Apparent contours in Minkowski 3-space and first order ordinary differential equations

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Abstract

We consider projections of a smooth and regular surface M in the Minkowski 3-space $\mathbb{R}^3_1$ along lightlike directions to a fixed transverse plane. The lightlike directions in $\mathbb{R}^3_1$ can be parametrised by a circle on the lightcone and the resulting 1-parameter family of projections can be considered as viewing M along a special "camera motion". The associated 1-parameter families of contour generators and apparent contours reveal some aspects of the extrinsic and intrinsic geometry of M. We characterise geometrically the generic $\mathcal{A}_e$-codimension ≤ 1 singularities of a given projection and consider their bifurcations in the family of projections. We show that the families of contour generators and apparent contours are solutions of certain first order ordinary differential equations and obtain their generic local configurations.

1 Introduction

We consider in this paper the projections of smooth and regular surfaces M embedded in the Minkowski 3-space $\mathbb{R}^3_1$ along lightlike directions to a fixed transverse plane. The singularities of projections are affine invariant ([3]) so do not depend on the metric in $\mathbb{R}^3$. Therefore, from the singularity theory point of view, the situation here is identical to that of surfaces in the Euclidean 3-space. The projections are members of a 1-parameter family of maps from the surface to a plane. For a generic surface we expect most projections to have stable singularities of ($\mathcal{A}$) type fold or cusp and isolated

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projections in the family to have singularities of type swallowtails or lips/beaks (§3). We also expect the family of projections to be an $\mathcal{A}_e$-versal unfolding of the swallowtails and lips/beaks singularities.

We show that the family of projections along the lightlike directions picks up information about the extrinsic and intrinsic geometry of M. The induced metric on M may be degenerate at some points on M (this is indeed the case on any closed surface in $\mathbb{R}^3_1$). We label the locus of such points the *Locus of Degeneracy* (LD). At a point $p \in M \setminus LD$, there is a well defined shape operator. For a generic M, there is a curve (which could be empty) in the Lorentzian part of M which separates M into a region where at each point the shape operator has two eigenvectors, called the principal directions, and a region where it has none. This curve is labelled the *Lightlike Principal Locus* (LPL) and coincides with points where the unique principal direction is lightlike ([14]).

The set of critical points of a projection is called a *contour generator* and its image under the projection the *apparent contour* or *profile* of the projection. The contour generators of the projections along the lightlike directions are located in the Lorentzian part of M and their envelope is precisely the LD of M (Theorem 4.5). In the language of computer vision (see for example [10]), the LD is the *frontier* of the family of lightlike projections and the closure of the Lorentzian part of M is the *visible part* of M under this family of projections or *camera motion*. Each point p_t on the LD belongs to a single contour generator of a projection P_t. The locus of points $P_t(p_t)$, $p_t \in LD$ is labeled *the image of the* LD.

The locus of points where the projections along lightlike directions have a singularity of type cusp or worse (the *cusp generator curve*, see [4]) is precisely the LPL of M (§3). The images of such points trace the *cusp curve* of the apparent contours in the plane of projections. It turns out that the envelope of the apparent contours is the union of the image of the LD together with the cusp curve.

Another key result in this paper is that the families of contour generators and of apparent contours are solutions of certain first order ordinary differential equations (§4). (This is also valid for projections of surfaces in the Euclidean 3-space along a given camera motion.) We then use the results in [6, 8, 12] to deduce the generic local configurations of the families of contour generators and apparent contours. We observe that the family of apparent contours in the Euclidean space are also studied in [15] using divergent diagrams.

We recall in §2 some notions of the geometry of surfaces in the Minkowski 3-space. In §3 we analyse the singularities of the projections and characterise them geometrically. We obtain in §4 the local configurations of the families of contour generators and apparent contours.

2 Preliminaries

The *Minkowski space* $(\mathbb{R}^3_1, \langle , \rangle)$ is the vector space $\mathbb{R}^3$ endowed with the metric induced by the pseudo-scalar product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -u_0v_0 + u_1v_1 + u_2v_2$, for any vectors $\boldsymbol{u} = (u_0, u_1, u_2)$ and $\boldsymbol{v} = (v_0, v_1, v_2)$ in $\mathbb{R}^3$ (see for example [16], p55). We say that a non-zero vector $\boldsymbol{u} \in \mathbb{R}^3_1$ is *spacelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$, *lightlike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and *timelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$. The norm of a vector $\boldsymbol{u} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$.

We have the following pseudo-spheres in $\mathbb{R}^3_1$ with centre $p \in \mathbb{R}^3_1$ and radius $r > 0$,

$$\begin{array}{rcl} H^2(p, -r) & = & \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = -r^2\} \text{ (2-sheeted hyperboloid)}, \\ S^2_1(p, r) & = & \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = r^2\} \text{ (1-sheeted hyperboloid)}, \\ LC^*(p) & = & \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = 0\} \text{ (cone, called light cone)}. \end{array}$$

We denote by $H^2(-r)$, $S^2_1(r)$ and LC^* the pseudo-spheres centred at the origin in $\mathbb{R}^3_1$.

We consider embeddings $\boldsymbol{i} : M \to \mathbb{R}^3_1$ of a smooth and regular surface M. To simplify notation, we shall identify $\boldsymbol{i}(M)$ with M and write $\boldsymbol{i}(M) = M$. The set of embeddings $\boldsymbol{i}$ is endowed with the Whitney C^∞-topology. We say that a property is *generic* if it is satisfied in a residual subset of embeddings of M in $\mathbb{R}^3_1$.

Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3_1$ be a local parametrisation of M. As our analysis of the singularities of the projections is local in nature, we shall simplify notation further and write $\boldsymbol{x}(U) = M$. Let

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle, \quad G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle$$

denote the coefficients of the first fundamental form of M with respect to $\boldsymbol{x}$ (the subscripts denote partial derivatives). The integral curves of the lightlike directions on M are the solution curves of the binary quadratic differential equation (BDE)

$$Edu^2 + 2Fdudv + Gdv^2 = 0. \tag{1}$$

We identify the LD on M and its pre-image in U by $\boldsymbol{x}$. Then the LD (in U) is given by

$$LD = \{(u, v) \in U \,|\, (F^2 - EG)(u, v) = 0\}.$$

The LD is the *discriminant curve* of the BDE (1) (the discriminant curve of a BDE is the set of points where the equation determines a unique solution direction). The LD is also the locus of points where the surface is tangent to a light cone.

For a generic surface, and we shall assume this to be the case in this paper, the LD is either empty or is a smooth curve that splits the surface locally into a Riemannian and a Lorentzian region. If the unique lightlike direction at a given point on the LD is transverse to the LD then the configuration of the lightlike curves is locally smoothly equivalent to Figure 1 left, i.e., the curves consist of a family of cusps. The unique lightlike direction on the LD can be tangent to the LD at isolated points on this set. We say in this case that the BDE (1) has a singularity. For a generic surface, the singularities of the BDE (1) are *well-folded* (see for example [7] for terminology and [23] for a survey paper on BDEs). This means that, at the singular point, the configuration of the lightlike curves is locally topologically equivalent to one of the last three cases in Figure 1.

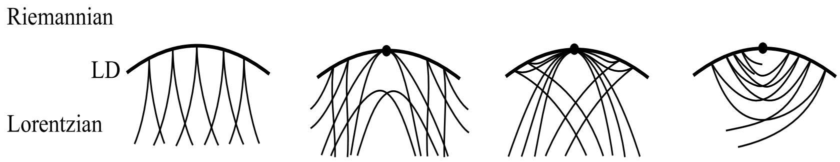

Figure 1: Stable local topological configurations of the lightlike curves at points on the LD from left to right: family of cusps, folded saddle, folded node and folded focus.

The following special local parametrisations simplify considerably the calculations and make the algebraic conditions involved easier to interpret geometrically. (The proof is standard and is omitted.)

Theorem 2.1 (1) *At any point p on the Lorentzian part of M there is a local parametrisation $\boldsymbol{x} : U \to V \subset M$ of a neighbourhood V of p, such that for any $p' \in V$, the coordinate curves through p' are tangent to the lightlike directions. Equivalently, there exists a local parametrisation with $E \equiv 0$ and $G \equiv 0$ on U.*

(2) *Let p be a point on the LD of a generic surface M. Then there exists a local parametrisation $\boldsymbol{x} : U \to V \subset M$ of a neighbourhood V of p, such that for any $p' = \boldsymbol{x}(q') \in V \cap LD$, the lightlike directions in $T_{p'}M$ are parallel to $\boldsymbol{x}_u(q')$, i.e., $E = F = 0$ on the LD.*

Pei [19] defined an $\mathbb{R}P^2$-valued Gauss map on M. This is simply the map $PN : M \to \mathbb{R}P^2$ which associates to a point $p = \boldsymbol{x}(q)$ the projectivisation of the vector $\boldsymbol{x}_u \times \boldsymbol{x}_v(q)$, where "$\times$" denotes the (Minkowski) vector product in $\mathbb{R}^3_1$. Away from the LD, the $\mathbb{R}P^2$-valued Gauss map can be identified with the de Sitter Gauss map $M \to S^2_1(1)$ on the Riemannian part of the surface and with the hyperbolic Gauss map $M \to H^2(-1)$ on its Lorentzian part. Both maps are given by $\boldsymbol{N} = \boldsymbol{x}_u \times \boldsymbol{x}_v / ||\boldsymbol{x}_u \times \boldsymbol{x}_v||$. The map $A_p = -d\boldsymbol{N}_p : T_pM \to T_pM$ is a self-adjoint operator on $M \setminus LD$. We denote by

$$
\begin{array}{rcccl}
l & = & -\langle \boldsymbol{N}_u, \boldsymbol{x}_u \rangle & = & \langle \boldsymbol{N}, \boldsymbol{x}_{uu} \rangle, \\
m & = & -\langle \boldsymbol{N}_u, \boldsymbol{x}_v \rangle & = & \langle \boldsymbol{N}, \boldsymbol{x}_{uv} \rangle, \\
n & = & -\langle \boldsymbol{N}_v, \boldsymbol{x}_v \rangle & = & \langle \boldsymbol{N}, \boldsymbol{x}_{vv} \rangle
\end{array}
$$

the coefficients of the second fundamental form on $M \setminus LD$. When A_p has real eigenvalues, we call them the *principal curvatures* and their associated eigenvectors the *principal directions* of M at p. (There are always two principal curvatures at each point on the Riemannian part of M but this is not always true on its Lorentzian part.) The lines of principal curvature, which are the integral curves of the principal directions, are solutions of the BDE

$$(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)du^2 = 0. \tag{2}$$

The discriminant of the BDE (2) is labelled the *Lightlike Principal Locus* (LPL) in [13, 14].

On the Riemannian part of a generic surface, the LPL consists of isolated points labelled *spacelike umbilic points* (these are points where A_p is a multiple of the identity map). At non spacelike umbilic points, there are always two orthogonal spacelike principal directions.

On the Lorentzian part of a generic surface, we consider the parametrisation in Theorem 2.1(1) so the LPL is given by $ln = 0$. One can deduce from this that the LPL is either empty or is a smooth curve except at isolated points where it has Morse singularities of type node. Such points are labelled *timelike umbilic points* (these are also points where A_p is a multiple of the identity map). The LPL consists of points where the principal directions coincide and become lightlike. There are two principal directions at each point on one side of the LPL and none on the other.

One can extend the lines of principal curvature across the LD as follows ([14]). As equation (2) is homogeneous in l, m, n, we can multiply these coefficients by $||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ and substitute them in the equation by

$$\bar{l} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uu} \rangle, \quad \bar{m} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uv} \rangle, \quad \bar{n} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{vv} \rangle.$$

This substitution does not alter the pair of foliations on $M \setminus LD$. The new equation is defined on the LD and defines the same pair of foliations associated to the de Sitter (resp. hyperbolic) Gauss map on the Riemannian (resp. Lorentzian) part of M. The extended lines of principal curvature are the solution curves of the BDE

$$(G\bar{m} - F\bar{n})dv^2 + (G\bar{l} - E\bar{n})dudv + (F\bar{l} - E\bar{m})du^2 = 0. \tag{3}$$

We observe that one of the principal directions at a point p on the LD is the unique lightlike direction and the other is spacelike if p is not also on the LPL.

The Gaussian curvature K and the mean curvature H of M at $p = \boldsymbol{x}(q) \in M \setminus LD$ are defined as

$$\begin{aligned} K(q) &= \det(-d\boldsymbol{N}_p) = \frac{ln - m^2}{EG - F^2}(q) \\ H(q) &= \tfrac{1}{2}\mathrm{trace}(-d\boldsymbol{N}_p) = \frac{lG - 2mF + nE}{2(EG - F^2)}(q). \end{aligned}$$

The *parabolic set* of $M \setminus LD$ is defined as the set of points where $K(q) = 0$, i.e., as the set of points $\boldsymbol{x}(q)$ where $(ln - m^2)(q) = 0$. The closure of the parabolic set is given by the set of points $\boldsymbol{x}(q)$ with $(\bar{l}\bar{n}-\bar{m}^2)(q)=0$. (We also call the parabolic set the set of points in the parameter space where $K(q)=0$.)

A direction $\boldsymbol{u}\in T_pM$, $p\in M\setminus LD$, is said to be *asymptotic* if $\langle d\boldsymbol{N}_p(\boldsymbol{u}),\boldsymbol{u}\rangle=0$. An asymptotic curve is one whose tangent direction at all points is asymptotic. Asymptotic curves are given by the BDE

$$ndv^2+2mdvdu+ldu^2=0.$$

These extend across the LD to curves given by the solutions of

$$\bar{n}dv^2+2\bar{m}dvdu+\bar{l}du^2=0.$$

Remarks 2.2 Suppose that M is a generic surface. Then, the LD and the LPL can intersect at isolated points and the two curves meet tangentially at such points. These points are exactly the folded singularities of the BDE (1) of the lightlike foliations ([14], see Figure 1).

The LPL and the parabolic set can intersect at isolated points on the Lorentzian part of M and the two curves meet tangentially at such points. The curve $H^{-1}(0)$ passes through such points and is transverse to both curves.

3 The singularities of the projections

We consider projections in $\mathbb{R}^3_1$ along lightlike directions. As the orthogonal plane to a given lightlike direction contains the direction itself, the concept of orthogonal projections does not make sense in this case. This is why, given a lightlike direction $\boldsymbol{v}=(v_0,v_1,v_2)\in LC^*$ (see §2 for definition), we consider the projection along $\boldsymbol{v}$ to a transverse plane which we fix to be

$$\mathbb{R}^2_+=\{(u_0,u_1,u_2)\in\mathbb{R}^3_1\,|\,u_0=0\}.$$

A point $p\in\mathbb{R}^3_1$ is then projected to the point

$$q=p-\frac{\langle p,\boldsymbol{e}_0\rangle}{\langle \boldsymbol{v},\boldsymbol{e}_0\rangle}\boldsymbol{v},$$

where $(\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2)$ denote the canonical basis in $\mathbb{R}^3_1$. We have $v_0\neq 0$ as the vector $\boldsymbol{v}$ is lightlike, so we can substitute $\boldsymbol{v}$ by the vector $-\frac{1}{\langle\boldsymbol{v},\boldsymbol{e}_0\rangle}\boldsymbol{v}=(1,\frac{v_1}{v_0},\frac{v_2}{v_0})$ and parametrise the family of projections along the lightlike directions by

$$S^1_+=\{\boldsymbol{v}=(v_0,v_1,v_2)\in LC^*\,|\,v_0=1\}.$$

We parametrise the circle S^1_+ by $\boldsymbol{v}(t)=(1,\cos(t),\sin(t))$, $t\in\mathbb{R}$. Given a surface M in $\mathbb{R}^3_1$, we denote by $P:M\times\mathbb{R}\to\mathbb{R}^2_+$ the family of lightlike projections of M to $\mathbb{R}^2_+$, given by

$$P(p,t)=p+\langle p,\boldsymbol{e}_0\rangle\boldsymbol{v}(t).$$

We denote by P^t the map $M \to \mathbb{R}^2_+$ given by $P^t(p) = P(p, \boldsymbol{v}(t))$. The singular set (*contour generator*) of P^t is denoted by Σ_t and its image (the discriminant or *apparent contour*) by $\Delta_t = P^t(\Sigma_t)$.

It follows from [17] and from the fact that the family of lightlike projections P on the ambient space $\mathbb{R}^3_1$ is a stable map that, for a residual set of embeddings of M in $\mathbb{R}^3_1$, the family P is a generic family of mappings. (The term generic is defined in terms of transversality to submanifolds of multi-jet spaces; see for example [11].) This means, in particular, that for any $t \in \mathbb{R}$ and at any point p on a generic M, the germ of the projection P^t at p (which can be viewed as a map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$) is $\mathcal{A}$-equivalent to one of the normal forms in Table 1. (Recall that two map-germs f, g are $\mathcal{A}$-equivalent if $g = k \circ f \circ h^{-1}$, where h, k are germs of diffeomorphisms.) We have the following geometric characterisations of the singularities of P^t.

Table 1: $\mathcal{A}_e$-codimension ≤ 1 local singularities of map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ ([20]).

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	(u, v)	0
Fold	(u, v^2)	0
Cusp	$(u, uv + v^3)$	0
Swallowtail	$(u, uv + v^4)$	1
Lips/beaks	$(u, v^3 \pm u^2 v)$	1

Theorem 3.1 *Let M be a smooth and regular surface in $\mathbb{R}^3_1$.*

(1) *The projection P^t is singular at $p \in M$ if and only if $\boldsymbol{v} \in T_pM$. As a consequence, P^t is a local diffeomorphism on the Riemannian part of M; at a point p on the Lorentzian part of M there are two directions $t_i, i = 1, 2$ in S^1_+ for which P^{t_i} is singular at p; at a point p on the LD there is a unique direction for which P^t is singular at p.*

(2) *Suppose that p is on the Lorentzian part of M. Then P^t has a singularity at p of type:*

(i) cusp if and only if $p \in LPL$, $\boldsymbol{v}$ is the unique principal lightlike direction at p and $\boldsymbol{v}$ is transverse to the LPL at p,

(ii) swallowtail if and only if $p \in LPL$, $\boldsymbol{v}$ is the unique principal lightlike direction at p, $\boldsymbol{v}$ is tangent to the LPL at p and the LPL is not inflectional at p,

(iii) lips/beaks singularity if and only if p is on both the LPL and on the parabolic set of M and the curve $H^{-1}(0)$ is transverse to the LPL at p (see Remark 2.2).

(3) *If $p \in LD$ but $p \notin LPL$ (i.e., p is not a folded singularity of the lightlike curves), then P^t has generically a fold singularity at p. Otherwise it has generically a cusp singularity at p.*

Proof (1) It is straightforward to check that P^t is singular at p if and only if $\boldsymbol{v}(t)$ is a tangent vector to M at p and the rest follows from the fact that there are 2/1/0 lightlike directions in T_pM if p is in the Lorentzian/LD/Riemannian component of M.

(2) We take a local parametrisation $\boldsymbol{x} : U \to \mathbb{R}^3_1$ of M as in Theorem 2.1(1) and suppose that

$$\boldsymbol{v}(t) = \alpha \boldsymbol{x}_u(u_0, v_0), \alpha \neq 0 \tag{4}$$

at $p_0 = \boldsymbol{x}(u_0, v_0)$. The LPL is given in this case by $(ln)(u, v) = 0$ and the branch of interest is $l(u, v) = 0$ as the unique principal lightlike direction on this branch is along $\boldsymbol{x}_u$ and is also an asymptotic direction.

It follows from (1) that P^t is singular if and only if

$$g(u, v) = \langle \boldsymbol{v}(t), \boldsymbol{N}(u, v)\rangle = 0. \tag{5}$$

(Here we drop the parameter t in g to simplify notation.) The singular set Σ_t is given by $g^{-1}(0)$. Dropping the arguments of functions, we have $\boldsymbol{N}_u = -\frac{m}{F}\boldsymbol{x}_u - \frac{l}{F}\boldsymbol{x}_v$ and $\boldsymbol{N}_v = -\frac{n}{F}\boldsymbol{x}_u - \frac{m}{F}\boldsymbol{x}_v$. Then $g_u = -\alpha l$ and $g_v = -\alpha m$. Therefore the critical set Σ_t is singular if and only if $l = m = 0$ at (u_0, v_0), equivalently, if and only if p is a point of tangency of the LPL with the parabolic set (Remark 2.2). Using Saji's recognition criteria in [21], the singularity of P^t at p is of type lips/beaks if and only if g has a Morse singularity at (u_0, v_0). This occurs if and only if $l_u m_v - l_v m_u \neq 0$ at (u_0, v_0), equivalently, if and only if $H^{-1}(0)$ is transverse to the LPL at p.

Suppose now that Σ_t is a regular curve parametrised by $\gamma(s) = (u(s), v(s))$ with $\gamma(0) = (u_0, v_0)$ and $\gamma'(0) = (-m, l)$.

The singularity of P^t at p is a fold if and only if $\frac{d}{ds}P^t(u(s), v(s))|_{s=0} \neq 0$ ([24]). Using (4), we get $P_u = \boldsymbol{x}_u + \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle \boldsymbol{v}(t) = 0$ on Σ_t, so

$$\frac{dP^t}{ds} = v'(\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}(t)) = l(\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}(t)).$$

Thus, the singularity is a fold if and only if $l \neq 0$ at (u_0, v_0), equivalently, if and only if $p \notin LPL$. If $p \in LPL$ and $\boldsymbol{v}(t)$ is parallel to $\boldsymbol{x}_u$ (i.e., $\boldsymbol{v}(t)$ is the unique lightlike principal direction at p) then Σ_t is a regular curve if and only if $m \neq 0$ at (u_0, v_0), i.e., p is not on the parabolic set of M. Suppose this to be the case. As $g_v = m \neq 0$, we can take $\gamma(s) = (s, v(s))$. Then the singularity of P^t at p is a cusp if and only if

$$\frac{d^2P^t}{ds^2} = v''(\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}(t))$$

is not zero at $s = 0$ ([24]). Now $v'' \neq 0$ if and only if $g_{uu} \neq 0$, equivalently, if and only if $l_u \neq 0$ at (u_0, v_0). This is precisely the condition for $\boldsymbol{x}_u$ to be transverse to the LPL at p. When $l_u = 0$ at (u_0, v_0), P^t has a swallowtail singularity if and only if $\frac{d^3P^t}{ds^3}|_{s=0} \neq 0$ ([21]). Similar calculations to the above show that this is the case if and only if $l_{uu} \neq 0$ at (u_0, v_0). This is precisely the condition for the LPL (given by $l = 0$) to have an ordinary contact with its tangent line at p.

(3) Here we take a local parametrisation of M as in Theorem 2.1(2) and take, as in (4), $\boldsymbol{v}(t) = \alpha \boldsymbol{x}_u$ at $p_0 = \boldsymbol{x}(u_0, v_0)$. Then P^t is singular if and only if

$$\bar{g}(u, v) = \langle \boldsymbol{v}(t), \boldsymbol{x}_u \times \boldsymbol{x}_v(u, v)\rangle = 0. \tag{6}$$

(Again, we drop here the parameter t in $\bar{g}$ to simplify notation.) We have $\bar{g}_u = -\alpha\bar{l}$ and $\bar{g}_v = -\alpha\bar{m}$, so for a generic surface, the critical sets Σ_t are smooth curves (Σ_t is singular at p if and only if p is a point of intersection of the following three curves: the LD, the LPL and the closure of the parabolic set). Following similar calculations as in (2), we have a fold singularity at points on the LD unless $\bar{l} = 0$ and this occurs exactly at points of tangency of the LD with the LPL. These points are precisely the folded singularities of the lightlike curves. At such points, the projection has generically a cusp singularity. □

Remark 3.2 Timelike umbilic points may occur at isolated points on the LPL of a generic surface. If the local parametrisation of the surface is as in Theorem 2.1(1), then a timelike umbilic point occurs when $l = 0$ and $n = 0$, so the LPL has generically a Morse singularity of type A_1^-. At such points both asymptotic directions are lightlike and we expect the lightlike projections along these directions to have a cusp singularity and not worse.

We turn now to the family P. Of course for a generic M, the family P is an $\mathcal{A}_e$-versal deformation of the codimension 1 singularities of P^{t_0} (which, according to Theorem 3.1, occur on the Lorentzian part of M). We give below the precise geometric conditions for P to be an $\mathcal{A}_e$-versal deformation of the singularities of P^{t_0}. In view of Remark 3.2, we assume that the point of interest is not a timelike umbilic point.

Theorem 3.3 *The family of projections P is a versal unfolding of the swallowtail singularity of P^{t_0} at $p_0 \in LPL$ if and only if the LPL is a smooth curve at p_0. The family P is always an $\mathcal{A}_e$-versal deformation of a lips/beaks singularity of P^{t_0} at $p_0 \in LPL$.*

Proof We follow the criteria in [18] for recognition of versal deformations of codimension 1 singularities of map-germs from the plane to the plane. (The criteria in [18] are for map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ but these apply to our case too.) Consider the following sets

$$\begin{array}{rcl} \Sigma^1 & = & \{(u,v,t)\,|\,P^t \text{ is singular at } (u,v)\}, \\ \Sigma^{1,1} & = & \{(u,v,t) \in \Sigma^1\,|\,P^t \text{ has a cusp singularity or worse at } (u,v)\}. \end{array}$$

Then the family P is an $\mathcal{A}_e$-versal deformation of the swallowtail singularity of P^{t_0} at (u_0, v_0) if and only if $\Sigma^{1,1}$ is a smooth curve in Σ^1. The set Σ^1 is given by $g(u,v,t) = 0$ and $\Sigma^{1,1}$ by $g_u(u,v,t) = 0$. Consider the map germ $H = (g, g_u)$ at (u_0, v_0, t_0) (so $g_u = g_{uu} = 0$ and $g_t \neq 0$ at (u_0, v_0, t_0)). We have, at (u_0, v_0, t_0),

$$dH = \begin{pmatrix} g_u & g_v & g_t \\ g_{uu} & g_{uv} & g_{ut} \end{pmatrix} = \begin{pmatrix} 0 & g_v & g_t \\ 0 & g_{uv} & g_{ut} \end{pmatrix}.$$

Also, at (u_0, v_0, t_0),

$$\begin{array}{rcl} g_{ut} & = & \langle \boldsymbol{v}', \boldsymbol{N}_u \rangle \\ & = & \langle \boldsymbol{v}', -\frac{m}{F}\boldsymbol{x}_u - \frac{l}{F}\boldsymbol{x}_v \rangle \\ & = & -\frac{m}{F}\langle \boldsymbol{v}', \boldsymbol{x}_u \rangle \text{ (as } l = 0) \\ & = & 0 \text{ (as } \boldsymbol{x}_u = \alpha \boldsymbol{v} \text{ and } \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \Rightarrow \langle \boldsymbol{v}, \boldsymbol{v}' \rangle = 0). \end{array}$$

Thus, H is a regular map if and only if $g_{uv} \neq 0$ at (u_0, v_0, t_0). Equivalently, if and only if $l_v \neq 0$ at (u_0, v_0), that is, if and only if the LPL is a smooth curve (we assumed p_0 not to be a timelike umbilic point, so $n \neq 0$).

The family P is a versal deformation of a lips/beaks singularity of P^{t_0} if and only if the family g is a versal deformation of the Morse singularity of g^{t_0}. This is the case if and only if $g_t \neq 0$. But this is always true as $\boldsymbol{N}$ is not a lightlike vector. □

4 The contour generators and apparent contours

By varying t, we obtain a family of contour generators (Σ_t) in the closure of the Lorentzian part of M (see Theorem 3.1) and a family of apparent contours (Δ_t) in $\mathbb{R}^2_+$. We consider in this section the local configurations of these families. The points of interest are on the LD and on the LPL as shown in Figure 2. We treat points on the LD, points in the $LD \cap LPL$ and points on the LPL separately.

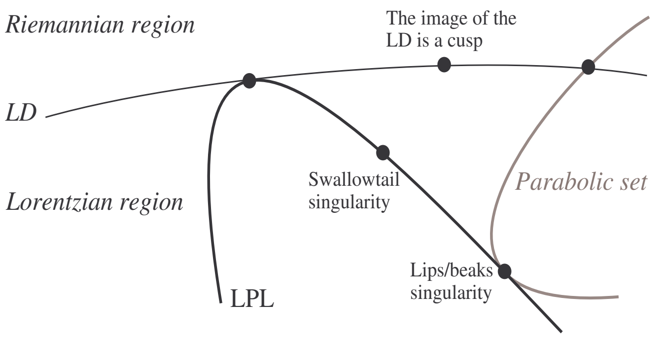

Figure 2: Special curves and points on the surface picked up by the family of projections along lightlike directions.

We stack the curves Σ_t (resp. Δ_t) together to form a surface Σ (resp. Δ) in $\mathbb{R}^2 \times \mathbb{R}, 0$ (we ignore the metric here). Let S denote Σ or Δ and let $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^2 \times \mathbb{R}, 0$ be a local parametrisation of this surface. Let $\pi : \mathbb{R}^2 \times \mathbb{R}, 0 \to \mathbb{R}^2$ be the projection to the first component and set $k = \pi \circ \phi$. The curves Σ_t or Δ_t in the parameter space $\mathbb{R}^2, 0$ are the fibres of some germ of a function h. To obtain the configuration of the curves Σ_t or Δ_t in the plane $\mathbb{R}^2, 0$, one can consider the divergent mapping diagram (k, h)

$$(k,h): \ \mathbb{R}, 0 \ \xleftarrow{h} \ \mathbb{R}^2, 0 \ \xrightarrow{k} \ \mathbb{R}^2, 0.$$

The above diagrams are studied by Dufour in [8]. Two germs (k, h), (k', h') of divergent mapping diagrams are equivalent if the diagram

$$\begin{array}{ccccc} \mathbb{R}, 0 & \xleftarrow{h} & \mathbb{R}^2, 0 & \xrightarrow{k} & \mathbb{R}^2, 0 \\ \downarrow \kappa_1 & & \downarrow \kappa_2 & & \downarrow \kappa_3 \\ \mathbb{R}, 0 & \xleftarrow{h'} & \mathbb{R}^2, 0 & \xrightarrow{k'} & \mathbb{R}^2, 0 \end{array}$$

commutes for some germs of diffeomorphisms $\kappa_i, i = 1, 2, 3$.

Theorem 4.1 ([8, 9]) *There are six generic types of divergent mapping diagrams and these are characterised as follows* (Figure 3)*:*

(1) *k is a diffeomorphism, h is a submersion;*
(2) *k is a diffeomorphism, h has a Morse singularity;*
(3) *k has a fold singularity, h restricted to the singular set S_k of k is regular and $(k, h) : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ is regular;*
(4) *k has a fold singularity, $h|_{S_k}$ has a Morse singularity, and (k, h) is regular;*
(5) *k has a fold singularity, (k, h) is a cross-cap whose double points is transverse at 0 to the direction $\{0\} \times \mathbb{R}$ in $\mathbb{R}^3$;*
(6) *k has a cusp singularity and (k, h) is regular.*

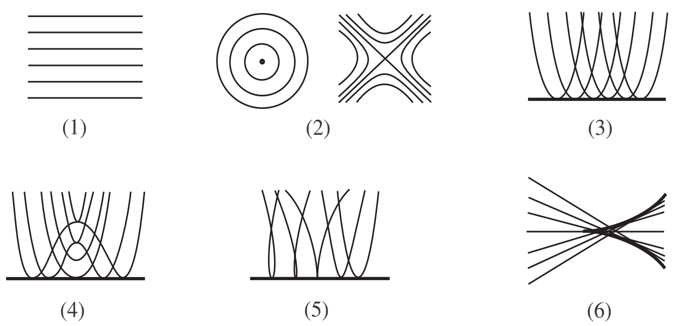

Figure 3: The generic types of divergent mapping diagrams.

Remark 4.2 1. For a generic surface, the configuration in Figure 3(v) occurs neither in the family of contour generators nor in the family of apparent contours of the projections along the lightlike directions. Indeed, the generic singularities of P^t are as in Table 1, so Σ_t cannot have a cusp singularity. As for the apparent contours, the "camera motion" we consider here is a special one and the cusp generator curve is the LPL (Theorem 3.1(2)(i)), which is generically not an isolated point when it is not empty. Therefore, the cusps on the apparent contours cannot occur on an isolated single apparent contour as in Figure 3(v).

2. For a generic surface, the configuration in Figure 3(vi) does not occur in the family of contour generators of the projections along the lightlike directions. We show in Theorem 4.5 that the envelope of contour generators is the LD and this set is a smooth curve when not empty.

The discriminants Δ_t can form a family of cusps and this case is not covered by the classification of generic divergent mapping diagrams. However, we show that the curves Σ_t and Δ_t are solutions of certain first order ordinary differential equations, so we can proceed as in [12].

In [12], the authors studied germs of first order ordinary differential equations (or, briefly, equations) with independent first integral. An equation is defined to be the germ of the surface $N = F^{-1}(0)$, with $F : PT^*\mathbb{R}^2, z \to \mathbb{R}, 0$ a germ of a smooth function. Here the projectivised cotangent bundle $PT^*\mathbb{R}^2$ of $\mathbb{R}^2$ is endowed with the canonical contact structure given by the 1-form $\alpha = dy - pdx$. The surface N is supposed to be smooth in [12], so is locally the image of a germ of an immersion $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$. The equation is then represented by the germ f.

Let $\pi : PT^*\mathbb{R}^2 \to \mathbb{R}^2$ be the natural projection. Two germs of immersions (equations) $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$ and $f' : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z'$ are said to be equivalent if there exist germs of diffeomorphisms $\psi : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ and $\phi : \mathbb{R}^2, \pi(z) \to \mathbb{R}^2, \pi(z')$ such that $\hat{\phi} \circ f = f' \circ \pi$, where $\hat{\phi} : PT^*\mathbb{R}^2, z \to PT^*\mathbb{R}^2, z'$ is the lift of ϕ.

Suppose that the equation f has a first integral, that is, there exists a germ of a submersion $\mu : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ such that $d\mu \wedge f^*\alpha = 0$. As the solutions of the equation in the plane are the images under $\pi \circ f$ of the level sets of μ, it is natural to consider the divergent mapping diagram $\mathbb{R}, 0 \xleftarrow{\mu} \mathbb{R}^2, 0 \xrightarrow{\pi\circ f} \mathbb{R}^2, 0$. Consider in general a divergent mapping diagram (g, μ)

$$\mathbb{R}, 0 \xleftarrow{\mu} \mathbb{R}^2, 0 \xrightarrow{g} \mathbb{R}^2, 0$$

where g is a smooth map germ and μ is a germ of a submersion. The diagram (g, μ) is called an *integral diagram* ([12]) if there exists a germ of an immersion $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$ such that $d\mu \wedge f^*\alpha = 0$ and $g = \pi \circ f$. Then (g, μ) is said to be induced by f. Suppose given two germs of equations f and f' with first integrals and with the set of critical points of $\pi \circ f$ and $\pi \circ f'$ nowhere dense. Then f and f' are equivalent as equations if and only if the diagrams $(\pi \circ f, \mu)$ and $(\pi \circ f', \mu')$ are equivalent as mapping diagrams ([12, Proposition 2.8]). (The "if" part of [12, Proposition 2.8] remains true if in the definition of f having a first integral one allows μ to have singularities. The "only if" part holds when μ is a submersion.)

Theorem 4.3 ([12, Theorem B]) *An integral diagram of generic type is equivalent to one of the following integral diagrams (g, μ):*

(1) Non-singular: $g = (u, v)$, $\mu = v$.
(2) Regular fold: $g = (u^2, v)$, $\mu = v - \frac{1}{3}u^3$.
(3) Clairaut fold: $g = (u, v^2)$, $\mu = v - \frac{1}{2}u$.
(4) Regular cusp: $g = (u^3 + uv, v)$, $\mu = \frac{3}{4}u^4 + \frac{1}{2}u^2 v + \beta \circ g$,
where $\beta(x, y)$ is a germ of a smooth function with $\beta(0) = 0$ and $\beta_y(0) = \pm 1$.
(5) Clairaut cusp: $g = (u, v^3 + uv)$, $\mu = v + \beta \circ g$,
where $\beta(x, y)$ is a germ of a smooth function with $\beta(0) = 0$.
(6) Mixed fold: $g = (u, v^3 + uv^2)$, $\mu = \frac{1}{2}v^2 + \beta \circ g$,
where $\beta(x, y)$ is a germ of a smooth function with $\beta(0) = 0$ and $\beta_x(0) = 1$.

The configurations of the solutions of the associated equations are as shown in Figure 4, (1)–(6).

The case when the surface N of the equation has a cross-cap singularity is studied in [6]. The generic model is the Clairaut cross-cap $g = (u, \frac{1}{4}v^2)$, $\mu = v - \frac{1}{2}u^2$ ([6, Theorem 2.7]); see Figure 4(7).

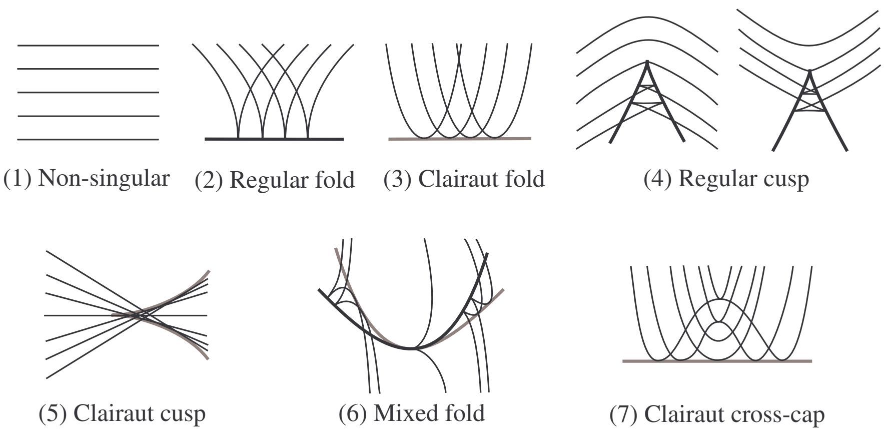

Figure 4: Configurations of generic integral diagrams (1)–(6) and of the Clairaut cross-cap (7).

Remark 4.4 It is worth observing that there are pairs (g, μ) which are generic as mapping diagrams but not as integral diagrams and vice-versa (compare Figure 3 and Figure 4).

4.1 Configurations of Σ_t and Δ_t on the LD

We start with the configurations of the contour generators Σ_t on M.

Theorem 4.5 (1) *The envelope of the family of contour generators Σ_t is the LD.*

(2) *At a point of intersection of the LD with the closure of the parabolic set, the contour generators Σ_t are solutions of a differential equation with an integral diagram of type Clairaut cross-cap and their configuration is as in* Figure 4(7). *Away from such points, they are solutions of a differential equation with an integral diagram of type Clauraut fold and their configuration is as in* Figure 4(3).

Proof (1) Let $\boldsymbol{x} : U \to \mathbb{R}^3_1$ be a local parametrisation of M. The contour generators Σ_t of P^t are given by

$$\bar{g}(u, v, t) = \langle \boldsymbol{v}(t), \boldsymbol{x}_u \times \boldsymbol{x}_v(u, v)\rangle = 0 \tag{7}$$

(this is valid at points on the LD as well as at points on the Lorentzian part of M). The envelope of Σ_t is given by

$$\mathcal{D} = \{(u, v) \in U \,|\, \exists t \in \mathbb{R} \text{ with } \bar{g}(u, v, t) = \frac{\partial \bar{g}}{\partial t}(u, v, t) = 0\}.$$

We set $\boldsymbol{x}_u \times \boldsymbol{x}_v(u, v) = (a, b, c)$. Then

$$\left\{\begin{array}{rcl} \bar{g}(u, v, t) &=& 0 \\ \frac{\partial \bar{g}}{\partial t}(u, v, t) &=& 0 \end{array}\right. \iff \left\{\begin{array}{rcl} -a + b\cos(t) + c\sin(t) &=& 0 \\ -b\sin(t) + c\cos(t) &=& 0 \end{array}\right.$$

which implies that $\boldsymbol{x}_u \times \boldsymbol{x}_v(u, v) = a(1, \cos(t), \sin(t))$, so

$$\boldsymbol{x}_u \times \boldsymbol{x}_v = a\boldsymbol{v}, a \neq 0. \tag{8}$$

That is, $\boldsymbol{x}_u \times \boldsymbol{x}_v(u, v)$ is a lightlike vector and therefore $\boldsymbol{x}(u, v) \in LD$. Conversely, if $p = \boldsymbol{x}(u, v) \in LD$, $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is a (non-zero) lightlike vector so $\langle \boldsymbol{x}_u \times \boldsymbol{x}_v(u, v), \boldsymbol{e}_0\rangle \neq 0$. We take $\boldsymbol{v}(t) = -\frac{1}{\langle \boldsymbol{x}_u \times \boldsymbol{x}_v(u,v), \boldsymbol{e}_0\rangle}\boldsymbol{x}_u \times \boldsymbol{x}_v(u, v)$ and this shows that $(u, v) \in \mathcal{D}$.

(2) We define the map $f : \Sigma \to PT^*(U) = U \times \mathbb{P}^1$ by

$$f(u, v, t) = (u, v, [\bar{g}_u(u, v, t) : \bar{g}_v(u, v, t)]),$$

where $\bar{g}$ is as (7). (The surface Σ is smooth for a generic surface M.) The canonical contact structure on $PT^*(U)$ is given by the one-form $\theta = pdu+qdv$, where $(u, v, [p : q])$ are the homogeneous coordinates of $PT^*(U)$. Since $\Sigma = (\bar{g})^{-1}(0)$, we have $d(\bar{g})|\Sigma = 0$. Thus, we have

$$f^*\theta = \left(\frac{\partial \bar{g}}{\partial u}du + \frac{\partial \bar{g}}{\partial v}dv\right)\Big|\Sigma = \left(d(\bar{g}) - \frac{\partial \bar{g}}{\partial t}dt\right)\Big|\Sigma = -\frac{\partial \bar{g}}{\partial t}dt\Big|\Sigma. \tag{9}$$

The above equation means that $f^*\theta \wedge dt|\Sigma = 0$. Here, we remark that the function $\mu = t|\Sigma = \pi_{\mathbb{R}}|\Sigma$, where $\pi_{\mathbb{R}} : U \times \mathbb{R} \to \mathbb{R}$ is the the canonical projection.

We consider points on the LD. We suppose, without loss of generality, that the point of interest p_0 is not a point of tangency of the LD with the LPL and take a parametrisation of the surface as in Theorem 2.1. Then $\bar{g}_u = -\alpha\bar{l} \neq 0$ and $\bar{g}_u = -\alpha\bar{m}$ at p_0 (α as in (4)), and the surface Σ can be parametrised by $(u(v,t), v, t)$ for some germ of a smooth function $u(v,t)$. We take an affine chart in the projective line and consider f as the map-germ

$$f(v,t) = (u(v,t), v, \psi(v,t)),$$

with $\psi(v,t) = \frac{\bar{g}_v}{\bar{g}_u}(u(v,t), v, t)$. We have (after dropping the arguments),

$$f_v = (u_v, 1, \psi_v) \text{ and } f_u = (u_t, 0, \psi_t).$$

As p_0 is on the LD, we have $\bar{g} = \bar{g}_t = 0$ at p_0, so it follows from $u_t\bar{g}_u + \bar{g}_t = 0$ that $u_t = 0$. Therefore, f is an immersion at p_0 if and only if

$$\psi_t = \frac{(u_t\bar{g}_{vu} + \bar{g}_{vt})\bar{g}_u - (u_t\bar{g}_{uu} + \bar{g}_{ut})\bar{g}_v}{\bar{g}_u^2} = 0.$$

We have,

$$\begin{array}{rcl} \bar{g}_{ut} & = & \langle \boldsymbol{v}', \boldsymbol{x}_{uu} \times \boldsymbol{x}_v + \boldsymbol{x}_u \times \boldsymbol{x}_{uv} \rangle, \\ \bar{g}_{vt} & = & \langle \boldsymbol{v}', \boldsymbol{x}_{uv} \times \boldsymbol{x}_v + \boldsymbol{x}_u \times \boldsymbol{x}_{vv} \rangle. \end{array}$$

As $\langle \boldsymbol{v}, \boldsymbol{v}' \rangle = 0$, $\boldsymbol{v}' \in T_{p_0}M$ (and is not a lightlike vector), so $\boldsymbol{v}' = a\boldsymbol{x}_u + b\boldsymbol{x}_v$ with $b \neq 0$. Therefore,

$$\bar{g}_{ut} = \langle a\boldsymbol{x}_u + b\boldsymbol{x}_v, \boldsymbol{x}_{uu} \times \boldsymbol{x}_v + \boldsymbol{x}_u \times \boldsymbol{x}_{uv} \rangle = -a\bar{l} - b\bar{m}$$

and

$$\bar{g}_{vt} = \langle a\boldsymbol{x}_u + b\boldsymbol{x}_v, \boldsymbol{x}_{uv} \times \boldsymbol{x}_v + \boldsymbol{x}_u \times \boldsymbol{x}_{vv} \rangle = -a\bar{m} - b\bar{n}.$$

It follows that

$$\bar{g}_{vt}\bar{g}_u - \bar{g}_{ut}\bar{g}_v = \alpha b(\bar{l}\bar{n} - \bar{m}^2).$$

Therefore, f fails to be an immersion at p_0 if and only if p_0 is the point of intersection of the LD with the closure of the parabolic set. At such points $f(\Sigma)$ is generically a surface with a cross-cap singularity. (The genericity condition depends on the coefficients of the 3-jet of the parametrisation $\boldsymbol{x}$.)

The function $\mu = t|_\Sigma = \pi_{\mathbb{R}}|_\Sigma$ is given by $(v,t) \to t$ and is clearly a submersion.

We can now apply the results in [6, 12]. Away from the point of intersection of the LD with the closure of the parabolic set the map f is an immersion. The map-germ $\pi \circ f : \Sigma \to \mathbb{R}^2$ is a fold map ($u_t = 0$ and $u_{tt} = -\bar{g}_{tt}/\bar{g}_u = -1/\alpha^2\bar{l} \neq 0$), μ is regular when restricted to the critical set of $\pi \circ f$ and $(\pi \circ f, \mu)$ is a regular map. Therefore, $(\pi \circ f, \mu)$ is equivalent to an integral diagram of type Clairaut fold (Theorem 4.3(3)), so the configuration of the family of contour generators is as in Figure 4(3).

At points of intersection of the LD with the closure of the parabolic set, $f(\Sigma)$ is a cross-cap. The map-germ $\pi \circ f$ is a fold map and the restriction of μ to its critical set has generically a Morse singularity. Also, $(\pi \circ f, \mu)$ is a regular map. Therefore, $(\pi \circ f, \mu)$ is equivalent to an integral diagram of type Clairaut cross-cap ([6, Theorem 2.7]), so the configuration of the family of contour generators is as in Figure 4(7). $\square$

Example 4.6 We draw in Figure 5 examples of contour generators at points of the LD using Maple. We take a surface patch parametrised by $(x, x + f(x, y), y)$, with $f(x, y) = xy + y^2 + x^3$ and (x, y) near the origin. Then the origin is a point on the LD but not on the closure of the parabolic set and the contour generators are as in Figure 5, left. We take $f(x, y) = x^2 + 2xy + y^2 + x^3$ so that the origin is a point of intersection of the LD with the closure of the parabolic set. The contour generators are those in Figure 5, right.

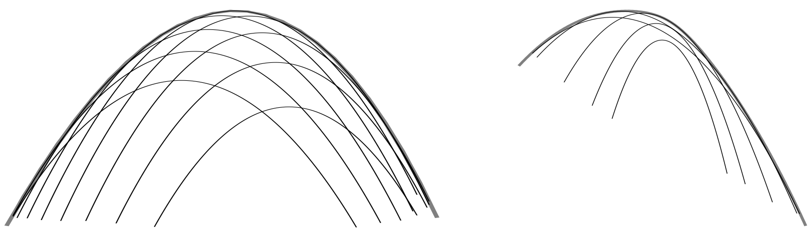

Figure 5: Maple generated figures of contour generators at a generic point on the LD left, and at a point of intersection of the LD with the closure of the parabolic set right. The envelope (LD) is shown in grey.

We consider now the family of apparent contours. The singularity type of an individual apparent contour and the way it changes in the family is well known. What we seek is the configuration of the family of apparent contours in the plane of projection. We consider the surface

$$\Delta := \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R} \,|\, (x, y) \in \Delta_t\}.$$

As the family P is an $\mathcal{A}_e$-versal unfolding of the singularities of a given member P^{t_0} of the family, it follows that the surface Δ is a regular surface at a fold singularity of P^{t_0}, a cuspidaledge at a cusp or a lips/beaks singularity of P^{t_0}, and a swallowtail surface at a swallowtail singularity of P^{t_0}, see Figure 6.

The locus of projections of points on the LD along the unique lightlike tangent direction at such points is labelled *the image of the LD*. We define similarly the image of the LPL, which is the cusp curve.

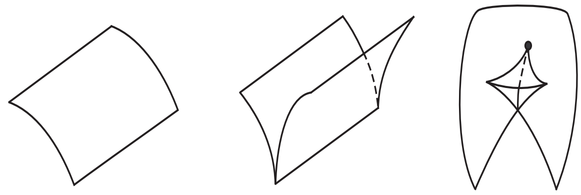

Figure 6: The surface Δ (from left to right) when the projection has respectively a fold singularity, a cusp or lips/beaks singularity, a swallowtail singularity.

Theorem 4.7 (1) *The envelope of the family of contours Δ_t is the image of the LD together with the cusp curve.*

(2) *The apparent contours are locally the solutions of a differential equation with an integral diagram of type:*

Clairaut fold:	*at images of generic points on the LD;* Figure 4(3).
Clairaut cusp:	*at images of some isolated points on the LD;* Figure 4(5).
Mixed fold:	*at images of the points of tangency of the LD with the LPL;* Figure 4(6).
Clairaut cross-cap:	*at images of the points of intersection of the LD with the closure of the parabolic set;* Figure 4(7).

Proof (1) Suppose, without loss of generality, that the critical sets are smooth curves near a point $p \in M$ (i.e., p is not on the closure of the parabolic set). We can parametrise Σ_t, again without loss of generality, by $(u(v,t),v)$ and the apparent contours by

$$\tilde{P}(v,t) = P(u(v,t),v,t) = \boldsymbol{x}(u(v,t),v)) + \langle \boldsymbol{x}(u(v,t),v), \boldsymbol{e}_0\rangle \boldsymbol{v}(t).$$

A point (v,t) contributes to the envelope of the apparent contours if and only if

$$\frac{\partial \tilde{P}}{\partial v}(v,t) \parallel \frac{\partial \tilde{P}}{\partial t}(v,t).$$

This is of course the case at points where the discriminant Δ_t are singular (i.e., where $\frac{\partial \tilde{P}}{\partial v}(v,t) = 0$), so the cusp curve is part of the envelope of the apparent contours. We have

$$\begin{array}{rcl} \dfrac{\partial \tilde{P}}{\partial v}(v,t) & = & u_v(\boldsymbol{x}_u + \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle \boldsymbol{v}) + \boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v} \\ \\ \dfrac{\partial \tilde{P}}{\partial t}(v,t) & = & u_t(\boldsymbol{x}_u + \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle \boldsymbol{v}) + \langle \boldsymbol{x}, \boldsymbol{e}_0\rangle \boldsymbol{v}' \end{array}$$

We also have $\boldsymbol{x}_u + \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle \boldsymbol{v} = P_u^t$ and $\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v} = P_v^t$ and these two vectors are parallel as $(u(v,t),v) \in \Sigma_t$. It follows that $\frac{\partial \tilde{P}}{\partial v} \parallel \frac{\partial \tilde{P}}{\partial t}$ if and only if $\langle \boldsymbol{x}, \boldsymbol{e}_0\rangle = 0$ or $\boldsymbol{v}' \parallel \boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}$. The condition $\langle \boldsymbol{x}, \boldsymbol{e}_0\rangle = 0$ can be avoided by moving the plane of projection away from the surface, so we shall assume that $\langle \boldsymbol{x}, \boldsymbol{e}_0\rangle \neq 0$.

If we write $\boldsymbol{x}_v = (x,y,z)$, then $\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v} = (0, y - x\cos(t), z - x\sin(t))$. Thus $\boldsymbol{v}' \parallel \boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}$ if and only if $-x + y\cos(t) + z\sin(t) = 0$, that is, if and only if $\langle \boldsymbol{x}_v, \boldsymbol{v}\rangle = 0$. As $\boldsymbol{x}_u + \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle \boldsymbol{v}$ is parallel to $\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}$, if $\langle \boldsymbol{x}_v, \boldsymbol{v}\rangle = 0$ we also get $\langle \boldsymbol{x}_u, \boldsymbol{v}\rangle = 0$. Therefore $\langle \boldsymbol{x}_v, \boldsymbol{v}\rangle = 0$ is equivalent to $\boldsymbol{x}_u \times \boldsymbol{x}_v$ being parallel to $\boldsymbol{v}$, which is equivalent to $p \in LD$. (Observe that if $p \in LD$ then $\boldsymbol{v}' \parallel \boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}$.)

(2) We deal with points on the $LD \setminus LPL$ and points in $LD \cap LPL$ separately. We start with the former. At such points, we take a parametrisation of Σ in the form $(u(v,t),v,t)$ (we have $\bar{l} \neq 0$).

We denote by $\xi(v,t)$ the direction of the intersection of the tangent plane T_pM with $\mathbb{R}^2_+$, where $p = \boldsymbol{x}(u(v,t),v)$. The direction $\xi(v,t)$ is a tangent (or is a limiting tangent) direction to Δ_t at $\tilde{P}(v,t)$ (with $\tilde{P}$ as in part (1) of the proof). We have $\xi = \lambda \boldsymbol{x}_u + \mu \boldsymbol{x}_v$ for some $\lambda, \mu \in \mathbb{R}$, and as it is also in $\mathbb{R}^2_+$, we can take it in the form

$$\xi = \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{x}_u - \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle \boldsymbol{x}_v.$$

(Observe that ξ is never zero.) We define the map $h : \Sigma \to PT^*(U) = U \times \mathbb{P}^1$ by

$$h(v,t) = (P(u(v,t),v), [\xi(v,t)]),$$

where $P(u(v,t),v) = \boldsymbol{x}(u(v,t),v) + \langle \boldsymbol{x}(u(v,t),v), \boldsymbol{e}_0\rangle \boldsymbol{v}(t)$. Then, for t fixed, the image of the map-germ $\pi \circ h(-,t)$ is Δ_t.

The map $\mu = t|_\Sigma$ is a submersion, and the curves $h_t(v) = h(v,t)$ for t fixed are, by construction, Legendrian curves. We need to check now when h is an immersion. We write $[\xi] = [\xi_1 : \xi_2]$ and suppose, without loss of generality, that $\xi_1 \neq 0$. Then we take an affine chart and write $h(v,t) = (P(u(v,t),v), (\xi_2/\xi_1)(v,t))$. We have, after dropping the arguments,

$$h_v = (u_v P_u + P_v, (\frac{\xi_2}{\xi_1})_v) \text{ and } h_t = (u_t P_u + P_t, (\frac{\xi_2}{\xi_1})_t).$$

At a point p_0 on the LD, $u_v \bar{g}_u + \bar{g}_v = 0$, $\bar{g}_u = -\alpha \bar{l}$ and $\bar{g}_v = -\alpha \bar{m}$, so $u_v = -\bar{m}/\bar{l}$. We also have $u_t = 0$ (see proof of Theorem 4.5).

Differentiating P gives $P_u = \boldsymbol{x}_u + \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle \boldsymbol{v} = \boldsymbol{x}_u + \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle (-\frac{1}{\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle} \boldsymbol{x}_u) = 0$, and $P_t = \langle \boldsymbol{x}, \boldsymbol{e}_0\rangle \boldsymbol{v}' \neq 0$ (we take $\langle \boldsymbol{x}, \boldsymbol{e}_0\rangle \neq 0$). Differentiating ξ with respect to t yields

$$\xi_t \quad = \quad \langle u_t \boldsymbol{x}_{uv}, \boldsymbol{e}_0\rangle \boldsymbol{x}_u + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle (u_t \boldsymbol{x}_{uu}) - \langle u_t \boldsymbol{x}_{uu}, \boldsymbol{e}_0\rangle \boldsymbol{x}_v - \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle (u_t \boldsymbol{x}_{uv}),$$

and this is the zero vector at p_0 as $u_t = 0$, so $(\xi_2/\xi_1)_t = 0$ at p_0. Therefore, at p_0,

$$h_v = (P_v, (\frac{\xi_2}{\xi_1})_v) \text{ and } h_t = (P_t, 0).$$

The vectors P_t and P_v are parallel at p_0, so h is an immersion if and only if $(\xi_2/\xi_1)_v \neq 0$. We have $(\xi_2/\xi_1)_v = 0$ if and only if $(\xi_2)_v\xi_1 - (\xi_1)_v\xi_2 = 0$, that is, if and only if

$$\langle \xi_v, \xi^{\perp}\rangle = 0,$$

where $\xi^{\perp}$ is an orthogonal vector to ξ in $\mathbb{R}^2_+$. It is not difficult to show that the vector $\boldsymbol{v} - \boldsymbol{e}_0$ is an orthogonal vector to ξ in $\mathbb{R}^2_+$. Thus, h fails to be an immersion if and only if $\langle \xi_v, \boldsymbol{v} - \boldsymbol{e}_0\rangle = 0$. We have,

$$\begin{array}{rcl}\xi_v &=& \langle u_v\boldsymbol{x}_{uv} + \boldsymbol{x}_{vv}, \boldsymbol{e}_0\rangle\boldsymbol{x}_u + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle(u_v\boldsymbol{x}_{uu} + \boldsymbol{x}_{uv})\\ && -\langle u_v\boldsymbol{x}_{uu} + \boldsymbol{x}_{uv}, \boldsymbol{e}_0\rangle\boldsymbol{x}_v - \langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle(u_v\boldsymbol{x}_{uv} + \boldsymbol{x}_{vv}).\end{array}$$

As $\boldsymbol{v} = \alpha\boldsymbol{x}_u$, $\langle \boldsymbol{x}_u, \boldsymbol{x}_u\rangle = \langle \boldsymbol{x}_u, \boldsymbol{x}_v\rangle = 0$ and $\boldsymbol{x}_u = \gamma\boldsymbol{x}_u \times \boldsymbol{x}_v$ on the LD ($\alpha\gamma \neq 0$),

$$\begin{array}{rcl}\langle \xi_v, \boldsymbol{v} - \boldsymbol{e}_0\rangle &=& \alpha\langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle(u_v\langle \boldsymbol{x}_{uv}, \boldsymbol{x}_u\rangle + \langle \boldsymbol{x}_{uv}, \boldsymbol{x}_u\rangle)\\ && -\alpha\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle(u_v\langle \boldsymbol{x}_{uv}, \boldsymbol{x}_u\rangle + \langle \boldsymbol{x}_{vv}, \boldsymbol{x}_u\rangle)\\ && -u_v\langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0\rangle\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle - \langle \boldsymbol{x}_{vv}, \boldsymbol{e}_0\rangle\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle\\ && -u_v\langle \boldsymbol{x}_{uu}, \boldsymbol{e}_0\rangle\langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle - \langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0\rangle\langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle\\ && +u_v\langle \boldsymbol{x}_{uu}, \boldsymbol{e}_0\rangle\langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle + \langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0\rangle\langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle\\ && +u_v\langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0\rangle\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle + \langle \boldsymbol{x}_{vv}, \boldsymbol{e}_0\rangle\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle\\ &=& -\alpha\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle(u_v\langle \boldsymbol{x}_{uv}, \boldsymbol{x}_u\rangle + \langle \boldsymbol{x}_{vv}, \boldsymbol{x}_u\rangle)\\ && +\alpha\langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle(u_v\langle \boldsymbol{x}_{uv}, \boldsymbol{x}_u\rangle + \langle \boldsymbol{x}_{uv}, \boldsymbol{x}_u\rangle)\\ &=& -\alpha\gamma\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle(-\frac{\bar{m}}{\bar{l}}\bar{m} + \bar{n}) + \alpha\gamma\langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle(-\frac{\bar{m}}{\bar{l}}\bar{l} + \bar{m})\\ &=& -\frac{\alpha\gamma}{\bar{l}}\langle \boldsymbol{x}_u, \boldsymbol{e}_0\rangle(\bar{l}\bar{n} - \bar{m}^2).\end{array}$$

It follows that h fails to be an immersion at precisely the points of intersection of the LD with the closure of the parabolic set. At such points, the image of h is generically a surface with a cross-cap singularity. (The genericity condition depends on the coefficients of the 3-jet of the parametrisation $\boldsymbol{x}$.)

The function $\mu = t|_{\Sigma}$ is given by $(v, t) \to t$ and is clearly a submersion.

We consider the projection $\pi \circ h = \tilde{P}$. Away from the LPL, its critical set is the set of points (v, t) such that $\boldsymbol{x}(u(v, t), v) \in LD$ and $\boldsymbol{x}_u(u(v, t), v) \parallel \boldsymbol{v}(t)$. This is the projection of the inverse image of $H = (\bar{g}, \bar{g}_t) : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ to the (v, t) plane. The map H is regular as $\bar{g}_u\bar{g}_{tt} \neq 0$, so $H^{-1}(0)$ is a smooth curve which can be parametrised in the form $(u(v), v, t(v))$. Following standard calculations, we find that

$$u' = -\frac{\bar{m}}{\bar{l}} \text{ and } t' = -\frac{\bar{m}^2 - \bar{l}\bar{n}}{\bar{l}\beta^2}$$

where we set $\boldsymbol{x}_u \times \boldsymbol{x}_v = \beta\boldsymbol{x}_u$. The tangent direction to the discriminant curve $\delta(v) = (\pi \circ h)(v, t(v))$ of the map-germ $\pi \circ h$ (which is the image of the LD) is along $\delta'(t) = \tilde{P}_v(v, t(v)) + t'(v)\tilde{P}_t(v, t(v)) = (\lambda + t'(v)\langle \boldsymbol{x}, \boldsymbol{e}_0\rangle)\boldsymbol{v}'(t(v))$, where λ satisfies $\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle\boldsymbol{v} = \lambda\boldsymbol{v}'$ (so $\lambda = \langle \boldsymbol{x}_v, \boldsymbol{v}'\rangle$ and is distinct from zero). This is the zero direction if and only if $\lambda + t'(v)\langle \boldsymbol{x}, \boldsymbol{e}_0\rangle = 0$. This can occur at isolated points on the

LD, and these points are distinct from the points of intersection of the LD with the closure of the parabolic set. At such points the projection $\pi \circ h$ has generically a cusp singularity and the map $(\pi \circ h, \mu)$ is a regular map. Therefore, $(\pi \circ h, \mu)$ is equivalent to an integral diagram of type Clairaut cusp (Theorem 4.3(5)) and the configuration of the family of apparent contours is as in Figure 4(5).

Suppose now that the image of the LD is not singular. The map h is an immersion and $\pi \circ h$ has fold singularities at points on the LD which are not in the closure of the parabolic set. Thus, $(\pi \circ h, \mu)$ is equivalent to an integral diagram of type Clairaut fold (Theorem 4.3(3)) and the configuration of the family of apparent contours is as in Figure 4(3). At points of intersection of the LD with the closure of the parabolic set, $h(\Sigma)$ is a surface with a cross-cap singularity. The map-germ $\pi \circ h$ is a fold map and the restriction of μ to its singular set has a Morse singularity. Therefore, $(\pi \circ h, \mu)$ is equivalent to an integral diagram of type Clairaut cross-cap ([6, Theorem 2.7]), so the configuration of the apparent contours is as in Figure 4(7).

We consider now points on the $LD \cap LPL$ and proceed as in the previous case. However, here $\bar{l} = 0$ so we take a parametrisation of Σ in the form $(u, v(u,t), t)$. The map $h : \Sigma \to PT^*(U) = U \times \mathbb{P}^1$ is now given by

$$h(u,t) = (P(u, v(u,t)), [\xi(u,t)]).$$

The function $\mu = t|_\Sigma$ is a submersion and the curves $h_t(u) = h(u,t)$ for t fixed are, by construction, Legendrian curves. We need to check when h is an immersion. Differentiating as above in an affine chart, we get

$$h_u = (P_u + v_u P_v, (\frac{\xi_2}{\xi_1})_u) \text{ and } h_t = (P_t + v_t P_v, (\frac{\xi_2}{\xi_1})_t).$$

At a point p_0 in $LPL \cap LD$, $\bar{g}_u + v_u \bar{g}_v = 0$, and $\bar{g}_u = -\alpha \bar{l} = 0$ (point on the LPL), so $v_u = 0$. Similarly, $\bar{g}_t + v_t \bar{g}_v = 0$, and $\bar{g}_t = 0$ (point on the LD), so $v_t = 0$. Also, we have $P_u = \boldsymbol{x}_u + \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle \boldsymbol{v} = 0$ and $P_t = \langle \boldsymbol{x}, \boldsymbol{e}_0 \rangle \boldsymbol{v'} \neq 0$ (we take $\langle \boldsymbol{x}, \boldsymbol{e}_0 \rangle \neq 0$), so

$$h_u = (0, 0, (\frac{\xi_2}{\xi_1})_u) \text{ and } h_t = (P_t, (\frac{\xi_2}{\xi_1})_t).$$

These two vectors are linearly dependent at p_0 if and only if $(\xi_2/\xi_1)_u = 0$ at p_0, that is, if and only if $\langle \xi_u, \boldsymbol{v} - \boldsymbol{e}_0 \rangle = 0$ (see argument above). We have $v_u = 0$ at p_0, so

$$\begin{aligned}
\xi_u &= \langle \boldsymbol{x}_{uv} + \boldsymbol{x}_{vv} v_u, \boldsymbol{e}_0 \rangle \boldsymbol{x}_u + \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle (\boldsymbol{x}_{uu} + v_u \boldsymbol{x}_{uv}) \\
&\quad - \langle \boldsymbol{x}_{uu} + v_u \boldsymbol{x}_{uv}, \boldsymbol{e}_0 \rangle \boldsymbol{x}_v - \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle (\boldsymbol{x}_{uv} + v_u \boldsymbol{x}_{vv}) \\
&= \langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0 \rangle \boldsymbol{x}_u + \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle \boldsymbol{x}_{uu} - \langle \boldsymbol{x}_{uu}, \boldsymbol{e}_0 \rangle \boldsymbol{x}_v - \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle \boldsymbol{x}_{uv}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle \xi_u, \boldsymbol{v} - \boldsymbol{e}_0 \rangle &= \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_{uu}, \boldsymbol{v} \rangle - \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_{uv}, \boldsymbol{v} \rangle \\
&\quad - \langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle - \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_{uu}, \boldsymbol{e}_0 \rangle \\
&\quad + \langle \boldsymbol{x}_{uu}, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle + \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0 \rangle \\
&= \alpha \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_{uu}, \boldsymbol{x}_u \rangle - \alpha \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_{uv}, \boldsymbol{x}_u \rangle \\
&= \alpha\gamma \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle \bar{l} - \alpha\gamma \bar{m} \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle \\
&= -\alpha\gamma \bar{m} \langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle,
\end{aligned}$$

where, as before, $\boldsymbol{v} = \alpha \boldsymbol{x}_u$ and $\boldsymbol{x}_u = \gamma \boldsymbol{x}_u \times \boldsymbol{x}_v$ (with $\alpha\gamma \neq 0$). For a generic surface, the point $p_0 \in LD \cap LPL$ is not in the closure of the parabolic set, i.e., $\bar{m} \neq 0$, so the above expression is not zero. Therefore, h is an immersion. The image of $(\pi \circ h, \mu)(u,t) = (P(u, v(u,t), t), t)$ is Δ which is a cuspidal edge. The singular set of the map-germ $\pi \circ h$ consists of the singular set of Δ together with another curve meeting it transversally (so the singular set has an A_1^--singularity). The fibre of μ at p_0 is tangent to the singular set of Δ. It is not hard to show that these are the geometric criteria for $(\pi \circ h, \mu)$ to be equivalent to an integral diagram of type mixed fold (Theorem 4.3(6)), so the configuration of the apparent contours is as in Figure 4(7). $\square$

Example 4.8 We use Maple to draw a family of apparent contours at $p_0 \in LD \cap LPL$. We proceed as in Example 4.6 and take $f(x,y) = xy + x^3 + x^4$. The contour generators and the apparent contours are drawn in Figure 7.

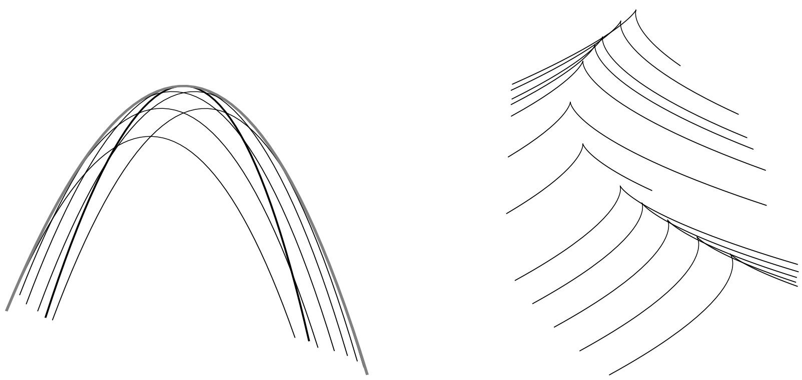

Figure 7: Maple generated figures of the contour generators left and of the apparent contours right at a point of intersection of the LD (in grey) with the LPL (in thick black). The cusp curve and the image of the LD are omitted in the figure on the right to make the apparent contours more visible.

4.2 Configurations of Σ_t and Δ_t at points on the LPL

We start with the configurations of the contour generators Σ_t at points on the LPL (and away from the LD).

Theorem 4.9 *The contour generators are solutions of a first order ordinary differential equation at points on the LPL. The associated integral diagram is of type Non-singular if the point is not parabolic;* Figure 4(1)). *If the point is parabolic, the associated integral diagram is not generic as an integral diagram but is generic as a divergent mapping diagram and has the type* (2) *of* Theorem 4.1 (Figure 3(2)).

Proof We follow the proof of Theorem 4.5(2). Here we can consider the function g in (5) instead of $\bar{g}$ as the point p_0 in consideration is in the Lorentzian part of the surface. We have $g_t \neq 0$ at p_0, so we can parametrise Σ locally in the form $(u, v, \psi(u, v))$. Then the map-germ $f : \Sigma \to U \times \mathbb{P}^1$ is given by

$$f(u, v) = (u, v, [g_u(u, v, \psi(u, v)) : g_v(u, v, \psi(u, v))]),$$

and clearly $\pi \circ f$ is a submersion. The map-germ $\mu = t|_\Sigma$ is $\psi(u, v)$. The point p_0 is on the LPL, so $l = 0$, that is $\psi_u = 0$ at p_0. We have $\psi_v = 0$ at p_0 if and only if $m = 0$ at p_0, equivalently, if and only if p_0 is a parabolic point. Therefore, if p_0 is not a parabolic point, ψ is a submersion so the integral diagram is of type Non-singular (Theorem 4.3(1)) and the configuration of the contour generators is as in Figure 3(1). (This is also the configuration at a swallowtail singularity of the projection. At such points the LPL is tangent the contour generator at p_0, Theorem 3.1(2(ii)).)

For a generic surface, the map $\mu = \psi$ has a Morse singularity at a parabolic point (this is a necessary and sufficient condition for the projection to have a lips/beaks singularity). Thus, the diagram $(\pi \circ f, \mu)$ is equivalent as a divergent mapping diagram to the case (2) of Theorem 4.1. The configurations of the contour generators are as in Figure 3(2). We have the closed loops configuration at the lips singularity of the projection P and the other configuration at the beaks singularities. As μ is not a submersion, $(\pi \circ f, \mu)$ is not generic as an integral diagram. □

We consider now the configurations of the family of apparent contours at points on the image of the LPL.

Theorem 4.10 *At the image of a point p_0 on the LPL, the apparent contours are solutions of a differential equation with an integral diagram of type*

Regular fold: *if the lightilke projection has cusp singularity at p_0* (Figure 4(2)).
Regular cusp: *if the lightilke projection has swallowtail singularity at p_0. Both instances of* Figure 4(4) occur.

If the lightilke projection has a lips/beaks singularity at p_0, the integral diagram of the differential equation is neither generic as an integral diagram nor as a divergent mapping diagram. The configurations of the apparent contours are as in Figure 9.

Proof We follow the same steps of the proof of Theorem 4.7(2) and parametrise Σ locally in the form $(u, v, \phi(u, v))$ (as in the proof of Theorem 4.9 above). Then $h : \Sigma \to U \times \mathbb{P}^1$ is given by $h(u, v) = (P(u, v, \psi(u, v)), [\xi(u, v)])$ and is locally an immersion (the arguments are similar to those in the proof of Theorem 4.7(2)).

Away from the parabolic and swallowtail points, the map $\pi \circ h$ is a fold map and the function $\mu = \psi(u, v)$ is a submersion, so $(\pi \circ h, \mu)$ is equivalent to an integral diagram of type regular fold (Theorem 4.3(2)) and the configuration of the apparent contours is as in Figure 4(2).

At a swallowtail singularity of the projection, $\pi \circ h$ is a cusp map and $\mu = \psi(u, v)$ is a submersion (p_0 is not a parabolic point). Therefore, $(\pi \circ h, \mu)$ is equivalent to an integral diagram of type regular cusp (Theorem 4.3(4)). We have two possible configurations of the apparent contours in Figure 4(4) (determined by the sign $\beta_y(0) = \pm 1$ in Theorem 4.3(4)). It is shown in [22] that the two configurations are determined geometrically by what is here the cusp curve and of the apparent contour Δ_{t_0} which passes through the cusp of the cusp curve. The configuration is as in Figure 4(4), left, if the cusp curve and Δ_{t_0} are in the same semi-plane delimited by the limiting tangent line to Δ_{t_0} and as in Figure 4(4), right, if they are in different semi-planes.

We take a local parametrisation of the surface M as in Theorem 2.1 (1). We have $g(u, v) = \langle \boldsymbol{v}(t_0), \boldsymbol{N}(u, v)\rangle$, and we take $l = l_u = 0$ and $l_{uu} \neq 0$ at (u_0, v_0), with $p_0 = \boldsymbol{x}(u_0, v_0)$. Thus, the critical set Σ_{t_0} can be parametrised by $(u, v_1(u))$ with $v_1'(u_0) = 0$. Now $\Delta_{t_0}(u) = \boldsymbol{x}(u, v_1(u)) + \langle \boldsymbol{x}(u, v_1(u)), \boldsymbol{e}_0\rangle \boldsymbol{v}(t_0)$. We know that this curve has a singularity $\mathcal{A}$-equivalent to (u^3, u^4). At (u_0, v_0), we have $\Delta_{t_0}''' = v_1'''(\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}(t_0))$ and $\Delta_{t_0}^{(4)} = v_1^{(4)}(\boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0\rangle \boldsymbol{v}(t_0)) + v_1'''(\boldsymbol{x}_{uv} + \langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0\rangle \boldsymbol{v}(t_0))$.

We also have $\boldsymbol{x}_{uv} + \langle \boldsymbol{x}_{uv}, \boldsymbol{e}_0\rangle \boldsymbol{v}(t_0) = m(\boldsymbol{N} + \langle \boldsymbol{N}, \boldsymbol{e}_0\rangle \boldsymbol{v}(t_0))$ at (u_0, v_0). We set

$$\begin{array}{rcl} \boldsymbol{w}_1 & = & \boldsymbol{x}_v(u_0, v_0) + \langle \boldsymbol{x}_v(u_0, v_0), \boldsymbol{e}_0\rangle \boldsymbol{v}(t_0) \\ \boldsymbol{w}_2 & = & \boldsymbol{N}(u_0, v_0) + \langle \boldsymbol{N}(u_0, v_0), \boldsymbol{e}_0\rangle \boldsymbol{v}(t_0) \end{array}$$

as the basis of a new coordinate system in the plane of projection. Then

$$\begin{array}{rcl} \Delta_{t_0}(u) & = & (\frac{1}{3!}v_1'''(u_0)(u-u_0)^3 + \frac{1}{4!}v_1^{(4)}(u_0)(u-u_0)^4 + h.o.t)\boldsymbol{w}_1 \\ & & + m(u_0, v_0)(\frac{3}{4!}v_1'''(u_0)(u-u_0)^4 + h.o.t)\boldsymbol{w}_2. \end{array}$$

A short calculation shows that $v_1'''(u_0) = -(l_{uu}/l_v)(u_0, v_0)$, so the position of the curve Δ_{t_0} with respect to the $\boldsymbol{w}_1$-axis (which is along its limiting tangent direction) is determined by the sign of

$$-\frac{ml_{uu}}{l_v}(u_0, v_0). \tag{10}$$

We turn now to the image of the LPL (the cusp curve). The LPL is the projection to the (u, v)-plane of the set of points (u, v, t) such that $g = g_u = 0$. The solution of this system of equations form a smooth curve $(u, v_2(u), t(u))$ in $\mathbb{R}^3$. The cusp curve is then parametrised by

$$\gamma(t) = \boldsymbol{x}(u, v_2(u)) + \langle \boldsymbol{x}(u, v_2(u)), \boldsymbol{e}_0 \rangle \boldsymbol{v}(t(u)).$$

We have $\gamma'(u_0) = 0$ so $\gamma(u) = \frac{1}{2!}\gamma''(u_0)(u - u_0)^2 + \frac{1}{3!}\gamma'''(u_0)(u - u_0)^3 + h.o.t$ with

$$\gamma''(u_0) = v_2''(u_0)\boldsymbol{w}_1 + t''(u_0)\langle \boldsymbol{x}(u_0, v_0), \boldsymbol{e}_0 \rangle \boldsymbol{v}'(t_0).$$

We are seeking the position of γ with respect to the $\boldsymbol{w}_1$-axis. This is determined by the position of the vector $\gamma''(u_0)$ with respect to the $\boldsymbol{w}_1$-axis, which in turn is determined by the sign of $\langle \gamma''(u_0), \boldsymbol{w}_2 - (\langle \boldsymbol{w}_2, \boldsymbol{w}_1 \rangle / \langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle)\boldsymbol{w}_1 \rangle$.

We have

$$v_2''(u_0) = -\frac{l_{uu}}{l_v}(u_0, v_0) \text{ and } t''(u_0) = -\frac{l_v(u_0, v_0)v_2''(u_0)}{\langle \boldsymbol{x}(u_0, v_0), \boldsymbol{e}_0 \rangle \langle \boldsymbol{N}(u_0, v_0), \boldsymbol{v}'(t_0) \rangle},$$

so the sign we are seeking is the same as that of

$$-\frac{l_{uu}}{l_v}\left(\frac{\langle \boldsymbol{N}, \boldsymbol{e}_0 \rangle}{\langle \boldsymbol{x}_u, \boldsymbol{e}_0 \rangle}F - (1 - \frac{\langle \boldsymbol{N}, \boldsymbol{e}_0 \rangle \langle \boldsymbol{x}_v, \boldsymbol{v}' \rangle}{\langle \boldsymbol{N}, \boldsymbol{v}' \rangle \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle})l_v\right) \tag{11}$$

evaluated at (u_0, v_0). Clearly, the product of the expression in (10) with that in (11) can have positive or negative sign, so both configurations in Figure 4(4) can occur. (See Example 4.11 for a Maple plot of examples of these configurations.)

At a lips/beaks singularity of the projection, the image of $(\pi \circ h, \mu)$ is Δ so is cuspidal-edge (in particular, Theorem 4.1 does not apply to this case). The function $\mu = \psi(u, v)$ has a Morse singularity (not a submersion, in particular, Theorem 4.3 does not apply to this case). Here, we have sections of the cuspidal edge by the smooth fibres of μ, with μ restricted to the singular set of the cuspidal edge having a Morse singularity. This is studied in [2], and we get the usual lips and beaks transitions. But we need to consider the projections of the sections to the plane and find how these are stacked together. We proceed as follows.

The apparent contours are the images of the level sets of $\mu = \psi(u, v)$ by the map-germ

$$\tilde{P}(u, v) = P(u, v, \psi(u, v)) = \boldsymbol{x}(u, v) + \langle \boldsymbol{x}(u, v), \boldsymbol{e}_0 \rangle \boldsymbol{v}(\psi(u, v))$$

We have $\tilde{P}_u = \psi_u \langle \boldsymbol{x}, \boldsymbol{e}_0 \rangle \boldsymbol{v}'$ and $\tilde{P}_v = \boldsymbol{x}_v + \langle \boldsymbol{x}_v, \boldsymbol{e}_0 \rangle \boldsymbol{v} + \psi_v \langle \boldsymbol{x}(u, v), \boldsymbol{e}_0 \rangle \boldsymbol{v}'$, so the map $\tilde{P}$ is singular if and only if $\psi_u = 0$, that is, if and only if $g_u = 0$ (equivalently, if and only if $\boldsymbol{x}(u, v) \in LPL$). The LPL is a smooth curve and can be parametrised in the form $(u(v), v)$. The vector $\tilde{P}_v$ is not zero along the LPL, so the map $\tilde{P}$ is a fold map. Observe that the critical sets of the projections P^t are tangent to the kernel of the map $\tilde{P}$.

The problem of the configuration of the images of the level sets of ψ by the map $\tilde{P}$ can then be formulated as follows. We take $\sigma(u, v) = (u, v^2)$ as a model of a fold map with the axis $v=0$ as the set of its fixed points. We then classify germs of functions $f:\mathbb{R}^2,0\to\mathbb{R},0$ up changes of coordinates in the source that preserve the involution σ and any changes of coordinates in the target. These changes of coordinate form the group $\mathcal{A}^{\mathbb{Z}_2}=\mathcal{R}^{\mathbb{Z}_2}\times\mathcal{L}$ which acts on the set of germs f. The germs of functions f of interests are those of submersions or of Morse functions. Also, we want the regular fibres of f to have vertical tangents along the fixed set of the involution σ, so we can write $f(u,v)=v^2h(u,v)$ for some germ of a smooth function h.

Using the standard classification techniques from singularity theory (see for example [1]), we find that the $\mathcal{A}^{\mathbb{Z}_2}$-finitely determined germs of interest are $u+v^3$ and $u^2\pm v^2+v^3$. Applying the involution σ to the fibres of $u^2\pm v^2+v^3$ gives the model of the configurations of the apparent contours at a lips/beaks singularity, Figure 9. (Observe that in Figure 9 there is a segment of a curve where the apparent contours are tangential. The configurations in Figure 9 are distinct from those of the folded saddle and focus in Figure 1.) Applying the involution σ to the fibres of $u+v^3$ gives a family of cusps which is the model at images of points on the LPL which are not swallowtail or lips/beaks singularities of the projection. $\square$

Example 4.11 Consider a surface parametrised in the form $(1+x-y,x+y,f(x,y))$, with (x,y) near the origin. We take $f(x,y)=y^2+x^3+xy^2$ for the case of the lips singularity and $f(x,y)=y^2+x^3-xy^2$ for the beaks (see Figure 9). For the swallowtail singularity we take $f(x,y)=xy+y^2+x^2y+x^4$ for case 1; Figure 8, center, and $f(x,y)=xy+y^2-x^2y-x^4$ for case 2; Figure 8, right.

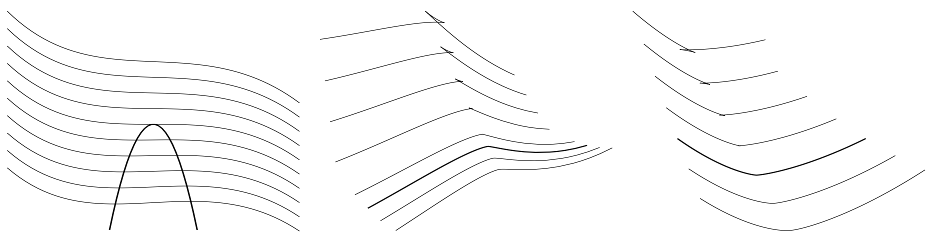

Figure 8: Maple generated figures of ontour generators and apparent contours at a swallowtail singularity. The cusp curve is omitted in the last two figures to make the apparent contours more visible.

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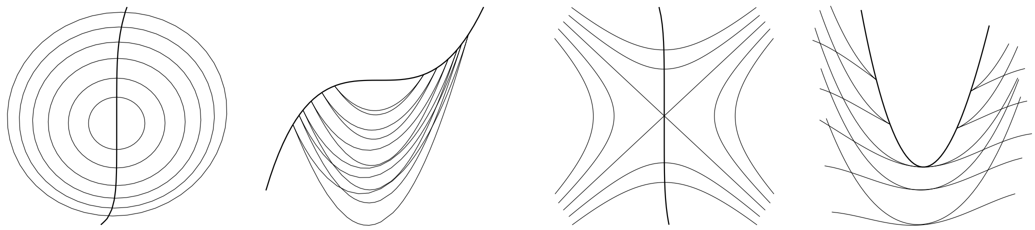

Figure 9: Maple generated figures of contour generators and apparent contours at a lips singularity (first two figures) and at a beaks singularity (last two figures).

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