empty, is a smooth curve. We consider now generic self-adjoint operators.

**Proposition 3.5** (1) *An $A$-asymptotic direction at a point $p$ on a Lorentzian surface $(M, g)$ is also an $A$-principal direction at $p$ if and only if $p$ is an $A$-parabolic point or a point on the $LPL$. On the $LPL$, the $A$-asymptotic direction is lightlike.*

(2) *The $A$-parabolic set and the $LPL$ are tangential at their points of intersection. On one side of such points the unique $A$-asymptotic direction on the $A$-parabolic set is spacelike and on the other side it is timelike.*

*Proof.* We take a local parametrisation as in Theorem 3.1. Then, the $LPL$ is given by $nl = 0$ and the $A$-parabolic set by $m^2 - nl = 0$.

(1) The BDE of the lines of $A$-principal curvature is given by $ndv^2 - ldu^2 = 0$. Subtracting this from equation (5) yields $du(ldu + mdv) = 0$. The lightlike direction $du = 0$ is both $A$-asymptotic and $A$-principal if and only if $n(q) = 0$, if and only if $p = \boldsymbol{x}(q)$ is on the $LPL$. The direction $ldu + mdv = 0$ is both $A$-asymptotic and $A$-principal if and only if $(m^2 - nl)(q) = 0$, if and only if $p$ is an $A$-parabolic point.

(2) The singular points of the $LPL$ are generically not on the $A$-parabolic curve (otherwise $n = l = m = 0$). Suppose, without loss of generality, that $n(q) = 0$ and $l(q) \neq 0$ at the point in consideration. Then, the $LPL$ is given locally by $n = 0$. The point $p = \boldsymbol{x}(q)$ is also an $A$-parabolic point if and only if $m(q) = 0$. The gradient of $m^2 - nl$ at $q$ is $(-n_u l, -n_v l)(q)$ and is parallel to the gradient of $n$ at $q$. Therefore, the two curves are tangential at their intersection point. Near $p$ on the $A$-parabolic set, the $A$-asymptotic direction is along $(m, -n)$ in the parameter space and along $m\boldsymbol{x}_u - n\boldsymbol{x}_v$ on the surface. We have $g(m\boldsymbol{x}_u - n\boldsymbol{x}_v, m\boldsymbol{x}_u - n\boldsymbol{x}_v) = -2nmF$, and this changes sign at $q$ for a generic $A$. □

**Theorem 3.6** *On the $A$-parabolic curve, the $A$-asymptotic curves form a family of cusps with the cusps tracing the $A$-parabolic curve, except may be at some isolated points on this curve. At such points, the equation of the $A$-asymptotic curves has generically a folded singularity of type saddle, node or focus* (*Figure* 3).

The proof is similar to that of Theorem 3.4(1) and is omitted.

**Remark 3.7** The point of tangency of the $A$-parabolic curve with the $LPL$ is not, in general, a folded singularity of the $A$-asymptotic curves BDE.

## 3.3 $A$-Characteristic curves

We use the results in [4] to define an $A$-characteristic direction. A BDE (1) can be viewed as a quadratic form and represented at each point in $U$ by the point $(a(u, v) : 2b(u, v) : c(u, v))$ in the projective plane. Let $\Gamma$ denote the conic of degenerate quadratic forms. To a point in the projective plane is associated a unique polar line with respect to $\Gamma$, and vice-versa. A triple of points is called a self-polar triangle if the polar line of any point of the triple contains the remaining two points. We first observe that the $A$-principal curves BDE belongs to the polar line of the $A$-asymptotic curves BDE. In fact, it is the only BDE on this polar line whose solutions, when they exist, are orthogonal (this follows in the same way as the proof of Theorem 2.2 in [21]). We call the intersection of the polar lines of the $A$-principal and the $A$-asymptotic curves BDEs the *$A$-characteristic curves BDE* (so, the three equations represent at each point on $M$ a self-polar triangle in the projective plane).

The $A$-characteristic curves BDE is given as the jacobian of the $A$-asymptotic and $A$-principal curves BDE and has the following expression

$$(2m(mG-nF)-n(lG-nE))dv^2+2(m(lG+nE)-2lnF)dvdu+(l(lG-nE)-2m(lF-mE))du^2=0.$$

If we take a parametrisation of the surface as in Theorem 3.1, then the above BDE becomes

$$mndv^2+2lndvdu+mldu^2=0. \qquad (6)$$

The discriminant of this equation is $ln(ln-m^2)=0$, which is the union of the $A$-parabolic set and the $LPL$. We analyse now the configurations of the $A$-characteristic curves of a generic self-adjoint operator $A$.

**Theorem 3.8** (1) *On the A-parabolic curve and away from its points of tangency with the LPL, the A-characteristic curves form a family of cusps with the cusps tracing the A-parabolic curve, except maybe at some isolated points on this curve. At such points, the BDE of the A-characteristic curves has generically a folded singularity of type saddle, node or focus* (*Figure* 3). *The folded singularities of the A-characteristic and A-asymptotic BDEs coincide and have opposite indices.*

(2) *At a point of tangency of the A-parabolic curve and the LPL, the A-characteristic BDE is topologically equivalent to*

$$vdv^2+2udvdu+v^3du^2=0 \quad \textit{Figure 5 C, or to}$$
$$vdv^2-2udvdu+v^3du^2=0 \quad \textit{Figure 5 D.}$$

(3) *At a A-timelike umbilic point, the characteristic BDE has generically a Morse Type 2 singularity of type $A_1^-$. All the generic five topological models of such singularities can occur* (*Figure* 2).

*Proof.* (1) We follow the same setting as in the proof of Theorem 3.6 and consider equation (6) of the $A$-characteristic curves. We take $q_0$ to be the origin and suppose that $l(q_0)n(q_0)\neq 0$ and $(m^2-ln)(q_0)=0$. We make the change of variable $v\to v-m_0/n_0u$ in order to transform the 2-jet of the BDE to the form given in §2.1. We can then read the conditions for the equation to a have a folded singularity. We do the same for the equation of the $A$-asymptotic curves BDE. We find that, at the folded singularity, the $A$-characteristic BDE is topologically equivalent to $dy^2+(-y+\lambda x^2)dx^2=0$ and the $A$-asymptotic BDE to $dy^2+(-y-\lambda x^2)dx^2=0$, where $\lambda$ depends on the 2-jets of $l,m,n$. It is clear that the two BDEs have opposite indices at their folded singularities.

(2) The point $q_0$ is on both the $LPL$ and the $A$-parabolic set if and only if $l(q_0)=m(q_0)=0$ or $m(q_0)=n(q_0)=0$. We can assume that $l(q_0)=m(q_0)=0$ and $n(q_0)\neq 0$, otherwise the point $p_0$ is a timelike umbilic point. Then, the discriminant of the BDE

of the $A$-characteristic BDE has generically an $A_3^-$-singularity (i.e., it is $\mathcal{R}$-equivalent to $\pm(u^2 - v^4)$) . Let $(a, b, c) = (mn, nl, ml)$ denote the coefficients of the BDE (6). We can make a change of coordinates in the source and multiply by non-zero constant so that $j^1 a = \alpha u + v$, $j^1 b = \pm u$ and $j^1 c = 0$, for some $\alpha$ depending on the coefficients of the 1-jets of $l, m, n$. The result follows by applying Theorem 3.9 below.

(3) If $p_0 = \boldsymbol{x}(q_0)$ is a timelike umbilic point, $l(q_0) = n(q_0) = 0$, and generically $m(q_0) \neq 0$. Then, the coefficients of the BDE (6) vanish at $q_0$. As the $LPL$, which is locally the discriminant of the equation, has a Morse singularity of type $A_1^-$, the equation has generically a Morse Type 2 singularity of type $A_1^-$. We assume again that $q_0$ is the origin. The 1-jet of the BDE is equivalent to $j^1 n dv^2 + j^1 l du^2$. Following the notation in §2.1, the singularities of the lifted field $\xi$ are the roots of the cubic $\phi = n_2 p^3 + n_1 p^2 + l_2 p + l_1$. The eigenvalues of the linear part of $\xi$ at the roots of $\phi$ are $-\phi'(p)$ and $\alpha_1(p)$, where $\alpha_1(p) = 2p(n_2 p + n_1)$. We need $\phi$ to have simple roots and $\alpha_1$ not to vanish at these roots, which is the case at timelike umbilic points of generic self-adjoint operators. It is clear that we can have all the five possible generic cases in Figure 2. $\square$

In the proof of Theorem 3.8, we require the topological models of BDEs with 1-jet $(\alpha u + v, \pm u, 0)$. The following result provide these models. The proof uses the blowing-up technique in [13, 19, 22, 23] and is omitted.

**Theorem 3.9** *A BDE with 1-jet equivalent to* $(\alpha u + v, \pm u, 0)$ *and with a discriminant with an* $A_3^\pm$*-singularity is generically topologically equivalent to one of the following cases.*

*(i) Discriminant has an* $A_3^-$*-singularity:*

$$\begin{array}{ll} v dv^2 + 2u dv du + v^3 du^2 = 0 & \textit{Figure 5 A, or to} \\ v dv^2 - 2u dv du + v^3 du^2 = 0 & \textit{Figure 5 B.} \end{array}$$

*(ii) Discriminant has an* $A_3^+$*-singularity*

$$\begin{array}{ll} v dv^2 + 2u dv du - v^3 du^2 = 0 & \textit{Figure 5 C, or to} \\ v dv^2 - 2u dv du - v^3 du^2 = 0 & \textit{Figure 5 D.} \end{array}$$

## 3.4 Surfaces in the de Sitter space $S_1^3$

The *Minkowski space* $(\mathbb{R}_1^{n+1}, \langle , \rangle)$ is the vector space $\mathbb{R}^{n+1}$ endowed with the metric $g = \langle , \rangle$ of signature 1, given by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^n x_i y_i$, where $\boldsymbol{x} = (x_0, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}_1^{n+1}$. (We will restrict to the case $n = 3$ in this subsection and consider the case $n = 2$ in §4.4.)

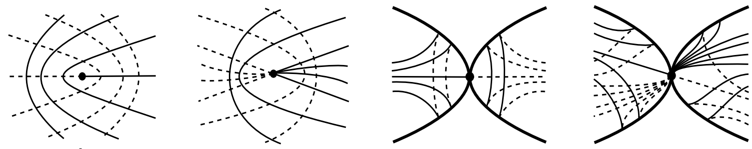


Figure 5: Topological models of BDEs with 1-jets equivalent to $(\alpha u+v, \epsilon u, 0)$, $\epsilon = \pm 1$, and with discriminant with an $A_3$-singularity.

Given a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number $c$, the hyperplane with pseudo normal $\boldsymbol{v}$ is defined by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v}\rangle = c\}.$$

We say that $HP(\boldsymbol{v}, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We also say that a two dimensional vector space is *spacelike* if all its vectors are spacelike, *timelike* if it has a spacelike and a timelike vector and *lightlike* otherwise.

The de Sitter space is the pseudo-sphere $S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$.

We consider timelike embedded surfaces $M$ in $S^3_1$ (so we take $n = 3$ above). These are embeddings with the property that the restriction of the metric $\langle , \rangle$ to $M$ is Lorentzian. Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S^3_1$ be a local parametrisation of $M$. Since $\langle \boldsymbol{x}, \boldsymbol{x}\rangle \equiv 1$, we have $\langle \boldsymbol{x}_u, \boldsymbol{x}\rangle \equiv 0$ and $\langle \boldsymbol{x}_v, \boldsymbol{x}\rangle \equiv 0$. We define the spacelike unit normal vector $\boldsymbol{e}(q)$ to $M$ at $p = \boldsymbol{x}(q)$ by

$$\boldsymbol{e}(q) = \frac{\boldsymbol{x}(q) \wedge \boldsymbol{x}_u(q) \wedge \boldsymbol{x}_v(q)}{\|\boldsymbol{x}(q) \wedge \boldsymbol{x}_u(q) \wedge \boldsymbol{x}_v(q)\|},$$

where $\wedge$ denotes the wedge product in $\mathbb{R}^4_1$. The *de Sitter Gauss map* of $M$ (of $\boldsymbol{x}(U)$ to be precise) is the map

$$\begin{array}{rccc} \mathbb{E}: & U & \to & S^3_1 \\ & q & \mapsto & \boldsymbol{e}(q) \end{array}$$

(see [14]). For any $p = \boldsymbol{x}(q) \in M$ and $\boldsymbol{v} \in T_pM$, one can show that $D_{\boldsymbol{v}}\mathbb{E}(q) \in T_pM$, where $D_{\boldsymbol{v}}$ denotes the covariant derivative with respect to the tangent vector $\boldsymbol{v}$. One can also show that $A_p(\boldsymbol{v}) = -D_{\boldsymbol{v}}\mathbb{E}(q)$ is a self-adjoint operator, called the *de Sitter shape operator*. We shall refer to the foliations associated to $A_p$ as the de Sitter foliations.

To apply the results in this section to the de Sitter shape operator, we require a modification of the notion of genericity. We consider the space of timelike embeddings of a surface $M$ into $S^3_1$, endowed with the Whitney $C^\infty$-topology. We say that a property is *generic* if it holds for a residual set of timelike embeddings of the surface. We can now apply the results in §3.1, 3.2, 3.3 to define and to obtain the local configurations of the de Sitter lines of principal curvature, asymptotic and characteristic curves (substituting in the statements "generic self-adjoint operator" with "generic timelike embedding").

**Remark 3.10** Let $\mathcal{C}(M, S^3_1)$ denote the space of timelike embeddings of an orientable smooth surface $M$ in the de Sitter space, endowed with the Whitney $C^\infty$-topology. It follows by Thom's transversality theorem that the set of embeddings with the properties (a)-(f) below form a residual subset of $\mathcal{C}(M, S^3_1)$.

(a) The de Sitter parabolic set, when not empty, is a smooth curve.

(b) The $LPL$, when not empty, is a smooth curve except maybe at isolated points points where it has Morse singularities of type $A_1^-$.

(c) The de Sitter parabolic set and the $LPL$ have ordinary tangency at their points of intersection.

(d) The local singularities of the BDE of the lines of de Sitter principal curvature are those described in Theorem 3.4.

(e) The local singularities of the BDE of the de Sitter asymptotic curves are those described in Theorem 3.6.

(f) The local singularities of the BDE of the de Sitter characteristic curves are those described in Theorem 3.8.

One can always construct a patch of a timelike surface in $S^1_3$ with one of the pairs of foliations described in this paper having one of its possible stable local singularities. In the following example, we find explicit expressions for the above foliations on a special surface. Consider the timelike surface $M$ given by $\boldsymbol{x} : \mathbb{R}^2 \to S^3_1$ with

$$\boldsymbol{x}(u,v) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), \cos(v), \sin(v)).$$

We have $\boldsymbol{x}_u = \frac{1}{\sqrt{2}}(\cosh(u), \sinh(u), 0, 0)$ and $\boldsymbol{x}_v = \frac{1}{\sqrt{2}}(0, 0, -\sin(v), \cos(v))$, so the coefficients of the first fundamental form are given by

$$E = -\frac{1}{2}, \quad F = 0, \quad G = \frac{1}{2}.$$

The de Sitter Gauss map is given by

$$\boldsymbol{e}(u,v) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), -\cos(v), -\sin(v)).$$

Therefore the coefficients of the second fundamental form are given by

$$l = \frac{1}{2}, \quad m = 0, \quad n = \frac{1}{2}.$$

It follows that the BDEs for the pairs of foliations described in this paper are as follows.

| | |
|---|---|
| The lines of de Sitter principal curvature: | $dvdu = 0,$ |
| The de Sitter asymptotic curves: | $dv^2 + du^2 = 0,$ |
| The de Sitter characteristic curves: | $dv^2 - du^2 = 0.$ |

Therefore the lines of de Sitter principal curvature are given $u = constant$ and $v = constant$; there are no de Sitter asymptotic curves, and the de Sitter characteristic curves are given by $v \pm u = constant$.

**Remark 3.11** The de Sitter principal directions can be detected via singularity theory. It is shown in [15] that, away from the LPL, the folding map of the surface with respect to a hyperplane has a degenerate singularity if and only if the normal to the hyperplane is parallel to a de Sitter principal direction. The *LPL* and the folded singularities of the lines of de Sitter principal curvature can also be detected via the singularities of projections along lightlike geodesics [16].

For the de Sitter asymptotic directions, it is shown in [16] that a projection along parallel geodesics to an orthogonal quadric has a singularity of type cusp or worse at $p$ if and only if the tangent direction to the geodesic at $p$ is along a de Sitter asymptotic direction.

# 4 Surfaces with degenerate metrics

Let $(M, g)$ be a smooth surface endowed with a singular metric. (For example, the restriction of the metric $g$ in the Minkowski space $\mathbb{R}^3_1$ to any closed smooth surface is degenerate at some points on the surface.) We assume that the *Locus of Degeneracy* (LD) of the metric is a smooth curve and splits the surface locally into regions where the metric is Riemannian on one side of the LD and Lorentzian on the other side.

The lightlike directions, whose integral curves we call here the *lightlike curves*, are given by

$$Gdv^2 + 2Fdvdu + Edu^2 = 0.$$

The discriminant of the above BDE is the *LD*. The unique lightlike direction on the *LD* is in general transverse to the *LD* but can be tangent to it at isolated points. We apply the results in §2.1 to obtain the following information.

**Proposition 4.1** *For a residual set of singular metrics on $M$, the lightlike curves form a family of cusps along the LD, except maybe at some isolated points on this curve where they have a folded singularity of type saddle, node or focus* (*Figure 3*).

We suppose from now on that the singular metric is generic, i.e., the *LD* is a smooth curve and the lightlike curves have isolated folded saddle, node or focus singularities. The following result follows using standard techniques, so the proof is omitted.

**Theorem 4.2** *There is a local parametrisation of $M$ at $p \in LD$, such that the set of lightlike directions in $T_pM$ is given by $\mathbb{R}.\boldsymbol{x}_u$, i.e., $E = F = 0$ on the LD.*

Let $A$ be a self-adjoint operator on $M$ and denote by $l, m, n$ its coefficients with respect to a local parametrisation $\boldsymbol{x}$ (see §2.2).

**Theorem 4.3** *For a parametrisation as in Theorem 4.2, $l = 0$ and $m = 0$ on the LD. Consequently, the unique lightlike direction at $p$ on the LD is an eigenvector at $p$ of any self-adjoint operator on $M$.*

*Proof.* We take a local parametrisation as in Theorem 4.2. Then, $g(\boldsymbol{x}_u, \boldsymbol{v}) = 0$ at any point $p$ on the $LD$ and for any $\boldsymbol{v} \in T_pM$. In particular, $l = g(\boldsymbol{x}_u, A_p(\boldsymbol{x}_u)) = 0$ and $m = g(\boldsymbol{x}_u, A_p(\boldsymbol{x}_v)) = g(A_p(\boldsymbol{x}_u), \boldsymbol{x}_v) = 0$. Consequently, $g(\boldsymbol{v}, A_p(\boldsymbol{x}_u)) = 0$, for any $\boldsymbol{v} \in T_pM$. Therefore, $A_p(\boldsymbol{x}_u)$ is a lightlike direction. As the set of lightlike directions in $T_pM$ is 1-dimensional when $p \in LD$, we have $A_p(\boldsymbol{x}_u) = \lambda \boldsymbol{x}_u$, for some scalar $\lambda$. $\square$

We show below how to extend the lines of $A$-principal curvature, the $A$-asymptotic and $A$-characteristic curves across the LD and study the local behaviour of the resulting foliations in a neighbourhood of a point on the $LD$.

## 4.1 Lines of $A$-principal curvature

On the $LD$, we have $EG - F^2 = 0$, so we cannot solve the systems (2). It follows from Theorem 4.3 that $b = 0$ and $d = \frac{n}{G}$. However, we cannot express $a, c$ in terms of the coefficients $E, F, G, l, m, n$. We proceed as follows, and take a local parametrisation as in Theorem 4.2. On the $LD$, which we assume to be a smooth curve, we have $E = F = l = m = 0$. Therefore, there exists a smooth function $\lambda$ such that $E = \lambda \tilde{E}$, $F = \lambda \tilde{F}$, $l = \lambda \tilde{l}$ and $m = \lambda \tilde{m}$, with $\lambda = 0$ on the $LD$. As the $LD$, which is now given by $\lambda(\lambda \tilde{F}^2 - \tilde{E}G) = 0$, is assumed to be a smooth curve, $\tilde{E}G$ does not vanish on it. Then, we can rewrite the systems of equations (2) as follows

$$\begin{array}{rclcrcl} a\tilde{E} + b\tilde{F} & = & \tilde{l} & \qquad & c\tilde{E} + d\tilde{F} & = & \tilde{m} \\ a\lambda\tilde{F} + bG & = & \lambda\tilde{m} & \qquad & c\lambda\tilde{F} + dG & = & n. \end{array} \tag{7}$$

This yield the following matrix of $A_p$, with respect to the chosen parametrisation,

$$A_p = \frac{1}{\tilde{E}G - \lambda\tilde{F}^2} \begin{pmatrix} G & -\tilde{F} \\ -\lambda\tilde{F} & \tilde{E} \end{pmatrix} \begin{pmatrix} \tilde{l} & \tilde{m} \\ \lambda\tilde{m} & n \end{pmatrix}.$$

The equation of the eigenvectors of the above matrix ($A$-lines of curvature) becomes

$$\lambda \left( (G\tilde{m} - \tilde{F}n)dv^2 + (G\tilde{l} - \tilde{E}n)dvdu + \lambda(\tilde{F}\tilde{l} - \tilde{E}\tilde{m})du^2 \right) = 0.$$

This means that the BDE has the $LD$ as a line of singularities. Dividing by $\lambda$ we get

$$(G\tilde{m} - \tilde{F}n)dv^2 + (G\tilde{l} - \tilde{E}n)dvdu + \lambda(\tilde{F}\tilde{l} - \tilde{E}\tilde{m})du^2 = 0. \tag{8}$$

Observe that equation (8) can also be obtained by direct substitutions in (4). The discriminant of (8) is $(G\tilde{l}-\tilde{E}n)^2-4\lambda(\tilde{F}\tilde{l}-\tilde{E}\tilde{m})(G\tilde{m}-\tilde{F}n)=0$, which is the $LPL$. When $\lambda=0$ and $G\tilde{l}-\tilde{E}n\neq 0$ the equation determines two directions on the $LD$, one of which is the unique lightlike direction. When $\lambda=G\tilde{l}-\tilde{E}n=0$, the $LPL$ intersects the $LD$. The two curves have (generically) ordinary tangency at their points of intersection. At such points the unique direction determined by (8) is lightlike and is tangent to the $LD$. Therefore, these points are the folded singularities of the lightlike curves.

**Theorem 4.4** *Let $g$ be a generic singular metric on $M$ and $A$ a self-adjoint operator on $(M,g)$. The lines of $A$-principal curvature can be extended to the LD where the tangent to one of the foliation is lightlike. For a generic $A$, the extended pair of foliation have folded saddle, node or focus singularities on the LD, and these occur at exactly the folded singularities of the lightlike curves.*

## 4.2 A-Asymptotic curves

We take a parametrisation as in Theorem 4.2, so that $l=m=0$ on the $LD$ (Theorem 4.3). With notation as in §4.1, the equation of the $A$-asymptotic curves becomes

$$ndv^2+2\lambda\tilde{m}dudv+\lambda\tilde{l}du^2=0, \tag{9}$$

and its discriminant, which is the $A$-parabolic set, is given by $\lambda(\tilde{m}^2-\tilde{l}n)=0$.

**Theorem 4.5** *Let $g$ be a generic singular metric on $M$ and $A$ a self-adjoint operator on $(M,g)$.*

(1) *The LD is a component of the $A$-parabolic set. For a generic $A$ and at regular points on the $A$-parabolic set, the $A$-asymptotic curves form a family of cusps except at the folded singularities of the lightlike curves where they have the folded singularities in Figure 3.*

(2) *At the singular points of the $A$-parabolic set on the LD, the BDE (9) has generically either a Morse Type 1 $A_1^-$-singularity or a Morse Type 2 $A_1^-$- singularity. The first two (resp. all five) configurations in Figure 4 (resp. Figure 2) can occur.*

*Proof.* (1) The discriminant of (9) is $\lambda(\tilde{l}n-\lambda\tilde{m}^2)=0$. On the $LD$ ($\lambda=0$), the unique $A$-asymptotic direction is lightlike. We then apply the results in §2.1.

(2) For a generic $A$, the $A$-parabolic set is singular when (i) $\lambda=\tilde{l}=0$ or (ii) $\lambda=n=0$. We assume the singular point to be the origin and write $j^1\lambda=\lambda_1u+\lambda_2v$, $j^1m=m_0j^1\lambda$, $j^1n=n_0+n_1u+n_2v$ and $j^1\tilde{l}=\tilde{l}_0+\tilde{l}_1u+\tilde{l}_2v$.

In case (i) the 2-jet of (9) is equivalent to $dv^2-(\frac{\lambda_1}{n_0^2}(m_0^2\lambda_1-\tilde{l}_1n_0)u^2+Auv+Bv^2)du^2$, where $A$ and $B$ depend on the coefficients of the of the 1-jets of $\lambda,\tilde{l},n$ and $m_0$. The BDE has a Morse Type 1 ($A_1^-$) singularity and the type (saddle, resp. focus) is determined by the sign of the coefficient of $u^2du^2$ (negative, resp. positive); see [3] and the first two figures in Figure 4.

In case (ii) all the coefficients of (9) vanish at the origin. The 1-jet of the coefficients of (9) is $(n_1u + n_2v, m_0(\lambda_1 u + \lambda_2 v), l_0(\lambda_1 u + \lambda_2 v))$, so the singularity is in general a Morse Type 2 ($A_1^-$), and all the five cases in Figure 2 can occur (see 2.1). □

## 4.3 A-Characteristic curves

We proceed as for the $A$-asymptotic BDE at points on the $LD$. With notation as in §4.1, the BDE of the $A$-characteristic curves becomes

$$\begin{aligned} &(2\lambda\tilde{m}(\tilde{m}G - n\tilde{F}) - n(\tilde{l}G - n\tilde{E}))dv^2 + \\ &2\lambda(\tilde{m}(\tilde{l}G + n\tilde{E}) - 2n\tilde{l}\tilde{F})dvdu + \lambda(\tilde{l}(\tilde{l}G - n\tilde{E}) - 2\lambda\tilde{m}(\tilde{l}\tilde{F} - \tilde{m}\tilde{E}))du^2 = 0 \end{aligned} \qquad (10)$$

**Theorem 4.6** *Let $g$ be a generic singular metric on $M$ and $A$ a self-adjoint operator on $(M, g)$.*

(1) *The LD is a component of the discriminant of the A-characteristic curves BDE. For a generic A, on the LD and away from the singular points of the discriminant, the A-characteristic curves form a family of cusps.*

(2) *At the singular points of the A-parabolic set, the BDE of the A-characteristic curves of a generic A have either a Morse Type 1 ($A_1^-$) or Morse Type 2 ($A_1^-$)- singularity. The possible two (resp. five) configurations in Figure 4 (resp. Figure 2) can occur.*

(3) *At the folded singularities of the lightlike curves the discriminant of the A-characteristic curves BDE of a generic A has an $A_3^-$-singularity (formed by the tangential curves LPL and the LD). The configurations there are topologically equivalent to the Morse Type 2 ($A_1^-$) singularities in Figure 2.*

*Proof.* For (1) and (2) we proceed as for Theorem 4.5. For (3), the coefficients of the BDE (10) do not all vanish at the folded singularities of the lightlike curves. The 2-jet of the BDE (10) is equivalent, by smooth changes of coordinates in the parameter space and multiplication by non-zero functions, to $dv^2 \pm u^2du^2$. One can show, following the methods highlighted in [20], that the BDE (10) is generically topologically equivalent to $dv^2 + (\pm u^2 + v^4)du^2 = 0$. The solution curves of these BDEs are topologically equivalent to those of the Morse Type 1 ($A_1^-$)-singularities in Figure 2. □

## 4.4 Surfaces in $\mathbb{R}^3_1$

We consider smooth surfaces $M$ in the Minkowski 3-space space $(\mathbb{R}^3_1, \langle,\rangle)$ (see §3.4). Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$ be a local parametrisation of $M$. Pei [17] defined an $\mathbb{R}P^2$-valued Gauss map on $M$. In $\boldsymbol{x}(U)$, this is the map $PN : \boldsymbol{x}(U) \to \mathbb{R}P^2$ which associates to $p$ the projectivisation of the vector $\boldsymbol{x}_u \times \boldsymbol{x}_v$. (As $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is lightlike on the $LD$, the usual Gauss map $\boldsymbol{x}_u \times \boldsymbol{x}_v/||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ is not defined on the $LD$.) Away from the $LD$, the $\mathbb{R}P^2$-valued Gauss map can be identified with the de Sitter Gauss map $\boldsymbol{x}(U) \to S_1^2$ on the Riemannian part of the surface and with the hyperbolic Gauss map $\boldsymbol{x}(U) \to H_+^2(-1)$ on its Lorentzian part. Both maps are given by $\boldsymbol{x}_u \times \boldsymbol{x}_v/||\boldsymbol{x}_u \times \boldsymbol{x}_v||$.

Away from the $LD$, the map $A_p(\boldsymbol{v}) = -D_{\boldsymbol{v}}(PN)(q)$ is a self-adjoint operator on $\boldsymbol{x}(U) \setminus LD$. However, this map does not extend to a self-adjoint operator on $M$ (see Remark 4.7). Therefore, we cannot use the results in §4.1, §4.2 and §4.3 to extend to the $LD$ the pairs of foliations associated to this self-adjoint-operator. (We denote such pairs by $PN$-pairs.) We proceed as follows.

We consider the de Sitter and hyperbolic Gauss maps $N = \boldsymbol{x}_u \times \boldsymbol{x}_v/||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ on $\boldsymbol{x}(U) \setminus LD$ and denote as before $l = -\langle N_u, \boldsymbol{x}_u\rangle = \langle N, \boldsymbol{x}_{uu}\rangle, \quad m = -\langle N_u, \boldsymbol{x}_v\rangle = \langle N, \boldsymbol{x}_{uv}\rangle, \quad n = -\langle N_v, \boldsymbol{x}_v\rangle = \langle N, \boldsymbol{x}_{vv}\rangle$. As the equations of the $PN$-asymptotic, $PN$-characteristic and $PN$-principal curves are homogeneous in $l, m, n$, we can multiply these coefficients by $||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ and substitute them by

$$\bar{l} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uu}\rangle, \quad \bar{m} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uv}\rangle, \quad \bar{n} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{vv}\rangle.$$

This substitution does not alter the $PN$-pairs of foliations on $\boldsymbol{x}(U) \setminus LD$. The new equations are defined on the $LD$ and define the same pairs of foliations associated to the de Sitter (resp. hyperbolic) Gauss map on the Riemannian (resp. Lorentzian) part of $\boldsymbol{x}(U)$. The extended $PN$-principal curves are given by

$$(G\bar{m} - F\bar{n})dv^2 + (G\bar{l} - E\bar{n})dudv + (F\bar{l} - E\bar{m})du^2 = 0.$$

The coefficient of $du^2$ vanishes on the $LD$ if we take a local parametrisation as in Theorem 4.2. Therefore, one of the $PN$-principal directions on the $LD$ is the double lightlike direction. The $LPL$ meets tangentially the $LD$ at points where $\bar{l} = 0$. On the $LD$, we have $\boldsymbol{x}_u \times \boldsymbol{x}_v = \mu \boldsymbol{x}_u$ for some smooth nowhere vanishing function $\mu$. Then, $\bar{l} = 0$ implies $\langle \boldsymbol{x}_u, \boldsymbol{x}_{uu}\rangle = 0$, that is $E_u = 0$. This means that the points of tangency of the LPL with the LD are exactly the folded singular points of the lightlike curves on $M$ (compare with Theorem 4.4). At such points, the lines of $PN$-principal curvature have generically folded singularities of type saddle, node or focus.

The $PN$-asymptotic and $PN$-characteristic curves also extend to the $LD$ using the new coefficients $\bar{l}, \bar{m}, \bar{n}$.

**Remark 4.7** *The parabolic set $\bar{m}^2 - \bar{l}\bar{n} = 0$ is not, in general, part of the LD (compare Theorem 4.5). This shows that the extended $PN$-pairs of foliations on $M$ do not, in general, come from a self-adjoint operator on $M$.*

Suppose that extended $PN$-pairs come from a self-adjoint operator $A$ on $M$ with matrix $[A_p]$ with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$. Then, $[A_p]$ is, up to multiplication by a non-zero function, in the form $\frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} \bar{l} & \bar{m} \\ \bar{m} & \bar{n} \end{pmatrix}$. Suppose that the degenerate lightlike direction on the $LD$ is $\boldsymbol{x}_u$, so the $LD$ is given by $E = F = 0$. If we write $E = \lambda\tilde{E}$ and $F = \lambda\tilde{F}$, where $\lambda = 0$ gives the $LD$, then the coefficients of the matrix $[A_p]$ extend smoothly to the $LD$ if and only if $\bar{l} = \bar{m} = 0$ on the $LD$, that is, if and only if $\bar{l} = \lambda\bar{\bar{l}}$ and $\bar{m} = \lambda\bar{\bar{m}}$ for some smooth functions $\bar{\bar{l}}$ and $\bar{\bar{m}}$.

On the $LD$, $\boldsymbol{x}_u \times \boldsymbol{x}_v = \mu\boldsymbol{x}_u$, for some smooth nowhere vanishing function $\mu$, so $\bar{l} = \frac{1}{2}\mu E_u$ and $\bar{m} = \frac{1}{2}\mu E_v$ on the $LD$. It follows that $E_u = E_v = 0$ on the $LD$, so $E = \lambda^2\tilde{\tilde{E}}$ for some smooth function $\tilde{\tilde{E}}$. Then, the discriminant ($LD$) of the lightlike curves is given by $\lambda^2(\tilde{\tilde{E}}G - \tilde{F}^2) = 0$. If $\tilde{\tilde{E}}G - \tilde{F}^2 \neq 0$, $\boldsymbol{x}(U) \setminus LD$ is either Riemannian or Lorentzian so we do not have a mixed metric structure on $\boldsymbol{x}(U) \setminus LD$.

We consider now the following example. Let $S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3_1 | x_0^2 + x_1^2 + x_2^2 = 1\}$ be the "Euclidean sphere" in $\mathbb{R}^3_1$. A normal vector to $S^2$ at $(x_0, x_1, x_2)$ is $\eta = (-x_0, x_1, x_2)$. This is lightlike if and only if $x_0^2 = \frac{1}{2}$ and $x_1^2 + x_2^2 = \frac{1}{2}$. Therefore, the $LD$ is the union of two circles. The tangent spaces at $(\pm 1, 0, 0)$ are spacelike and the tangent space at $(0, 1, 0)$ is timelike, so the $LD$ separates the sphere into three regions, the "middle"is Lorentzian and the top and bottom parts are Riemannian (think of the $x_0$-axis as the vertical axis in $\mathbb{R}^3_1$). We seek the extended $PN$-pairs of foliations on $S^2$. Consider the parametrisation $\boldsymbol{x} : U = (0, 2\pi) \times (0, \pi) \to S^2$ given by

$$\boldsymbol{x}(u, v) = (\cos v, \cos u \sin v, \sin u \sin v),$$

which covers the sphere minus a semi-circle. Then, $E = \sin^2 v$, $F = 0$, $G = \cos^2 v - \sin^2 v$. The coefficients $\bar{l}, \bar{m}, \bar{n}$ are given, up to a multiple of $-\sin v$, by $\bar{l} = -\sin^2 v$, $\bar{m} = 0$, $\bar{n} = -1$. It follows that the $PN$-lines of curvature are given by $dudv = 0$, i.e., they are the meridians and parallels in $\boldsymbol{x}(U)$. Both components of the $LD$ (circles) are lines of $PN$-principal curvature and the lightlike $PN$-principal direction is transverse to them. We analyse the configuration at the poles $(\pm 1, 0, 0)$ using the parametrisation

$$\boldsymbol{y}(u, v) = (\sin u \sin v, \cos v, \cos u \sin v),$$

with $(u, v) \in (0, 2\pi) \times (0, \pi)$. We get $E = \sin^2 v(-\cos^2 u + \sin^2 u)$, $F = -\frac{1}{2}\sin 2u \sin 2v$, $G = \cos^2 v(\cos^2 u - \sin^2 u) + \sin^2 v$. The coefficients $\bar{l}, \bar{m}, \bar{n}$ are given, up to a multiple of $-\sin v$ by, $\bar{l} = -\sin^2 v$, $\bar{m} = 0$, $\bar{n} = -1$. The discriminant of the lines of $PN$-principal curvature consists of the poles $(\pm 1, 0, 0)$, which are spacelike umbilic points. Thus, $S^2$ has two spacelike umbilic points.

There are no $PN$-asymptotic curves on the sphere.

The equation for the $PN$-characteristic curves with respect to the parametrisation $\boldsymbol{x}$ is given by $dv^2 - \sin^2 v du^2 = 0$, which factorises into two ODEs with smooth solutions in $U$. The $PN$-characteristic curves are singular at the poles $(\pm 1, 0, 0)$.

The following result follows by applying the Poincaré-Hopf Theorem to one of the extended $PN$-principal direction field on $M$.

**Theorem 4.8** *Let $M$ be a smooth closed surface in $\mathbb{R}^3_1$ homeomorphic to the Euclidean sphere $S^2$. Suppose that the LD is a disjoint union of simple closed curves and that the lightlike direction is transverse to it. Then, $M$ has at least two spacelike umbilic points (of the $\mathbb{R}P^2$-valued Gauss map).*

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