Projections of surfaces in the hyperbolic space along horocycles

Shyuichi Izumiya and Farid Tari

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Abstract

We study in this paper orthogonal projections of embedded surfaces M in $H^3_+(-1)$ along horocycles to planes. The singularities of the projections capture the extrinsic geometry of M related to the lightcone Gauss map. We give geometric characterisations of these singularities and prove a Koenderink type theorem which relates the hyperbolic curvature of the surface to the curvature of the profile and of the normal section of the surface. We also prove duality results concerning the bifurcation set of the family of projections.

1 Introduction

The work of this paper is part of a wider project of investigating the extrinsic geometry of submanifolds embedded in the pseudo-spheres in the Minkowski space $\mathbb{R}^n_1$ via singularity theory. The extrinsic geometric information is obtained by considering the contact of the submanifold with degenerate objects in the given pseudo-sphere. By degenerate we mean, for instance, a flat object (that is, a submanifold with some curvature vanishing everywhere).

We studied in [27] the contact of surfaces in the hyperbolic space with geodesics. This is measured by the singularities of orthogonal projections of the surface along geodesics to hyperplanes and horospheres. The expression of an orthogonal projection along a geodesic is also given in [28] using a different method. The work in [27] is analogous to that on orthogonal projections of surfaces in the Euclidean and projective 3-spaces (these are well studied; see for example [1, 3, 4, 6, 7, 9, 11, 29, 30, 32, 33, 34,

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35]). Analogous in [27] means that we view a line in the Euclidean space as a geodesic. But a line in the Euclidean space can also be viewed as the limit of circles with radii tending to infinity. In the hyperbolic space, the limit of circles with radii tending to infinity is a horocycle. So it is natural to consider, as we do in this paper, projections in the hyperbolic space along horocycles.

Another reason we consider projections along horocycles is the following. Given $p \in M$, there is a well defined unit normal vector $e(p)$ to M at p; see §2. (If M is orientable, then $e(p)$ is globally defined. However, it is always locally defined and our investigation here is local in nature.) The vector $e(p)$ is in the de Sitter space S^3_1 and we have the de Sitter Gauss map

$$\begin{array}{rccc} \mathbb{E}: & M & \to & S^3_1 \\ & p & \to & e(p) \end{array}$$

The projections of M along geodesics pick up extrinsic geometric information about M related to the de Sitter Gauss map, see [27]. For example, the projection along a geodesic has a cusp singularity at p if and only if the tangent to the geodesic at p is a de Sitter asymptotic direction. The points on M where the projection has a swallowtail singularity is precisely the locus of points of geodesic inflections of the de Sitter asymptotic curves. Also, the projection has a lips/beaks singularity at p if and only if p is a de Sitter parabolic point and the tangent to the geodesic is along the unique de Sitter asymptotic direction at p.

There is another Gauss map on the surface introduced in [19] and called the lightcone Gauss map; see §2. The vector $p \pm e(p)$ is lightlike (i.e., belongs to the lighcone LC^*), so we have the lightcone Gauss maps

$$\begin{array}{rccc} \mathbb{L}^\pm: & M & \to & LC^* \\ & p & \to & p \pm e(p) \end{array}$$

Projecting along horocycles is a natural candidate to pick up extrinsic geometric information about the surface related to the lightcone Gauss map. We expect, for instance, the projection along a horocycle with tangent at p along a horo-asymptotic direction (i.e., a direction u satisfying $\langle d_p(\mathbb{L}^\pm)(u), u\rangle = 0$) to have a cusp singularity at p.

We give in §3 the expression of the family P of projections along horocycles to orthogonal planes. The planes of projection are arbitrary, that is, projecting to parallel orthogonal planes yields the same information. So we project to the planes that passe through the point $p_0 = (1, 0, 0, 0)$. Then the horocycles of interest are determined by a pair of vectors (l, v) in the set $\mathcal{C} = \{(l, v) \in S^2_+ \times S^2_0 \mid \langle l, v\rangle = 0\}$, where S^2_+ and S^2_0 are spheres in LC^* and S^3_1 respectively. The set $\mathcal{C}$ is in fact the parameter space of our family of projections. For (l, v) fixed, the map $P_{(l,v)}$ can be considered locally as a map-germ from $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$. We show in §4 that $P_{(l,v)}$ has a cusp singularity or worse if and only if $\kappa(v^*) = -\langle l, e(p)\rangle / \langle l, p\rangle$, where v^* is the tangent to the horocycle (l, v)

at p and $\kappa(v^*)$ is the de Sitter normal curvature at p along v^*. The above relation yields several interesting geometric properties of M (Theorem 4.2). For instance, $\kappa(v^*) = \pm 1$ if and only if v^* is a horo-asymptotic direction and $l = \widetilde{\mathbb{L}^{\pm}}(p)$, where $\widetilde{\mathbb{L}^{\pm}}(p)$ is the radial projection of $\mathbb{L}^{\pm}(p)$ to S^2_+. We also show that the points $p \in M$ where the projection along the horocycle $(\widetilde{\mathbb{L}^{\pm}}(p), v)$ has a swallowtail singularity at p is precisely the locus of geodesic inflections of the horo-asymptotic curves (Proposition 4.3). We then prove a Koenderink type theorem which relates the hyperbolic curvature of the surface to the curvature of the profile and of the normal section of the surface (Theorem 4.4).

We prove in §5 a duality result between some surfaces associated to M and some special curves on the Δ_2-dual surface $M^{(2,*)}$ of M; (see §6). The result is analogous to that of Shcherback in [34]. We also prove a duality result concerning the bifurcation sets of the families of projections, analogous to that of Bruce-Romero Fuster in [9]. In §5, we use the duality concepts introduced in [13, 14]; see §6 for details.

2 Preliminaries

The *Minkowski space* $(\mathbb{R}^4_1, \langle , \rangle)$ is the 4-dimensional vector space $\mathbb{R}^4$ endowed with the pseudo scalar product $\langle x, y\rangle = -x_0y_0 + \sum_{i=1}^3 x_iy_i$, where $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, x_1, x_2, y_3)$ in $\mathbb{R}^4_1$. We say that a vector x in $\mathbb{R}^4_1 \setminus \{0\}$ is *spacelike*, *lightlike* or *timelike* if $\langle x, x\rangle > 0$, $= 0$ or < 0 respectively. The norm of a vector $x \in \mathbb{R}^4_1$ is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$.

Given a vector $v \in \mathbb{R}^4_1$ and a real number c, the hyperplane with pseudo normal v is defined by

$$HP(v, c) = \{x \in \mathbb{R}^4_1 \mid \langle x, v\rangle = c\}.$$

We say that $HP(v, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if v is timelike, spacelike or lightlike respectively. We have the following three types of pseudo-spheres in $\mathbb{R}^4_1$:

$$\begin{array}{rrl} \textit{Hyperbolic } 3\textit{-space}: & H^3(-1) & = \{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = -1\}, \\ \textit{de Sitter } 3\textit{-space}: & S^3_1 & = \{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = 1\}, \\ (\textit{open}) \textit{ lightcone}: & LC^* & = \{x \in \mathbb{R}^4_1 \setminus \{0\} \mid \langle x, x\rangle = 0\}. \end{array}$$

We also define the *lightcone sphere*

$$S^2_+ = \{x \in LC^* \,|\, \langle x, x\rangle = 0,\ x_0 = 1\}$$

and the Euclidean sphere

$$S^2_0 = \{x \in S^3_1 \,|\, \langle x, x\rangle = 1,\ x_0 = 0\}.$$

For $x \in LC^*$, we have $x_0 \neq 0$ so

$$\widetilde{x} = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \in S^2_+.$$

The hyperbolic space has two connected components. We only consider embedded surfaces in $H^3_+(-1) = \{x \in H^3(-1) |\, x_0 \geq 1\}$ as the study is similar for those embedded in $H^3_-(-1) = \{x \in H^3(-1) |\, x_0 \leq -1\}$.

The wedge product of 4 vectors $a_1, a_2, a_3 \in \mathbb{R}^4_1$ is given by

$$a_1 \wedge a_2 \wedge a_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ a^1_0 & a^1_1 & a^1_2 & a^1_3 \\ a^2_0 & a^2_1 & a^2_2 & a^2_3 \\ a^3_0 & a^3_1 & a^3_2 & a^3_3 \end{vmatrix},$$

where $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4_1$ and $a_i = (a^i_0, a^i_1, a^i_2, a^i_3)$, $i = 1, 2, 3$. One can check that $\langle a, a_1 \wedge a_2 \wedge a_3 \rangle = \det(a, a_1, a_2, a_3)$, so the vector $a_1 \wedge a_2 \wedge a_3$ is pseudo orthogonal to all the vectors a_i, $i = 1, 2, 3$.

Some aspects of the extrinsic geometry of hypersurfaces in the hyperbolic space are studied in [13]–[23] and [26, 27]. Let M be a surface embedded in $H^3_+(-1)$. Given a local chart $\boldsymbol{i} : U \to M$, where U is an open subset of $\mathbb{R}^2$, we denote by $\boldsymbol{x} : U \to H^3_+(-1)$ such embedding, identify $\boldsymbol{x}(U)$ with U through the embedding $\boldsymbol{x}$ and write $M = \boldsymbol{x}(U)$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv -1$, we have $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle \equiv 0$, for $i = 1, 2$, where $u = (u_1, u_2) \in U$. We define the spacelike unit normal vector $e(u)$ to M at $\boldsymbol{x}(u)$ by

$$e(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \boldsymbol{x}_{u_2}(u)\|}.$$

It follows that the vector $\boldsymbol{x} \pm e$ is a lightlike vector. Let

$$\mathbb{E} : U \to S^n_1 \quad \text{and} \quad \mathbb{L}^\pm : U \to LC^*$$

be the maps defined by $\mathbb{E}(u) = e(u)$ and $\mathbb{L}^\pm(u) = \boldsymbol{x}(u) \pm e(u)$. These are called, respectively, the *de Sitter Gauss map* and the *lightcone Gauss map* (or *hyperbolic Gauss indicatrix*) of M ([19]). For any $p = \boldsymbol{x}(u_0) \in M$ and $v \in T_pM$, one can show that $D_v\mathbb{E} \in T_pM$, where D_v denotes the covariant derivative with respect to the tangent vector v. Since the derivative $d\boldsymbol{x}(u_0)$ can be identified with the identity mapping 1_{T_pM} on the tangent space T_pM, we have $d\mathbb{L}^\pm(u_0) = 1_{T_pM} \pm d\mathbb{E}(u_0)$, under the identification of U and M via the embedding $\boldsymbol{x}$.

The linear transformation $A_p = -d\mathbb{E}(u_0)$ is called the *de Sitter shape operator*. Its eigenvalues κ_i, $i = 1, 2$, are called the *de Sitter principal curvature* and the corresponding eigenvectors p_i, $i = 1, 2$, are called the *de Sitter principal directions*.

The linear transformation $S^\pm_p = -d\mathbb{L}^\pm(u_0)$ is labelled the *lightcone shape operator* of M at p. It has the same eigenvectors as A_p but its eigenvalues are distinct from those of A_p. In fact the eigenvalues $\bar{\kappa}^\pm_i$ of $S^\pm_p$ satisfy $\bar{\kappa}^\pm_i = -1 \pm \kappa_i$, $i = 1, 2$.

We call $K_e(p) = \det(A_p) = \kappa_1(p)\kappa_2(p)$ (resp. $K_h(p) = \det(S^\pm_p) = \bar{\kappa_1}(p)\bar{\kappa_2}(p)$) the *de Sitter* (resp. *hyperbolic*) *Gauss-Kronecker curvature* of M at p. The curvature K_e is also called the *extrinsic Gaussian curvature*. The set of points where $K_e(p) = 0$ (resp.

$K_h^{\pm}(p) = 0$) is labelled the *de Sitter* (resp. *horospherical*, horo- for short) *parabolic set* of M. The restriction of the pseudo-scalar product to the hyperbolic space is a scalar product, so $H^3_+(-1)$ is a Rimaniann manifold. We have the sectional curvature K_I of M which is also called the *intrinsic Gaussian curvature*. It is known that $K_e = K_I + 1$ (see §2.2 in [10]).

The operators A_p and $S_p^{\pm}$ are self-adjoint operators on M, so we can define the notion of asymptotic directions at p. We say that $u \in T_pM$ is a *de Sitter* (*resp. horo-*) *asymptotic direction* if and only if $\langle A_p(u), u\rangle = 0$ (resp. $\left\langle S_p^{\pm}(u), u\right\rangle = 0$). There are 0/1/2 de Sitter (resp. horo-) asymptotic directions at every point where $K_e(p)$ (resp. $K_h(p)$) $0 > / = / < 0$.

We also define the de Sitter *normal curvature* at p along a direction $u \in T_pM$ to be $\kappa(u) = \langle A_p(u), u\rangle / \langle u, u\rangle$. The horo-normal curvature at p along a direction $u \in T_pM$ is defined similarly and is given by $\kappa_h^{\pm}(u) = \left\langle S_p^{\pm}(u), u\right\rangle / \langle u, u\rangle$.

A surface given by the intersection of $H^3_+(-1)$ with a spacelike, timelike or lightlike hyperplane is called respectively *sphere*, *equidistant surface* or *horosphere*. The intersection of the surface with timelike hyperplane through the origin is called simply a *plane*. Planes are the only surfaces with everywhere zero de Sitter Gaussian curvature ([21]). Horospheres are the only surfaces with everywhere zero hyperbolic curvature ([21]).

3 The family of projections along horocycles

We need some preliminaries about curves in $H^3_+(-1)$. Let $\gamma : I \to H^3_+(-1)$ be a regular curve. Since $H^3_+(-1)$ is a Riemannian manifold, we can parametrise γ by arc-length and assume that $\gamma(s)$ is unit speed. Let $t(s) = \gamma'(s)$, with $||t(s)|| = 1$. The vector $t'(s)$ is not in the tangent space $T_{\gamma(s)}H^3_+(-1)$, so we project it along γ to this tangent space. The resulting vector is $t'(s) - \gamma(s)$. Now if $\langle t'(s), t'(s)\rangle \neq -1$, then $||t'(s) - \gamma(s)|| \neq 0$. In this case, we define the unit normal vector to the curve as the vector $n(s) = \frac{t'(s)-\gamma(s)}{||t'(s)-\gamma(s)||}$. If $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$, then we have a pseudo orthogonal frame $\{\gamma(s), t(s), n(s), e(s)\}$ in $\mathbb{R}^4_1$ along γ. Frenet-Serret type formulae, similar to those for a space curve in $\mathbb{R}^3$, can be proved for the curve γ ([22]) and are as follows

$$\left\{\begin{array}{rcl} \gamma'(s) &=& t(s) \\ t'(s) &=& \gamma(s) + \kappa_h(s)n(s) \\ n'(s) &=& -\kappa_h(s)t(s) + \tau_h(s)e(s) \\ e'(s) &=& -\tau_h(s)n(s) \end{array}\right.$$

where $\kappa_h(s) = ||t'(s) - \gamma(s)||$ and $\tau_h(s) = -\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))/\kappa_h(s)^2$. The quantities $\kappa_h(s)$ and $\tau_h(s)$ are called the *curvature* (resp. *torsion*) of the curve γ at $\gamma(s)$. The condition $\langle t'(s), t'(s)\rangle \neq -1$ above is in fact equivalent to $\kappa_h(s) \neq 0$. (See

[22] for more results on curves in the hyperbolic plane.)

When the curve γ is not parametrised by arc length, the formula for the curvature is

$$\kappa_h(\theta) = ||\frac{1}{l'(\theta)^3}\left(\gamma''(\theta)l'(\theta) - \gamma'(\theta)l''(\theta)\right) - \gamma(\theta)||,$$

where

$$l(\theta) = \int_0^\theta ||\gamma'(\alpha)||d\alpha.$$

A *horocycle* in $H^3_+(-1)$ is the intersection of a horosphere with a plane orthogonal to the horosphere. Let $HS^3(l,c) = H^3_+(-1) \cap HP(l,c)$, with $l \in S^2_+ \subset LC^*$ and $c \in \mathbb{R}$, denote a horosphere in $H^3_+(-1)$. Then a horocycle is given by $HS^3(l,c) \cap HP(v,0)$, with $v \in S^3_1$ and $\langle l, v\rangle = 0$. Fixing l and v and varying c gives parallel horocycles.

A smooth curve in $H^3_+(-1)$ is a horocycle if and only if $\kappa_h \equiv 1$ and $\tau_h \equiv 0$ ([18]). Also, for a curve in a hyperbolic plane (which is the case of a horocycle), $\kappa_h = \kappa_g$, where κ_g denotes its geodesic curvature, and the hyperbolic curvature $K_h = \kappa_g - 1 = \kappa_h - 1$. Therefore, for a horocycle we have $K_h \equiv 0$, which means that they are flat objects.

We are considering here orthogonal projections along horocycles to planes. Projecting to two parallel planes yields the same information, so we choose those that pass through the point $p_0 = (1, 0, 0, 0)$. These are parametrised by the sphere S^2_0. So parallel horocycles in $H^3_+(-1)$ that are of interest are those determined by the elements of the set

$$\mathcal{C} = \{(l, v) \in S^2_+ \times S^2_0 \mid \langle l, v\rangle = 0\}.$$

Given a plane $HP(v, 0) \cap H^3_+(-1)$, with $v \in S^2_0$, and a point $p \in H^3_+(-1)$, there is a unique horocycle through p, determined by a unique $(l, v) \in \mathcal{C}$, which intersects orthogonally the given plane at some point $q(p,(l, v))$. We call the point $q(p,(l, v))$ the orthogonal projection of p along the horocycle (l, v).

Let $\pi : \mathcal{C} \to S^2_0$ be the projection $\pi(l, v) = v$ and $i : S^2_0 \to S^3_1$ be the inclusion $i(v) = v$. Let

$$\mathcal{F} = \{(v, w) \in TS^3_1 | \langle w, w\rangle = -1\} \to S^3_1$$

be the timelike unit spherical bundle over S^3_1.

Theorem 3.1 *The family of projections along horocycles in $H^3_+(-1)$ is given by*

$$\begin{array}{rccc} P: & H^3_+(-1) \times \mathcal{C} & \to & (i \circ \pi)^*\mathcal{F} \\ & (p,(l, v)) & \mapsto & ((l, v), q(p,(l, v))) \end{array}$$

where

$$q(p,(l, v)) = p - \langle p, v\rangle\, v + \frac{1}{2}\frac{\langle p, v\rangle^2}{\langle p, l\rangle} l.$$

Proof. A parametrisation of the horocycle (l, v) which is orthogonal to the plane $HP(v, 0) \cap H^3_+(-1)$ at $q(p, (l, v))$ is given by

$$c(s) = q(p, (l, v)) + sv + \frac{s^2}{2}\lambda l.$$

for some scalar λ, see [24]. The horocycle $c(s)$ passes through p if $p = q(p, (l, v)) + s_0 v + \frac{s_0^2}{2}\lambda l$ for some $s_0 \in \mathbb{R}$. So $q(p, (l, v)) = p - s_0 v - \frac{s_0^2}{2}\lambda l$. We have

$$\langle q(p, (l, v)), v\rangle = \langle p, v\rangle - s_0 = 0,$$

which gives $s_0 = \langle p, v\rangle$. As $c(s_0) \in H^3_+(-1)$, $\langle c(s_0), c(s_0)\rangle = -1$, we have

$$\langle q(p, (l, v)), \lambda l\rangle = \langle p, \lambda l\rangle = -1,$$

therefore $l = -1/\langle p, l\rangle$. Hence $q(p, (l, v)) = p - \langle p, v\rangle\, v + \frac{1}{2}\frac{\langle p,v\rangle^2}{\langle p,l\rangle} l$. □

In this paper, the family of orthogonal projections of a given surface M in $H^3_+(-1)$ along horocycles refers to the restriction of the family P to M. We still denote this restriction by P. We have the following result where the term generic is defined in terms of transversality to submanifolds of multi-jet spaces (see for example [12]).

Theorem 3.2 *For a residual set of embeddings $x : M \to H^3_+(-1)$, the family P is a generic family of mappings.*

Proof. The theorem follows from Montaldi's result in [31] and the fact that $P : H^3_+(-1) \times \mathcal{C} \to (i \circ \pi)^*\mathcal{F}$ is a stable map. □

We denote by $P_{(l,v)}$ the map $H^3_+(-1) \to H^3_+(-1)$, given by $P_{(l,v)}(p) = q(p, (l, v))$, with $q(p, (l, v))$ as in Theorem 3.1. We also keep the same notation for the restriction of $P_{(l,v)}$ to the surface $M \subset H^3_+(-1)$.

For a given $(l, v) \in \mathcal{C}$ and a point $p_0 \in M$, one can choose local coordinates so that $P_{(l,v)}$ restricted to M can be considered locally as a map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$. These map-germs are extensively studied. We refer to [32] for the list of the $\mathcal{A}$-orbits with $\mathcal{A}_e$-codimension ≤ 6, where $\mathcal{A}$ denotes the Mather group of smooth changes of coordinates in the source and target. In Table 1, we reproduce from [32] the list of local singularities of $\mathcal{A}_e$-codimension ≤ 3. Some of these singularities are also called as follows: 4_2 (lips/beaks), 4_3 (goose), 5 (swallowtail), 6 (butterfly), 11_5 (gulls).

It follows from Theorem 3.2 that for generic embeddings of the surface only singularities of $\mathcal{A}_e$-codimension $\leq \dim(\mathcal{C}) = 3$ can occur in the members of the family of orthogonal projections. So we have the following result.

Proposition 3.3 *For a residual set of embeddings $x : M \to H^3_+(-1)$, the projections $P_{(l,v)} : M \to H^3_+(-1)$ in the family P have local singularities $\mathcal{A}$-equivalent to one in Table 1 whose $\mathcal{A}_e$-codimension ≤ 3. Moreover, these singularities are versally unfolded by the family P.*

Table 1: $\mathcal{A}_e$-codimension ≤ 3 local singularities of map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ ([32]).

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	(x, y)	0
Fold	(x, y^2)	0
Cusp	$(x, xy + y^3)$	0
4_k	$(x, y^3 \pm x^k y), k = 2, 3, 4$	$k - 1$
5	$(x, xy + y^4)$	1
6	$(x, xy + y^5 \pm y^7)$	2
7	$(x, xy + y^5)$	3
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1}), k = 2, 3$	k
12	$(x, xy^2 + y^5 + y^6)$	3
16	$(x, x^2 y + y^4 \pm y^5)$	3

The members of P can also have multi-local local singularities with $\mathcal{A}_e$-codimension ≤ 3, and these singularities are also versally unfolded by the family P. However, in this paper, we deal mainly with the geometry of the local singularities.

4 Characterisations of the singularities of $P_{(l,v)}$

Given $(l, v) \in \mathcal{C}$ and a point $p \in M$, we denote by v^* the tangent at p to the horocycle through p and $q(p, (l, v))$. Then

$$v^* = c'(s_0) = v - \frac{\langle p, v \rangle}{\langle p, l \rangle} l.$$

where $c(s)$ and s_0 are as in the proof of Theorem 3.1. Let $S^2_p = T_p H^3_+(-1) \cap S^3_1$, $\mathcal{C}_p = \{(l, w) \in S^2_+ \times S^2_p |\, \langle l, w \rangle = 0\}$ and

$$\begin{array}{rccc} F_p : & \mathcal{C} & \to & \mathcal{C}_p \\ & (l, v) & \mapsto & (l, v^*) \end{array}$$

Proposition 4.1 *The map F_p is a diffeomorphism.*

Proof. The map F_p is clearly of class C^∞. It is injective as the vector v is spacelike and l is lightlike. To show that it is surjective, let $(l, w) \in \mathcal{C}_p$ and consider the horocycle $c(s) = p + sw - s^2/(2 \langle p, l \rangle) l$ through p. We are seeking a vector $v = w - (s_1/\langle p, l \rangle) l$ tangent to the horocycle at $c(s_1)$ and orthogonal to a plane that passes through $p_0 = (1, 0, 0, 0)$. So we require $\langle p_0, v \rangle = 0$. This gives $s_1 = \langle p_0, w \rangle \langle p, l \rangle / \langle p_0, l \rangle$ and $v = w - \langle p_0, w \rangle / \langle p_0, l \rangle l$. The inverse map $F_p^{-1}(l, w) = (l, w - \langle p_0, w \rangle / \langle p_0, l \rangle l)$ is clearly of class C^∞. $\square$

Let p be a point on an embedded surface M in $H^3_+(-1)$ and $w \in T_pM$. Then as a consequence of Proposition 4.1, if $(l, w) \in \mathcal{C}_p$, there is a unique $v \in S^2_0$ such that $w = v^*$.

We have the following geometric characterisations of the singularities of $P_{(l,v)}$, where "worse" means more degenerate, alternatively, has a higher $\mathcal{A}_e$-codimension.

Theorem 4.2 *Let M be an embedded surface in $H^3_+(-1)$ and $(l, v) \in \mathcal{C}$.*

(1) *The projection $P_{(l,v)}$ is singular at $p \in M$ if and only if $v^* \in T_pM$.*

(2) *For $v \in S^2_0$ fixed and $p \in M$, there is a circle of directions $l \in S^2_+ \subset LC^*$ such that $v^* \in T_pM$. This circle contains the directions $\widetilde{\mathbb{L}}^{\pm}(p) = \widetilde{p \pm e(p)}$.*

(3) *The projection $P_{(l,v)}$ has a singularity of type cusp or worse at $p \in M$ if and only if $v^* \in T_pM$ and $\kappa(v^*) = -\lambda(p, l)$, where $\lambda(p, l) = \langle e(p), l\rangle \,/\, \langle p, l\rangle$. So, for l fixed, there are at most two directions $v^* \in T_pM$ for which the singularity of $P_{(l,v)}$ is of type cusp or worse.*

(4) *The projection $P_{(l,v)}$ has a singularity of type lips/beaks or worse at $p \in M$ if and only if v^* is a principal direction and $-\lambda(p, l)$ is its associated de Sitter principal curvature. There are two directions l that satisfy $\kappa_1 = -\lambda(p, l)$ (or $\kappa_2 = -\lambda(p, l)$).*

(5) *The curves $\kappa_i =$ constant, $i = 1, 2$, are the loci of points where the two directions in (3) coincide. They foliate the region $-1 \le \kappa_i \le 1$, $i = 1, 2$. (Recall that the curves $\kappa_i = \pm 1$, $i = 1, 2$, are part of the horo-parabolic set associated to $K^{\pm}_h$.)*

(6) *The equality $\kappa(v^*) = -\lambda(p, l) = \pm 1$ holds if and only if $l = \widetilde{\mathbb{L}}^{\pm}(p)$ and v^* is a horo-asymptotic direction. So the projection $P_{(\widetilde{\mathbb{L}}^{\pm}(p),v)}$ has a singularity of type cusp at p or worse if and only if v^* is a horo-asymptotic direction. The singularity is of type lips/beaks or worse if and only if p is a horo-parabolic point and v^* is the unique horo-asymptotic direction at p (which is also a principal direction).*

(7) *Let p be an umbilic point, $l \in S^2_+$ with $|\kappa_1| < 1$. Then the projection $P_{(l,v)}$ has a singularity of type lips/beaks or worse for any $v \in S^2_0$ with $v^* \in T_pM$ and $l \in S^2_+$ with $-\lambda(p, l) = \kappa_1$. For all directions $v^* \in T_pM$, except a finite number of them, the singularity is genuinely of type lips/beaks. There are generically $0, 2, 4, 6$ directions in T_pM where the singularity becomes of type 11_5 and $0, 2, 4, 6, 8$ where it becomes of type 4_3.*

Proof. The statements in the theorem are of local nature, so we shall take the surface M in hyperbolic Monge form (H-Monge form, see [19]) at the point in consideration. In fact, by hyperbolic motions, we can suppose that the point of interest is $p_0 = (1, 0, 0, 0)$ and the surface given locally by

$$\boldsymbol{x}(x, y) = \left(\sqrt{f^2(x, y) + x^2 + y^2 + 1}, f(x, y), x, y\right),$$

with (x, y) in some neighbourhood of the origin. Here f is a smooth function with $f(0, 0) = 0$ and $f_x(0, 0) = f_y(0, 0) = 0$. So a unit normal to M at p_0 is given by

$e(0,0) = (0,1,0,0)$. We shall write the Taylor expansion of f, at the origin, in the form

$$f(x,y) = a_{20}x^2 + a_{21}xy + a_{22}y^2 + \sum_{i=0}^{3} a_{3i}x^{3-i}y^i + \sum_{i=0}^{4} a_{4i}x^{4-i}y^i + \text{h.o.t.}$$

Let $v = (0, v_1, v_2, v_3) \in S^2_+$. In order to make the contribution of $\langle p_0, l\rangle$ apparent, we take l on a sphere in LC^* with $l_0 = constant$. Then $l_0 = \langle p_0, l\rangle$. (If $l \in S^2_+$, then $\langle p_0, l\rangle = 1$.) We assume that $\langle l, v\rangle = 0$. We have $v^* = v$ at p_0 and

$$\begin{aligned} \partial P_{(l,v)}/\partial x(0,0) &= (0, -v_1v_2, 1 - v_2^2, -v_2v_3), \\ \partial P_{(l,v)}/\partial y(0,0) &= (0, -v_1v_3, -v_2v_3, 1 - v_3^2). \end{aligned}$$

These two vectors are linearly dependent if and only if $v_1 = 0$, if and only if $v^* \in T_{p_0}M$, which proves (1).

(2) We suppose from now on that $v_1 = 0$. Then $v_2^2 + v_3^2 = 1$ and $l = (l_0, l_1, -tv_3, tv_2)$ with $l_1^2 + t^2 = l_0^2$. We still denote $\widetilde{\mathbb{L}}^{\pm}(p)$ the projection of $p + e(p)$ to the sphere $l_0 = constant$. We have $\widetilde{\mathbb{L}}^{\pm}(p_0) = (l_0, \pm l_1, 0, 0)$, so $\langle \widetilde{\mathbb{L}}^{\pm}(p_0), v\rangle = 0$ for any $v = v^* \in T_{p_0}M$.

(3), (4) and (5) We write $P_{(l,v)}[i]$ for the ith coordinate function of $P_{(l,v)}$. The tangent plane Π of $HP(v,0) \cap H^3_+(-1)$ at p_0 is generated by the two vectors $(0,1,0,0)$ and $(0,0,-v_3,v_2)$. Let $\pi : HP(v,0) \cap H^3_+(-1) \to \Pi$ denotes the linear projection along the vector v. Consider the composite map

$$\pi \circ P_{(l,v)} = (0, P_{(l,v)}[2], v_3(v_3P_{(l,v)}[3] - v_2P_{(l,v)}[4]), -v_2(v_3P_{(l,v)}[3] - v_2P_{(l,v)}[4])),$$

which is $\mathcal{A}$-equivalent to the map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ given by

$$\widetilde{P}_{(l,v)} = (P_{(l,v)}[2], v_3P_{(l,v)}[3] - v_2P_{(l,v)}[4]).$$

As the singularities of $P_{(l,v)}$ restricted to M and those of $\widetilde{P}_{(l,v)}$ are $\mathcal{A}$-equivalent, we work with the map-germ $\widetilde{P}_{(l,v)}$. We reduce the appropriate k-jets of $\widetilde{P}_{(l,v)}$ to the form $(x, g(x,y))$ and interpret geometrically the conditions for this germ to be $\mathcal{A}$-equivalent to one in Table 1. A calculation shows that

$$j^2\widetilde{P}_{(l,v)} \sim_{\mathcal{A}} \left(x, (2l_0(a_{20}v_2^2 + a_{21}v_2v_3 + a_{22}v_3^2) - l_1)y^2 + ((2a_{20}v_2 + v_3a_{21})l_0 - v_2l_1)xy\right).$$

The singularity of $P_{(l,v)}$ at p_0 is of type cusp or worse if and only if the coefficient of y^2 in the above expression vanishes. This is the case if and only if

$$2(a_{20}v_2^2 + a_{21}v_2v_3 + a_{22}v_3^2) = \frac{l_1}{l_0},$$

which can be written as

$$\kappa(v^*) = -\frac{\langle e(p_0), l\rangle}{\langle p_0, l\rangle}. \tag{1}$$

Since $l \in LC^*$, $l_1^2 + l_2^2 + l_3^2 = l_0^2$, so $|l_1/l_0| \leq 1$. (Alternatively, we can choose $p_0, e(p_0), v^*, v^{*\perp}$ as an orthonormal basis of $\mathbb{R}^4_1$, where $v^{*\perp}$ is a unit orthogonal vector to v^* in $T_{p_0}M$. We have $\langle l, v^* \rangle = 0$, so $l = \langle p_0, l \rangle\, p_0 + \langle e(p_0), l \rangle\, e(p_0) + \left\langle v^{*\perp}, l \right\rangle v^{*\perp}$. But $l \in LC^*$, so $- \langle p_0, l \rangle^2 + \langle e(p_0), l \rangle^2 + \left\langle v^{*\perp}, l \right\rangle^2 = 0$, which gives $|\langle e(p_0), l \rangle / \langle p_0, l \rangle| \leq 1$.)

Now, $\kappa(v^*) = \cos^2\theta \kappa_1 + \sin^2\theta\kappa_2$, with θ the angle v^* makes with the principal direction p_1 associated to κ_1. Therefore $\kappa_1 \leq \kappa(v^*) \leq \kappa_2$, so for equation (1) to have a solution (in v or l) we need $m = \min(|\kappa_1|, |\kappa_2|) \leq 1$. If $m > |\lambda(p_0, l)|$ where $\lambda(p_0, l) = \langle e(p_0), l \rangle \,/\, \langle p_0, l \rangle = -l_1/l_0$, then there are no solutions to $\kappa(v^*) = -\lambda(p_0, l)$. If $m < |\lambda(p_0, l)|$, for any λ there are two directions in the tangent space that satisfy the equation $\kappa(v^*) = -\lambda(p_0, l)$. A short calculation shows that the two directions coincide if and only if

$$(\frac{l_1}{l_0})^2 - 2(a_{22} + a_{20})\frac{l_1}{l_0} + 4a_{20}a_{22} - a_{21}^2 = 0.$$

This means that $-\lambda(p_0, l)$ is a de Sitter principal curvature, and the direction v^* is the associated principal direction. When this is the case, the coefficient of xy in the expression for the 2-jet of $\widetilde{P}_{(l,v)}$ also vanishes. So the singularity becomes of type lips/beaks or worse.

(6) In the above setting, if $|\lambda(p_0, l)| = 1$, then $\left\langle v^{*\perp}, l \right\rangle = 0$. This means that $l = \langle p_0, l \rangle\, (p_0 \pm e(p_0))$, therefore $\widetilde{l} = \widetilde{\mathbb{L}}^{\pm}(p_0)$.

We know that $\bar{\kappa}^{\pm}(v^*) = -1 \pm \kappa(v^*)$, so $\kappa(v^*) = \pm 1$ if and only if $\bar{\kappa}^{\pm}(v^*) = 0$, if and only if v^* is a horo-asymptotic direction.

(7) Follows by analysing the 3-jet of $\widetilde{P}_{(l,v)}$ at an umbilic point. □

Proposition 4.3 *The projection $P_{(\widetilde{\mathbb{L}}^{\pm}(p),v)}$ has a singularity of type swallowtail* (11_5) *or worse at p if and only if v^* is a horo-asymptotic direction and p is a geodesic inflection on the associated horo-asymptotic direction. The locus of such points form generically a smooth curve on M that we label* the horo-flecnodal curve of *M.*

Proof. We choose the setting of the proof of Theorem 4.2, take $p = p_0$ and analyse the 3-jet of $P_{(\widetilde{\mathbb{L}}^{\pm}(p_0),v)}$. We write this 3-jet in the form $(x, g(x, y))$. Then p_0 is a is swallowtail singularity (or worse) of $P_{(\widetilde{\mathbb{L}}^{\pm}(p_0),v)}$, if and only if $\partial^2 g/\partial y^2(0, 0) = \partial^3 g/\partial y^2(0, 0) = 0$. Calculations show that this is the case if and only if

$$(1 \pm 2a_{20})v_2^2 \pm 2a_{21}v_2v_3 + (1 \pm 2a_{22})v_3^2 = 0$$

(i.e., $v^* = (0, 0, v_2, v_3)$ is a horo-asymptotic direction) and

$$C(v_2, v_3) = a_{30}v_2^3 + a_{31}v_2^2v_3 + a_{32}v_2v_3^2 + a_{33}v_3^2 = 0$$

where C is the cubic part of f. To simplify the calculations, we rotate the axis in the tangent plane and suppose that $(0, 0, 1, 0)$ is a horo-asymptotic direction, so

$1 \pm 2a_{20} = 0$. Then the singularity is of type swallowtail or worse if and only if $a_{30} = 0$.

The equation of the horo-asymptotic curves in the parameter space is given by $\bar{l}dx^2 + 2\bar{m}dxdy + \bar{n}dy^2 = 0$, where $\bar{l}, \bar{m}, \bar{n}$ are the coefficients of the hyperbolic second fundamental form. Then the horo-asymptotic curve tangent to $v^* = (0,0,1,0)$ is parametrised by

$$\gamma(t) = (1 + \frac{1}{2}t^2, \mp\frac{1}{2}t^2, t, -\frac{3}{2}\frac{a_{30}}{a_{21}}t^2) + \text{h.o.t.}$$

The geodesic curvature of this asymptotic curve at p_0 is $3a_{30}/(2a_{21})$. This vanishes when $a_{30} = 0$, equivalently when the singularity of the projection is of type swallowtail or worse.

To show that the horo-flecnodal curve is generically a smooth curve, one can follow the method in [5] (see also [8]) and consider the Monge-Taylor map $\phi : U \to J^3(2)$ which associates to $(x,y) \in U$ the 3-jet of f at (x,y), where f is the function in the parametrisation of the surface $\boldsymbol{x}(U)$ in H-Monge form. The swallowtail singularities form a smooth variety in $J^3(2)$ and for generic surfaces $\phi(U)$ intersects this variety transversally. The horo-flecnodal curve is the pre-image of this intersection, so is generically a smooth curve. □

We call the image of the critical set of $P_{(l,v)}$ the *contour* (or *profile*) of M. We shall suppose here that it is a smooth curve and restrict to the case where $l = \widetilde{\mathbb{L}}^{\pm}(p_0)$ at some point $p_0 \in M$. We call the intersection of M with the 3-dimensional space generated by the vectors p_0, v^* and $e(p_0)$ the *normal section* of M at p_0 along v^*. Koenderink showed in [30] that for embedded surfaces in $\mathbb{R}^3$, the Gaussian curvature of the surface at a given point is the product of the curvature of the contour with the curvature of the normal section in the direction of projection. We have the following result for the projections $P_{(\widetilde{\mathbb{L}}^{\pm}(p_0),v)}$, where the curvature of a curve in $H^3_+(-1)$ is as given in §3.

Theorem 4.4 (Koenderink type theorem) *Let κ_c be the curvature of the contour and κ_n the curvature of the normal section. If the point on the surface is also on the plane of projection (alternatively, if $v \in T_pM$), then*

$$K_h^{\pm} = (\kappa_c + 1)(\kappa_n \mp 1).$$

Otherwise, the left hand side of the above equality depends on $\langle v^*, p\rangle$.

Proof. We take, without loss of generality, the surface in the H-Monge form as in the proof of Theorem 4.2, $v = v^* = (0,0,0,1)$ and $l = (1, \pm 1, 0, 0)$ at $p_0 = (1,0,0,0)$. We assume that the singularity of the projection is a fold at p_0, so $2a_{22} \pm 1 \neq 0$. Then the 2-jet of the profile is given by

$$(1 + \frac{1}{2}t^2, \frac{(4a_{20}a_{22} - 2a_{20} - a_{21}^2)}{2(2a_{22} - 1)}t^2, t, 0).$$

Following the formula in §3, its curvature at p_0 is given by

$$\kappa_c = \frac{4a_{20}a_{22} - a_{21}^2 \mp 2a_{20}}{2a_{22} \mp 1}.$$

The normal section of the surface along v is given by $(\sqrt{f(0,y)^2 + y^2 + 1}, f(0,y), 0, y)$ and its curvature at p_0 is given by $\kappa_n = 2a_{22}$. Given the fact that the hyperbolic curvature of the surface at p_0 is $K_h^{\pm} = 4a_{20}a_{22} - a_{21}^2 \mp 2a_{20} \mp 2a_{22} + 1$, it follows that

$$\kappa_c = \frac{K_h^{\pm}}{\kappa_n \mp 1} - 1,$$

equivalently,

$$K_h^{\pm} = (\kappa_c + 1)(\kappa_n \mp 1).$$

We now show that the above formula does not hold if p_0 does not belong to the plane of projection. Let $\lambda \in \mathbb{R}$ and consider the map

$$\begin{array}{rccc} hc: & HP(v,0) \cap H_+^3(-1) & \to & HP(v,\lambda) \cap H_+^3(-1) \\ & q & \mapsto & q + \lambda v - \frac{\lambda^2}{2\langle q,l\rangle} l \end{array}$$

that takes a point on the plane determined by $v \in S_0^2$ to a point on another parallel plane along the horocycles $(l,v) \in \mathcal{C}$. It is not difficult to show that the curvature of a curve $\gamma(t) \subset HP(v,0) \cap H_+^3(-1)$ is distinct from that of $hc(\gamma(t)) \subset HP(v,\lambda) \cap H_+^3(-1)$ when $\lambda \neq 0$. So if we project to $HP(v,\lambda) \cap H_+^3(-1)$ instead, the left hand side of the above equality remains the same while the value of κ_c changes as λ varies. So the equality does not hold in this case. □

5 Duality

We prove in this section duality result similar to those in [34] for central projections of surfaces in $\mathbb{R}P^3$ and to those in [9] for orthogonal projections of surfaces in $\mathbb{R}^3$.

Let M be an embedded surface in $H_+^3(-1)$. We shall use the duality concepts in [13, 14, 25], see §6 for details. In [24] is introduced the notion of a horocyclic surface which is defined to be a one-parmeter family of horocylces in $H_+^3(-1)$. We denote by $A_2^{h\text{-}par}$ the horocyclic surface in $H_+^3(-1)$ swept out by the horocycles in $H_+^3(-1)$ passing through a horo-parabolic point of M and with tangent direction there the unique horo-asymptotic direction. We also denote by $(A_2^{h\text{-}par})^{(1,*)}$ the Δ_1-dual of $A_2^{h\text{-}par}$.

A *bi-tangent horosphere* of M is a horosphere $HS^3(l,c)$ tangent to M at two distinct points. If there exist points $p_1, p_2 \in M$ such that $\mathbb{L}^{\pm}(p_1) = \mathbb{L}^{\pm}(p_2)$ and $\langle p_1, \mathbb{L}^{\pm}(p_1)\rangle = \langle p_2, \mathbb{L}^{\pm}(p_2)\rangle$, then we have the bi-tangent horosphere $HS^3(\mathbb{L}^{\pm}(p_1), \langle p_1, \mathbb{L}^{\pm}(p_1)\rangle)$ at p_1 and p_2. In this case, there exists a unique horocycle in $HS^3(\mathbb{L}^{\pm}(p_1), \langle p_1, \mathbb{L}^{\pm}(p_1)\rangle)$

passing through p_1 and p_2. We call such horocycle a *bi-tangent horocycle to M relative to* $\mathbb{L}^{\pm}(p_1) = \mathbb{L}^{\pm}(p_2)$. We denote by $A_1^h||A_1^h$ the horocyclic surface swept out by the bi-tangent horocycles to M relative to the lightcone normal at points on the bi-tangent locus of horospheres to M.

Theorem 5.1 *Let M be an embedded surface in $H^3_+(-1)$ and $M^{(2,*)}$ its Δ_2-dual. Then,*

(1) *The Δ_2-dual of the surface $A_2^{h\text{-}par}$ is the cuspidaledge of $M^{(2,*)}$.*

(2) *The Δ_2-dual of the surface $A_1^h||A_1^h$ is the self-intersection curve of $M^{(2,*)}$.*

(3) *The Δ_3-dual of the surface $(A_2^{h\text{-}par})^{(1,*)}$ is the cuspidaledge of $-M^{(2,*)}$, where $-M^{(2,*)}$ denotes the antipodal surface of $M^{(2,*)}$.*

Proof. We consider a local parametrisation $\boldsymbol{x} : U \to H^3_+(-1)$ of M (i.e., $M = \boldsymbol{x}(U)$). In this case $M^{(2,*)} = \mathbb{L}^{\pm}(U)$.

(1) We suppose that the horo-parabolic set $K_h^{\pm} = 0$ is a regular curve. This property holds for generic embeddings of surface M in $H^3_+(-1)$. Let $p(t)$, $t \in I$, be a parametrisation of the horo-parabolic set of M and $u(t)$ the unique unit horo-asymptotic direction at $p(t)$. There exists a curve $\gamma : I \to U$ such that $\boldsymbol{x}(\gamma(t)) = p(t)$ and $u(t) \in T_{\gamma(t)}M$. Then we have $e(t) = \mathbb{E}(\gamma(t))$, so that $\mathbb{L}^{\pm}(\gamma(t)) = p(t) \pm e(t)$. Since $p(t) = \boldsymbol{x}(\gamma(t))$ is a parametrisation of $K_h^{-1}(0)$ which is the singular locus of $\mathbb{L}^{\pm}$, $\mathbb{L}^{\pm}(\gamma(t)) = p(t) \pm e(t)$ is the cuspidaledge of $M^{(2,*)} = \mathbb{L}^{\pm}(U)$.

The horocyclic surface $A_2^{h\text{-}par}$ is parametrised by

$$y(s,t) = p(t) + su(t) + \frac{s^2}{2}(p(t) \pm e(t)).$$

We have

$$\frac{\partial y}{\partial s}(s,t) = u(t) + s(p(t) \pm e(t)) \quad \text{and} \quad \frac{\partial y}{\partial t}(s,t) = p'(t) + su'(t) + \frac{s^2}{2}(p'(t) \pm e'(t)).$$

Since $u(t) \in T_{\gamma(t)}M$ and $\mathbb{L}^{\pm}(\gamma(t)) = p(t) \pm e(t)$ is a lightlike normal to M, we have

$$\left\langle \frac{\partial y}{\partial s}(s,t), \mathbb{L}(\gamma(t)) \right\rangle = 0.$$

We remark that $p'(t) \in T_{\gamma(t)}M$ and $\langle p'(t) \pm e'(t), p(t) \pm e(t) \rangle = 0$. Therefore, we have

$$\left\langle \frac{\partial y}{\partial t}(s,t), p(t) \pm e(t) \right\rangle = s\langle u'(t), p(t) \pm e(t) \rangle.$$

Taking the derivative of the relation $\langle p(t) \pm e(t), u(t) \rangle = 0$, we have

$$\langle d\mathbb{L}^{\pm}(\gamma(t))p'(t), u(t) \rangle = \langle (\mathbb{L}^{\pm})'(\gamma(t)), u(t) \rangle = -\langle p(t) \pm e(t), u'(t) \rangle.$$

The fact that $p(t)$ is the parametrisation of horo-parabolic set means that the image of $d\mathbb{L}^{\pm}(\gamma(t))$ is one dimensional, so that there exists $\lambda \in \mathbb{R}$ such that

$$d\mathbb{L}^{\pm}(\gamma(t))p'(t) = \lambda d\mathbb{L}^{\pm}(\gamma(t))u(t).$$

Since $u(t)$ is the unique asymptotic direction, we have

$$\langle d\mathbb{L}^{\pm}(\gamma(t))p'(t), u(t)\rangle = \lambda \langle d\mathbb{L}^{\pm}(\gamma(t))u(t), u(t)\rangle = 0.$$

It follows that

$$\left\langle \frac{\partial y}{\partial t}(s,t), \mathbb{L}^{\pm}(\gamma(t))\right\rangle = 0.$$

So the lightlike normal to the horocyclic surface $A_2^{h\text{-}par}$ is constant along each horocycle. (This means that the image of the lightcone normal to $A_2^{h\text{-}par}$ is a curve in LC^*, which implies that $A_2^{h\text{-}par}$ is a horo-flat horocyclic surface in $H^3_+(-1)$; see [24].) Moreover, we have $\langle y(s,t), \mathbb{L}^{\pm}(\gamma(t))\rangle = -1$. Therefore $y(s,t)$ and $\mathbb{L}^{\pm}(\gamma(t))$ are Δ_2-dual.

(2) Suppose that there exists a bi-tangent horocycle to M at two points p_1 and p_2 on M. The surface $A_1^h||A_1^h$ is then a horocyclic surface generated by horocycles along a curve C_1 on M through p_1 (or a curve C_2 through p_2). The lightlike normals $\mathbb{L}^{\pm}$ of M along C_1 and C_2 coincide. Let $q(t)$ be a local parametrisation of the curve C_1 and $v(t)$ be the unit tangent direction to the horocycle in $A_1^h||A_1^h$ through $q(t)$. We also denote by $e(t)$ the de Sitter normal to M along C_1, so that the lightcone normal to M along C_1 is given by $\mathbb{L}^{\pm}(t) = q(t) \pm e(t)$. Then a local parametrisation of $A_1^h||A_1^h$ is given by

$$z(s,t) = q(t) + sv(t) + \frac{s^2}{2}(q(t) \pm e(t)).$$

It follows that

$$\frac{\partial z}{\partial s}(s,t) = v(t) + s(q(t) \pm e(t)) \quad \text{and} \quad \frac{\partial z}{\partial t}(s,t) = q'(t) + sv'(t) + \frac{s^2}{2}(q'(t) \pm e'(t)).$$

By the same arguments in case (1), we have

$$\left\langle \frac{\partial z}{\partial s}(s,t), q(t) \pm e(t)\right\rangle = 0 \quad \text{and} \quad \left\langle \frac{\partial z}{\partial t}(s,t), q(t) \pm e(t)\right\rangle = s\langle v'(t), q(t) \pm e(t)\rangle.$$

Since C_1 and C_2 are disjoint, C_2 can be parametrised by $q_2(t) = z(s(t), t)$ for some non-zero smooth function $s(t)$. On C_2, we have

$$0 = \left\langle \frac{\partial z}{\partial t}(s(t),t), q(t) \pm e(t)\right\rangle = s(t)\langle v'(t), q(t) \pm e(t)\rangle.$$

Since $s(t) \neq 0$, we have $\langle v'(t), q(t) \pm e(t)\rangle = 0$, so that

$$\left\langle \frac{\partial z}{\partial t}(s,t), q(t) \pm e(t)\right\rangle = 0.$$

So the lightlike normal to the horocyclic surface $A_1^h||A_1^h$ is constant along each horocycle. (This means that the image of the lightcone normal to $A_1^h||A_1^h$ is a curve in LC^*, which implies that $A_1^h||A_1^h$ is a horo-flat horocyclic surface in $H_+^3(-1)$; see [24].) Moreover, we have $\langle z(s,t), \mathbb{L}^\pm(t)\rangle = -1$. This means that $z(s,t)$ and $\mathbb{L}^\pm(t)$ are Δ_2-dual.

The maps $q(t) = z(t,0)$ and $z(t, s(t))$ are parametrisations of C_1 and C_2 respectively. Therefore, $\mathbb{L}^\pm(t)$ is the common lightcone normal to M at $q(t)$ and $z(t, s(t))$, which is the self intersection curve of $M^{(2,*)}$.

(3) We use here the notation in case (1) and define the contact diffeomorphism $\Psi_{31} : \Delta_3 \to \Delta_1$ by $\Psi_{31}(v,w) = (v-w, -w)$. We also define the mapping

$$w(s,t) = \pm e(t) - su(t) - \frac{s^2}{2}(p(t) \pm e(t)).$$

One can easily show that $w(s,t) \in S_1^3$ and $\langle p(t) \pm e(t), w(s,t)\rangle = 1$. By almost the same calculations as those in case (1), we can show that

$$\left\langle \frac{\partial w}{\partial s}(s,t), q(t) \pm e(t) \right\rangle = 0 \quad \text{and} \quad \left\langle \frac{\partial w}{\partial t}(s,t), q(t) \pm e(t) \right\rangle = 0.$$

This means that $\mathbb{L}^\pm(\gamma(t)) = p(t) \pm e(t)$ and $w(s,t)$ are Δ_3-dual. We also have

$$\Psi_{31}(\mathbb{L}^\pm(\gamma(t)), w(s,t)) = (y(s,t), -w(s,t)).$$

Therefore $y(s,t)$ is the Δ_1-dual of $-w(s,t)$. Since $y(s,t)$ gives a local parametrisation of $A_2^{h\text{-}par}$, $-w(s,t)$ is a local parametrisation of $(A_2^{h\text{-}par})^{(1,*)}$. By definition, Δ_3 is invariant under the antipodal action on $\mathbb{R}_1^4$, so $(A_2^{h\text{-}par})^{(1,*)}$ is the Δ_3-dual of $-\mathbb{L}^\pm(\gamma(t))$ which is the cuspidaledge of $-M^{(2,*)}$. $\square$

There are Euclidean analogues in [9] of the results in [34] (see also [3, 4, 7] for related results). Given an embedded surface M in the Euclidean space $\mathbb{R}^3$ the family of height functions on M is given by

$$\begin{array}{rccc} H_E: & M \times S^2 & \to & \mathbb{R} \times S^2 \\ & (q,v) & \mapsto & q.v \end{array}$$

and the family of orthogonal projections is given by

$$\begin{array}{rccc} P_E: & M \times S^2 & \to & TS^2 \\ & (q,v) & \mapsto & (q, q-(q.v)v) \end{array}$$

where S^2 denotes the unit sphere and "." the scalar product in $\mathbb{R}^3$. The local bifurcation set $Bif(H_E)$ of H_E (resp. $Bif(P_E)$ of P_E) is the set of $u \in S^2$ for which there exist $p \in M$ such that $H_E(-,u)$ (resp. $P_E(-,u)$) has a non-stable singularity at p. The A_2-stratum of $Bif(H_E)$ is the set of unit normals at the parabolic points of

M. The lips/beaks stratum of $Bif(P_E)$ is the set of unit asymptotic directions at the parabolic points of M. It is shown in [9] that the A_2-stratum of $Bif(H_E)$ is dual to the lips/beaks stratum of $Bif(P_E)$. The duality in [9] refers to the double Legendrian fibration $S^2 \xleftarrow{\pi_1} \Delta \xrightarrow{\pi_2} S^2$, where $\Delta = \{(u,v) \in S^2 \times S^2 \mid u.v = 0\}$. The contact structure on Δ is given by the 1-form $\theta = v.du|_\Delta$.

We seek an analogous duality result for the family of projections along horocycles. The family of lightcone height functions on $M \subset H^3_+(-1)$ is introduced in [19] and is given by

$$\begin{array}{rccc} H: & M \times S^2_+ & \to & \mathbb{R} \times S^2_+ \\ & (p,l) & \mapsto & \langle p, l\rangle \end{array}$$

The A_2-stratum of $Bif(H)$ is the set of lightcone vectors $\widetilde{\mathbb{L}}^\pm(p)$, with p a horo-parabolic point of M. We know from Theorem 4.2 that the projection $P_{(\widetilde{\mathbb{L}}^\pm(p),v)}$ has a lips/beaks singularity when p a horo-parabolic point and v^* is a horo-asymptotic direction at p.

The set $\mathcal{C} = \{(l,v) \in S^2_+ \times S^2_0 \mid \langle l, v\rangle = 0\}$ can be given a contact structure associated to the 1-form $\theta = \langle v, dl\rangle|_\mathcal{C}$. The smooth fibre bundles $\pi_1 : \mathcal{C} \to S^2_+$ and $\pi_2 : \mathcal{C} \to S^2_0$ are Legendrian fibrations. Therefore, given a Legendrian curve $\boldsymbol{i} : L \to \mathcal{C}$, $\pi_1(\boldsymbol{i}(L))$ is dual to $\pi_2(\boldsymbol{i}(L))$ and vice-versa. We call this duality $\mathcal{C}$-duality. (See §6 for more details on Legendrian duality.)

Suppose that the horo-parabolic set of $M \in H^3_+(-1)$ is smooth (which is generically the case) and is parametrised locally by $p(t)$, $t \in I$. Let $u(t)$ be the unique horo-asymptotic direction at $p(t)$ and $(\widetilde{\mathbb{L}}^\pm(p(t)), v(t)) = F^{-1}_{p(t)}(\widetilde{\mathbb{L}}^\pm(p(t)), u(t))$, with F_p the map in Proposition 4.1. We have the following consequence of the $\mathcal{C}$-duality.

Proposition 5.2 *The curve $L = \{(\widetilde{\mathbb{L}}^\pm(p(t)), v(t)), t \in I\}$ is a Legendrian curve in $\mathcal{C}$. Therefore $\pi_1(L)$ and $\pi_2(L)$ are $\mathcal{C}$-dual curves. That is, the A_2-stratum of the bifurcation set of the family of lightcone height functions is $\mathcal{C}$-dual to the set of unit horo-asymptotic directions at horo-parabolic points transported along horocycles to S^2_0.*

The set of unit horo-asymptotic directions at horo-parabolic points transported along horocycles to S^2_0 is not of course the lips/beaks stratum of the bifurcation set of the family of projections along horocycles. The lips/beaks stratum of this family is generically a 2-dimensional surface in $\mathcal{C}$. However, the curve $L = \{(\widetilde{L}^\pm(p(t)), v(t)), t \in I\}$ is special on this stratum. To show this, let

$$S(4_2) = \{(p,(l,v)) \in M \times \mathcal{C} \mid P_{(l,v)} \text{ has a singularity at } p \text{ of type lips/beaks}\},$$

The surface $S(4_2)$ is generically smooth. Consider the projections

$$\pi_1 : S(4_2) \subset M \times \mathcal{C} \to M \text{ and } \pi_2 : S(4_2) \subset M \times \mathcal{C} \to \mathcal{C}.$$

Proposition 5.3 *Suppose that p is not an umbilic point. Then the projection π_1 is singular if and only if p is a horo-parabolic point, $l = \widetilde{L}^\pm(p)$ and v^* is the unique horo-asymptotic direction at p. Therefore $L = \{(\widetilde{\mathbb{L}}^\pm(p(t)), v(t)), t \in I\} = \pi_2(\Sigma(\pi_1))$.*

Proof. The proof follows by direct calculations using the H-Monge form setting. □

6 Appendix

We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this paper (for more details see for example [2]). Let N be a $(2n+1)$-dimensional smooth manifold and K be a field of tangent hyperplanes on N. Such a field is locally defined by a 1-form α. The tangent hyperplane field K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point on N. The pair (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α a *contact form.*

A submanifold $\boldsymbol{i} : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if dim $L = n$ and $d\boldsymbol{i}_x(T_xL) \subset K_{\boldsymbol{i}(x)}$ at any $x \in L$. A smooth fibre bundle $\pi : E \to M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and the fibres of π are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration. For a Legendrian submanifold $\boldsymbol{i} : L \subset E$, $\pi \circ \boldsymbol{i} : L \to M$ is called a *Legendrian map.* The image of the Legendrian map $\pi \circ \boldsymbol{i}$ is called a *wavefront set* of $\boldsymbol{i}$ and is denoted by $W(\boldsymbol{i})$.

The duality concepts we use in this paper are those introduced in [13, 14, 25], where five Legendrian double fibrations are considered on the subsets Δ_i, $i = 1, \dots, 5$ below, of the product of two of the pseudo spheres $H^n(-1)$, S^n_1 and LC^*. The geometric ideas behind the choice of the subsets Δ_i and the Legendrian double fibrations are as follows (for more details see [13, 14, 25]).

To any hypersurface $\boldsymbol{x} : U \to H^n(-1)$ is associated the de Sitter Gauss map $\mathbb{E} : U \to S^n_1$. It is easy to show that the pair $(\boldsymbol{x}, \mathbb{E}) : U \to H^n(-1) \times S^n_1$ is a Legendrian embedding into the set $\Delta_1 = \{(v, w) \in H^n(-1) \times S^n_1 \mid \langle v, w\rangle = 0\}$. (The contact structure on Δ_1 is given below.) This means that $M = \boldsymbol{x}(U)$ and $M^{(1,*)} = \mathbb{E}(U)$ are dual. We call this duality the Δ_1-*duality.* This is a direct analogue of the spherical duality in the Euclidean space.

Consider now the lightcone Gauss map $\mathbb{L}^{\pm} : U \to H^n(-1) \times LC^*$ which satisfies $\langle \boldsymbol{x}(u), \mathbb{L}^{\pm}(u)\rangle = -1$. The pair $(\boldsymbol{x}, \mathbb{L}^{\pm}) : U \to H^n(-1) \times LC^*$ determines a Legendrian embedding into the set $\Delta_2 = \{(v, w) \in H^n(-1) \times LC^* \mid \langle v, w\rangle = -1\}$, so $M = \boldsymbol{x}(U)$ and $M^{(2,*)} = \mathbb{L}^{\pm}(U)$ are dual. We call this duality the Δ_2-*duality.*

Similarly, we have $\langle \mathbb{E}(u) \pm \boldsymbol{x}(u), \mathbb{E}(u)\rangle = 1$ and $\langle \mathbb{L}^+(u), \mathbb{L}^-(u)\rangle = -2$ and these lead to the concepts of Δ_3-*duality* and Δ_4-*duality* respectively.

For spacelike hypersurfaces embedded in one of the pseudo-spheres in the Minkowski space (i.e. surfaces whose tangent spaces at all points are spacelike), we need to consider only the above four Δ_i-dualities, $i = 1, \dots, 4$. However, if we consider timelike hypersurfaces in S^n_1, (i.e. surfaces whose tangent spaces at all points are timelike) we need the concept of Δ_5-*duality* below which is also a direct analogue to the spherical duality in the Euclidean space. To summarise, we have the following five Legendrian double fibrations.

(1) (a) $H^n(-1) \times S_1^n \supset \Delta_1 = \{(v,w) \mid \langle v,w\rangle = 0\}$,
    (b) $\pi_{11} : \Delta_1 \to H^n(-1)$, $\pi_{12} : \Delta_1 \to S_1^n$,
    (c) $\theta_{11} = \langle dv, w\rangle|\Delta_1$, $\theta_{12} = \langle v, dw\rangle|\Delta_1$.

(2) (a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(v,w) \mid \langle v,w\rangle = -1\ \}$,
    (b) $\pi_{21} : \Delta_2 \to H^n(-1)$,$\pi_{22} : \Delta_2 \to LC^*$,
    (c) $\theta_{21} = \langle dv, w\rangle|\Delta_2$, $\theta_{22} = \langle v, dw\rangle|\Delta_2$.

(3) (a) $LC^* \times S_1^n \supset \Delta_3 = \{(v,w) \mid \langle v,w\rangle = 1\ \}$,
    (b) $\pi_{31} : \Delta_3 \to LC^*$,$\pi_{32} : \Delta_3 \to S_1^n$,
    (c) $\theta_{31} = \langle dv, w\rangle|\Delta_3$, $\theta_{32} = \langle v, dw\rangle|\Delta_3$.

(4) (a) $LC^* \times LC^* \supset \Delta_4 = \{(v,w) \mid \langle v,w\rangle = -2\ \}$,
    (b) $\pi_{41} : \Delta_4 \to LC^*$,$\pi_{42} : \Delta_4 \to LC^*$,
    (c) $\theta_{41} = \langle dv, w\rangle|\Delta_4$, $\theta_{42} = \langle v, dw\rangle|\Delta_4$.

(5) (a) $S_1^n \times S_1^n \supset \Delta_5 = \{(v,w) \mid \langle v,w\rangle = 0\}$,
    (b) $\pi_{51} : \Delta_5 \to S_1^n$,$\pi_{52} : \Delta_5 \to S_1^n$,
    (c) $\theta_{51} = \langle dv, w\rangle|\Delta_5$, $\theta_{52} = \langle v, dw\rangle|\Delta_5$.

Above, $\pi_{i1}(v,w) = v$ and $\pi_{i2}(v,w) = w$ for $i = 1,\ldots,5$, $\langle dv, w\rangle = -w_0dv_0 + \sum_{i=1}^n w_idv_i$ and $\langle v, dw\rangle = -v_0dw_0 + \sum_{i=1}^n v_idw_i$. The 1-forms θ_{i1}^{-1} and θ_{i2}^{-1}, $i = 1,\ldots,5$, define the same tangent hyperplane field over Δ_i which is denoted by K_i.

We have the following duality theorem on the above spaces.

Theorem 6.1 ([13, 14, 25]) *The pairs (Δ_i, K_i), $i = 1,\ldots,5$, are contact manifolds and π_{i1} and π_{i2} are Legendrian fibrations.*

We have the following general remarks, some of which follow from the discussion proceeding Theorem 6.1.

Remark 6.2 1. Given a Legendrian submanifold $\boldsymbol{i} : L \to \Delta_i$, $i = 1,\ldots,5$, Theorem 6.1 states that $\pi_{i1}(\boldsymbol{i}(L))$ is the Δ_i-dual of $\pi_{i2}(\boldsymbol{i}(L))$ and vice-versa.

2. We have the following geometric properties for a Legendrian submanifold $L \subset \Delta_i$, $i = 1,\ldots,5$. Take the case $i = 1$. If $\pi_{11}(\boldsymbol{i}(L))$ is smooth at a point $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$, then $\pi_{12}(\boldsymbol{i}(\boldsymbol{u}))$ is the normal vector to the hypersurface $\pi_{11}(\boldsymbol{i}(L)) \subset H_+^n(-1)$ at $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$. Conversely, if $\pi_{12}(\boldsymbol{i}(L))$ is smooth at a point $\pi_{12}(\boldsymbol{i}(\boldsymbol{u}))$, then $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$ is the normal vector to the hypersurface $\pi_{12}(\boldsymbol{i}(L)) \subset S_1^n$. The same holds for the Δ_i-dualities, $i = 2,\ldots,5$, where we take the normal to a hypersurface $M \subset LC^*$ at $p \in M$ as the direction given by the intersection of the normal plane to T_pM in $\mathbb{R}_1^{n+1}$ with T_pLC^*.

3. Since the normal of a hypersurface in $H^n(-1)$ is always spacelike, we have no good duality relationship in $H^n(-1) \times H^n(-1)$.

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S. Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
e-mail: `izumiya@math.sci.hokudai.ac.jp`

F. Tari, Department of Mathematical Sciences, Durham University, Science Laboratories, South Road, Durham DH1 3LE, UK
e-mail: `farid.tari@durham.ac.uk`