Self-adjoint operators on surfaces in $\mathbb{R}^n$

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Abstract

Our aim in this paper is to define principal and characteristic directions at points on a smooth 2-dimensional surface in the Euclidean space $\mathbb{R}^4$ in such a way that their equations together with that of the asymptotic directions behave in the same way as the triple formed by their counterpart on smooth surfaces in the Euclidean space $\mathbb{R}^3$. The definitions we propose are derived from a more general approach, namely an analysis of self-adjoint operators on 2-dimensional smooth surfaces in the Euclidean space $\mathbb{R}^n$.

1 Introduction

To a smooth and oriented surface M in (the Euclidean space) $\mathbb{R}^3$ is associated a shape operator S. This is a self-adjoint operator defined on each tangent plane of M and describes the shape of M in $\mathbb{R}^3$. On M are defined three pairs of foliations that are intimately related to S. These are the lines of principal curvature defined away from umbilic points, the asymptotic curves defined in the hyperbolic region and the characteristic curves defined in the elliptic region of the surface. The three pairs of foliations are given, in a local chart, by binary differential equations (BDEs), also know as quadratic differential equations. These are equations in the form

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0. \tag{1}$$

It is shown in [7] that the equations of the asymptotic, characteristic and principal curves are related. A BDE can be viewed as a quadratic form and represented at each point in the plane by a point in the projective plane. If Γ denotes the set of degenerate quadratic forms, then the asymptotic, characteristic and principal BDEs represent a self-polar triangle with respect to Γ ([7]). In particular, any two of them determine the third one.

We show in this paper that the results in [7] and [4] extend to 2-dimensional surfaces in (the Euclidean space) $\mathbb{R}^n$. A key observation is that the geometric concepts above for surfaces

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in $\mathbb{R}^3$ are derived from the shape operator which is a self-adjoint operator on the surface. Given any smooth 2-dimensional surface M in $\mathbb{R}^n$ ($n \geq 3$) and a self-adjoint operator $\mathcal{S}$ on M, one can define at any point $p \in M$ the concepts of pseudo normal curvature, principal curvatures, Gaussian and normal curvatures, and pseudo principal/asymptotic/characteristic directions associated to $\mathcal{S}$ at p, in an analogous way to the concepts associated to a shape operator on a smooth surface in $\mathbb{R}^3$. The pseudo principal, asymptotic, characteristic curves are given by BDEs which form a self-polar triangle (§6). Furthermore, this triangle is determined by the pseudo asymptotic BDE (Theorem 2.2 and Remark 2.3).

The differential geometry of surfaces in $\mathbb{R}^4$ has been investigated previously in for example [5, 6, 18, 19, 21, 23, 27, 29, 31, 34]. However, it appears that there is no previous study that produces in a natural way orthogonal pairs of tangent (principal) directions at most points on the surface. (So in particular, no natural way of defining a pair of orthogonal principal foliation on the surface was given.) A definition of principal directions is given in [27] in terms of the curvature ellipse. However, there are four such directions at generic points on the surface. In [34] are defined the v-principal directions. These are the eigenvectors of the shape operator S_v along a smooth vector field v normal to M. There are two such directions at generic points on the surface, but these depend on the choice of v.

We provide in this paper a natural way of producing orthogonal pairs of (principal) tangent directions at most points on a generic smooth surface in $\mathbb{R}^4$. Our definition is derived from the study of pseudo shape operators on surfaces in $\mathbb{R}^n$. A key observation is that the asymptotic BDE is well defined on a smooth 2-dimensional surface in $\mathbb{R}^4$ ([27]; see also [5, 6, 21, 29]). Therefore, a self-adjoint operator can be recovered from this BDE and one can associate pseudo principal and characteristic directions to this operator. We define the principal and characteristic directions at a point on a surface in $\mathbb{R}^4$ to be these pseudo principal and characteristic directions. These are given by BDEs and the triple asymptotic, characteristic and principal BDEs behaves in the same way as its analogue on surfaces in $\mathbb{R}^3$. We show that there is a unique (up to multiplication by nowhere zero functions) normal vector field v on M such that the principal directions defined here are the v-principal directions of S_v.

The main results in this paper and the way it is organised are given in the next section.

2 Main results

A 2-dimensional surface M in the Euclidean space $\mathbb{R}^n$ inherits the scalar product of $\mathbb{R}^n$ and we have a well-defined first fundamental form on M. A second fundamental form is also defined on M, and for a given v in the normal space to M at p, there is a shape operator $S_v : T_pM \to T_pM$ which is a self-adjoint operator. We consider the following generalisation of S_v.

Definition 2.1 *A pseudo shape operator is a smooth map $\mathcal{S} : TM \to TM$ which defines a self-adjoint operator on each tangent plane T_pM, $p \in M$.*

We say that a pseudo shape operator $\mathcal{S}$ is a shape operator if there exists a smooth vector field v normal to M and a nowhere zero function λ such that $\mathcal{S} = \lambda S_v$.

To a pseudo shape operator are associated pseudo asymptotic, characteristic and principal curves which are given by BDEs. These form a self-polar triangle. The pseudo principal BDE has a unique property amongst the points on the polar line of the pseudo asymptotic BDE.

Theorem 2.2 *Let $\mathcal{S}$ be a pseudo shape operator on M. Then there is a unique BDE on the polar line of the pseudo asymptotic BDE of $\mathcal{S}$ whose solutions are orthogonal away from pseudo umbilic points. This BDE is precisely that of the pseudo lines of curvature.*

Remark 2.3 *It follows from Theorem 2.2 that the self-polar triangle formed by the pseudo asymptotic, characteristic and principal BDEs is completely determined by the pseudo asymptotic BDE.*

The pseudo asymptotic BDE defines completely the pseudo shape operator, so we have a 1-1 correspondence between pseudo shape operators and BDEs on M. The asymptotic BDE is well defined on a smooth surface M in $\mathbb{R}^4$, so it provides a pseudo shape operator on M.

Definition 2.4 *A curve on a smooth surface M in $\mathbb{R}^4$ is called a principal (resp. characteristic) curve if it is a pseudo principal (resp. characteristic) curve of the pseudo shape operator associated to the asymptotic BDE on M. A point on M is called an umbilic point if it is a singular point of the principal curves BDE. .*

The principal curves defined here are related as follows to the v-principal curves defined in [34] (which are the pseudo principal curves associated to the shape operator S_v).

Theorem 2.5 *Let M be a smooth compact oriented surface in $\mathbb{R}^4$ with isolated umbilic points.*

(1) *The only pseudo shape operators on M that are also shape operators are those whose associated BDEs lie on the polar line of the asymptotic BDE of M.*

(2) *There exists a smooth normal vector field v, unique up to multiplication by nowhere zero functions, such that the principal curves on M as defined here are the v-principal curves as defined in* [34].

The main result in [34] can now be applied here to conclude the following.

Corollary 2.6 *The principal curves are structurally stable on generic compact and oriented surfaces in $\mathbb{R}^4$.*

We also have the following global result, analogous to Theorem 1.13 in [27].

Theorem 2.7 *Any compact and oriented surface in $\mathbb{R}^4$ of non-vanishing Euler characteristic must have umbilic points.*

Theorem 2.2 is proved in section 4 and the results on surfaces in $\mathbb{R}^4$ are proved in §7. Other results in the paper are in the following sections. In §3 we define what we mean by generic properties and give the conditions for a pseudo shape operator to be a shape operator. In §4 we study in detail the BDEs associated to a given pseudo shape operator, and consider the configurations of the solution curves of these equations in §5. In §6 we look at some natural families of BDEs determined by pseudo shape operators.

3 Pseudo shape operators: generic properties

In all the paper, M denotes a smooth compact oriented 2-dimensional surface embedded in $\mathbb{R}^n$, and smooth means C^∞. We follow the approach in [2] to define generic geometric properties of M. Cover M with finitely many open sets U_i. At each point p in U_i we can choose an orthonormal frame $\mathbf{e} = \{e_1, e_2, e_3, \cdots, e_n\}$ such that $\mathbf{e}$ agrees with the orientation of $\mathbb{R}^n$ and $\{e_1, e_2\}$ generates the tangent space T_pM and agrees with the orientation of M. Denote by V_k the vector space of polynomial maps $\mathbb{R}^2 \to \mathbb{R}^{n-2}$ of degree d with $2 \le d \le k$. At each point $p \in U_i$, the surface can be written locally, in the system of coordinates determined by $\mathbf{e}$, in Monge form $(x, y, f_3(x, y), \cdots, f_n(x, y))$, where the smooth functions f_i depend on the point p and have zero 1-jets. This defines a map $\theta_i : U_i \to V_k$ given by $p \mapsto j^k(f_3, \cdots, f_n)(p)$, which is the Taylor expansion of $(f_3, \cdots, f_n)$ at p truncated to degree k. Different choices of the x, y axes are related by a change of coordinates via the special orthogonal group $SO(2)$. This groups acts on V_k via its variables. Suppose that $X \subset V_k$ is an $SO(2)$-invariant submanifold. Then Bruce's Theorem 1 in [2] for surfaces in $\mathbb{R}^3$ generalises to surfaces in $\mathbb{R}^n$. (The proof is identical to that in [2] and is omitted.)

Theorem 3.1 *Let $X \subset V_k$ be an $SO(2)$-invariant submanifold. For a dense set of embeddings of M in $\mathbb{R}^n$ the mappings $\theta_i : U_i \to V_k$ are transverse to X.*

In the cases treated in this paper X is of codimension > 2, so the maps θ_i miss X. Also X is closed and as M is compact the set of embeddings such that θ_i is transverse to X is open. An embedding of M (or the surface M for short) is said to be *generic* if $\theta_i : U_i \to V_k$ are transverse to X.

Each tangent space T_pM, $p \in M$, inherits the scalar product in $\mathbb{R}^n$, and given a pair of tangent vectors v, w in T_pM, we have the first fundamental form $I(v, w) = \langle v, w\rangle$ at p. Let $\mathbf{r} : U \to M \subset \mathbb{R}^n$ be a parametrisation of a patch in M, and denote the coefficients of I by

$$E = \langle \mathbf{r}_x, \mathbf{r}_x\rangle, \quad F = \langle \mathbf{r}_x, \mathbf{r}_y\rangle, \quad G = \langle \mathbf{r}_y, \mathbf{r}_y\rangle.$$

A second fundamental form is also defined on M (see for example [10] on how this is done using the Riemann connection in $\mathbb{R}^n$). Consider the frame $\mathbf{e}$, and let

$$a_i = \langle e_i, \mathbf{r}_{xx}\rangle, \quad b_i = \langle e_i, \mathbf{r}_{xy}\rangle, \quad c_i = \langle e_i, \mathbf{r}_{yy}\rangle, \quad i = 3, \cdots, n$$

be the coefficients of the second fundamental form. Given $v = (\lambda_3, \cdots, \lambda_n)$ in the normal space N_pM to M at p, the shape operator $S_v : T_pM \to T_pM$ is represented, with respect to the basis $\{\mathbf{r}_x, \mathbf{r}_y\}$, by the matrix

$$\begin{pmatrix} \sum_{i=3}^n \lambda_i a_i & \sum_{i=3}^n \lambda_i b_i \\ \sum_{i=3}^n \lambda_i b_i & \sum_{i=3}^n \lambda_i c_i \end{pmatrix}.$$

The shape operator S_v is a self-adjoint operator and depends on v. We consider a generalisation of S_v and define a pseudo shape operator as in Definition 2.1. We shall not distinguish between $\mathcal{S}$ and $\lambda\mathcal{S}$ where λ is a smooth function nowhere zero on M. A pseudo shape operator is represented, with respect to the basis $\{\mathbf{r}_x, \mathbf{r}_y\}$, by the matrix $\begin{pmatrix} l & m \\ m & n \end{pmatrix}$, where

$$l = \langle \mathcal{S}(\mathbf{r}_x), \mathbf{r}_x \rangle, \quad m = \langle \mathcal{S}(\mathbf{r}_x), \mathbf{r}_y \rangle = \langle \mathcal{S}(\mathbf{r}_y), \mathbf{r}_x \rangle, \quad n = \langle \mathcal{S}(\mathbf{r}_y), \mathbf{r}_y \rangle.$$

These shall be called the coefficients of $\mathcal{S}$ (they completely determine $\mathcal{S}$).

Proposition 3.2 *For an open dense subset of the space of smooth embeddings of M in $\mathbb{R}^n$, a pseudo shape operator with coefficients (l, m, n) is a shape operator for $n \geq 6$, and for the other dimensions it is a shape operator if and only if*

$n = 3$: $(l, m, n) = \alpha(a_3, b_3, c_3)$, α a nowhere zero function.

$n = 4$: $(b_3c_4 - c_3b_4)l - (a_3c_4 - c_3a_4)m + (a_3b_4 - b_3a_4)n = 0$.

$n = 5$: on an open dense subset of the surface M.

Proof It is enough to work on a chart and use the partition of unity to obtain the required v on M. We need to solve $\sum_{i=3}^n \lambda_i a_i = l, \sum_{i=3}^n \lambda_i b_i = m, \sum_{i=3}^n \lambda_i b_i = n$ in $(\lambda_3, \cdots, \lambda_n)$.

The case $n = 3$ is obvious.

When $n = 4$, we have the following system of linear equations

$$\begin{aligned} \lambda_3 a_3 + \lambda_4 a_4 &= l, \\ \lambda_3 b_3 + \lambda_4 b_4 &= m, \\ \lambda_3 c_3 + \lambda_4 c_4 &= n. \end{aligned}$$

Suppose that one of the determinants $a_3b_4 - b_3a_4$, $b_3c_4 - c_3b_4$, $a_3c_4 - a_4c_3$ is not zero, say the first one. Solving the first two equations yields $\lambda_3 = \frac{lb_4 - ma_4}{a_3b_4 - b_3a_4}$ and $\lambda_4 = \frac{ma_3 - lb_3}{a_3b_4 - b_3a_4}$. Substituting in the third equation yields the condition in the proposition. (The condition has a geometric interpretation; see Theorem 2.5(1).) We can multiply by the denominator and take $v = (lb_4 - ma_4, ma_3 - lb_3)$.

If two of the above determinants vanish at a point p then the remaining one also vanishes at p. We can write the surface locally in Monge form. Then, following the notation proceeding Theorem 3.1, the vanishing of two determinants determines a closed set $X \subset V_2$ of codimension 2. By Theorem 3.1 and the discussion that follows, this occurs only at isolated points on generic surfaces (called inflection points in [27]). We can extend smoothly the solution v at these points (it is generically not zero there).

When $n = 5$, we have a system of three equations with three unknowns. By Theorem 3.1, for a generic surface M, the set of points where the determinant is not zero is either empty or form a smooth curve on M. This curve is denoted by M_2 in [30]. We can solve the system on $M_3 = M \backslash M_2$ but may not be able to extend the solution to M_2.

When $n \geq 6$, we have more unknowns than equations. For generic surfaces, at least one of the 3×3 determinant of the system is not zero at each point on the surface. Therefore the system has always a solution for a generic embedding of the surface M in $\mathbb{R}^n$. $\square$

4 Pseudo shape operators and BDEs

Given a pseudo shape operator $\mathcal{S}$, we define the following notions, analogous to those associated to the shape operator on a smooth surface in $\mathbb{R}^3$. Let u be a tangent direction, i.e. a line in the direction of a vector $u \in T_pM$.

We call the number $\kappa(u) = \langle \mathcal{S}(u), u \rangle$ the *pseudo normal curvature* of M at p along the unit direction u. The maximum and minimum values of $\kappa(u)$ are called the *pseudo principal curvatures* and are denoted by κ_1 and κ_2. The invariants $\mathcal{K} = \det(\mathcal{S})$ and $\mathcal{H} = \mathrm{trace}(\mathcal{S})$ are called, respectively, the *pseudo Gaussian curvature* and *pseudo mean curvature.* A point is called pseudo hyperbolic if $\mathcal{K} < 0$, pseudo parabolic if $\mathcal{K} = 0$ and pseudo elliptic if $\mathcal{K} > 0$.

The directions along which $\kappa(u)$ is extremal are called *pseudo principal directions*. A point p is said to be a *pseudo umbilic point* when the pseudo normal curvature is constant on all unit tangent directions at p. Every direction is considered pseudo principal at a pseudo umbilic point.

A direction $\bar{u}$ satisfying $\langle \mathcal{S}(u), \bar{u} \rangle = 0$ is called a *conjugate direction* to u. The directions along which the pseudo normal curvature vanishes are called the *pseudo asymptotic directions*. These are also the directions that are self-conjugate. A direction is said to be *pseudo characteristic* if the angle it makes with its conjugate direction is extremal. These are also the directions along which the pseudo normal curvature is the harmonic mean of the pseudo principal curvatures, i.e. directions along which $\kappa(u) = \frac{\mathcal{K}}{\mathcal{H}} = 1/((1/\kappa_1 + 1/\kappa_2)/2)$. (Characteristic/harmonic directions on surfaces in $\mathbb{R}^3$ are studied in [16, 32, 33] and more recently in [4, 7, 20].)

A curve on M whose tangent direction at each point is pseudo asymptotic, characteristic or principal is called, respectively, a *pseudo asymptotic, characteristic,* or *principal curve.*

The *discriminant* Δ of a BDE (1) is the set of points (x, y) where $\delta = b^2 - ac$ vanishes, that is, the set of points where the equation determines a double direction.

We denote by (dx, dy) the coordinates of a vector in T_pM with respect to the basis $\{\mathbf{r}_x, \mathbf{r}_y\}$. The statements in the following theorem are classical results on real symmetric matrices (see for example [28]).

Theorem 4.1 (1) *If p is a pseudo umbilic point of M, then the pseudo shape operator $\mathcal{S}$ at p is just a scalar multiplication.*

(2) *If p is not a pseudo umbilic point, then there are exactly two pseudo principal directions, and these are orthogonal. Moreover, the pseudo principal directions e_i, $i = 1, 2$, are parallel to the eigenvectors of $\mathcal{S}$, that is $\mathcal{S}(e_i) = \kappa_i e_i$.*

(3) *The pseudo principal curves are given by*

$$(mG - nF)dy^2 + (lG - nE)dxdy + (lF - mE)dx^2 = 0.$$

(4) $\mathcal{K} = \kappa_1\kappa_2 = \dfrac{ln - m^2}{EG - F^2}$, $\mathcal{H} = \dfrac{\kappa_1 + \kappa_2}{2} = \dfrac{Gl + En - 2Fm}{2(EG - F^2)}$.

(5) *The discriminant of the pseudo principal curves BDE is the set of pseudo umbilic points.*

We now turn to the pseudo asymptotic and characteristic curves.

Theorem 4.2 (1) *The pseudo asymptotic curves are given by*

$$ndy^2 + 2mdxdy + ldx^2 = 0.$$

The discriminant is the pseudo parabolic set. There are 2/1/0 asymptotic directions at a pseudo hyperbolic/parabolic/elliptic point.

(2) *The pseudo characteristic curves are given by*

$$(2m(mG-nF)-n(lG-nE))dy^2+2(m(lG+nE)-2lnF)dydx+(l(lG-nE)-2m(lF-mE))dx^2=0,$$

which can be written in a determinant form

$$\begin{vmatrix} dy^2 & -2dxdy & dx^2 \\ l & 2m & n \\ lF-mE & lG-nE & mG-nF \end{vmatrix}=0.$$

The discriminant consists of the pseudo parabolic set together with the pseudo umbilic points. Away from pseudo umbilic points, there are 0/1/2 characteristic directions at a pseudo hyperbolic/parabolic/elliptic point.

Proof (1) A direction $u=(dx,dy)$ is pseudo asymptotic if and only if $\langle \mathcal{S}(u),u\rangle=0$, which can be expressed in the coordinate system $\{\mathbf{r}_x,\mathbf{r}_y\}$ as $ndy^2+2mdxdy+ldx^2=0$. This BDE determines 2/1/0 directions at a point on M if $\delta=m^2-ln>0/=0/<0$. The sign of δ is the same as that of $-\mathcal{K}$.

(2) We write, without loss of generality, $u=\mathbf{r}_x+p\mathbf{r}_y$ ($p=\frac{dy}{dx}$) and its conjugate direction as $\bar{u}=\mathbf{r}_x+\xi\mathbf{r}_y$. From $\langle \mathcal{S}(u),\bar{u}\rangle=0$ we get $l+(p+\xi)m+p\xi n=0$, so $\xi=-\dfrac{l+pm}{m+pn}$. The coordinates of u in an orthonormal system $\{\frac{\mathbf{r}_x}{\sqrt{E}},\frac{\sqrt{E}}{\sqrt{EG-F^2}}(\mathbf{r}_y-\frac{F}{E}\mathbf{r}_x)\}$ are $(\frac{E+pF}{\sqrt{E}},\frac{p\sqrt{EG-F^2}}{\sqrt{E}})$ and those of $\bar{u}$ are $(\frac{E+\xi F}{\sqrt{E}},\frac{\xi\sqrt{EG-F^2}}{\sqrt{E}})$. Therefore the angles θ_1 and θ_2 that u and $\bar{u}$ make with $\mathbf{r}_x$ are given by

$$\tan\theta_1=\frac{p\sqrt{EG-F^2}}{E+pF},\quad \tan\theta_2=\frac{\xi\sqrt{EG-F^2}}{E+\xi F}.$$

We consider $\tan(\theta_1-\theta_2)$ as a function of p (after substituting ξ by its expression in terms of p). Setting $p=\frac{dy}{dx}$ at the extrema of this function yields the equation in the statement of the theorem.

The discriminant of the characteristics BDE is given by $4(EG-F^2)^2(\mathcal{H}^2-\mathcal{K})\mathcal{K}=0$, so it consists of the pseudo parabolic set and the pseudo umbilic points. □

Remark 4.3 *It is clear from Theorem 4.2(1) that there is a 1-1 correspondence between pseudo shape operators on M and BDEs on M. We shall call a shape operator with coefficient of a given BDE the shape operator associated to the BDE. The BDE of the pseudo asymptotic curves of a given shape operator will be referred to as the BDE of the shape operator.*

At each non pseudo umbilic point p on M, we can choose a local coordinates system with axes along the principal directions and with $E=G=1$, $F=0$, $l=\kappa_1$, $m=0$ and $n=\kappa_2$ at p. Then we have the following.

Corollary 4.4 *With the above setting, the pseudo asymptotic (resp. characteristic) directions at p are given by $\kappa_2dy^2+\kappa_1dx^2=0$ (resp. $\kappa_2dy^2-\kappa_1dx^2=0$). So the pseudo principal directions bisect the angles formed by the pseudo asymptotic and characteristic directions.*

It is shown in [7] that the BDEs of the asymptotic, characteristic and principal curves on a smooth surface in $\mathbb{R}^3$ are related. The arguments in [7] can be used to show that the same relation holds for the pseudo asymptotic, characteristic and principal curves on a smooth surface in $\mathbb{R}^n$. As we do not distinguish between a BDE and its non-zero multiples, at each point (x, y), we can view a BDE (1) as a quadratic form in dx, dy and represent it by the point $(a : 2b : c)$ in $\mathbb{R}P^2$. In $\mathbb{R}P^2$ there is the conic $\Gamma = \{(X : Y : Z) \,|\, Y^2 - 4XZ = 0\}$ of degenerate quadratic forms. To a point $(a : 2b : c)$ is associated a polar line with respect to Γ, and is given by $aZ - bY + cX = 0$. Three points in $\mathbb{R}P^2$ form a *self-polar triangle* if the polar of any vertex is the line through the remaining two points. In our case the point $(a : 2b : c)$ is parametrised by $(x, y) \in U$ and so is its associated polar line. We shall refer to this parametrised line as the polar line of the BDE.

Proposition 4.5 *The triple pseudo asymptotic, characteristic and principal BDEs form a self-polar triangle. In particular, any two of the BDEs determine the third one.*

Proof The proof is identical to that in [7] for the asymptotic, characteristic and principal curves BDEs on a smooth surface in $\mathbb{R}^3$. □

We now prove Theorem 2.2 that shows that the pseudo principal BDE has a unique property amongst the elements of the polar line of the pseudo asymptotic BDE, namely that it is the unique BDE on this line that has orthogonal solutions.

Proof of Theorem 2.2

Denote, as before, by l, m, n the coefficients of $\mathcal{S}$. The BDE of the pseudo asymptotic curves determines the point $(n : 2m : l) \in \mathbb{R}P^2$.

Suppose that the BDE we are looking for is given by $Ady^2 + Bdxdy + Cdx^2 = 0$. Suppose also, without loss of generality, that $A \neq 0$ so its solutions at (x, y) are along $\mathbf{r}_x + s_i \mathbf{r}_y$, $i = 1, 2$. The solutions are orthogonal if and only if

$$\langle \mathbf{r}_x + s_1 \mathbf{r}_y, \mathbf{r}_x + s_2 \mathbf{r}_y \rangle = E + (s_1 + s_2)F + s_1 s_2 G = 0.$$

We have $s_1 + s_2 = -B/A$, $s_1 s_2 = C/A$, so the condition on the solutions to be orthogonal is equivalent to

$$EA - FB + GC = 0.$$

The BDE we are seeking is represented by the point $(A : B : C) \in \mathbb{R}P^2$. This point belongs to the polar line of $(n : 2m : l)$ if and only if

$$lA - mB + nC = 0.$$

We have then a system of two linear equations in $\mathbb{R}P^2$. As the point in consideration on the surface is not pseudo umbilic, the system has a unique solution given by (if for example $mG - nF \neq 0$)

$$\frac{B}{A} = \frac{lG - nE}{mG - nF}, \quad \frac{C}{A} = \frac{lF - mE}{mG - nF}.$$

Therefore the equation of the BDE we are looking for is given by

$$(mG - nF)dy^2 + (lG - nE)dxdy + (lF - mE)dx^2 = 0,$$

which is precisely that of the pseudo principal curves. $\square$

Let $\mathcal{S}_P$ (resp. $\mathcal{S}_C$) be the shape operator whose associated BDE is that of the pseudo principal (resp. characteristic) curves of a given pseudo shape operator $\mathcal{S}$. We relate below the triple of BDEs associated to $\mathcal{S}$, $\mathcal{S}_P$ and $\mathcal{S}_C$.

Proposition 4.6 *Let $\mathcal{S}$ be a pseudo shape operator with isolated pseudo umbilic points. The relations between the pseudo asymptotic, characteristic and principal curves of the pseudo shape operators $\mathcal{S}$, $\mathcal{S}_P$ and $\mathcal{S}_C$ are given below (where p. is short for pseudo).*

	p. asymptotic	*p. principal*	*p. characteristic*
$\mathcal{S}$	*p. asymptotic of* $\mathcal{S}$	*p. principal of* $\mathcal{S}$	*p. characteristic of* $\mathcal{S}$
$\mathcal{S}_C$	*p. characteristic of* $\mathcal{S}$	*p. principal of* $\mathcal{S}$	*p. asymptotic of* $\mathcal{S}$
$\mathcal{S}_P$	*p. principal of* $\mathcal{S}$	*rotation of the p. principal of* $\mathcal{S}$ *by* $\frac{\pi}{4}$	$\emptyset$

Proof The case $\mathcal{S}_C$ follows from Theorem 2.2 and the fact that the pseudo asymptotic, characteristic and principal BDEs of $\mathcal{S}$ form a self-polar triangle.

The coefficients of the pseudo shape operator $\mathcal{S}_P$ are those of the BDE of the pseudo principal curves of $\mathcal{S}$ (so the pseudo principal BDE of $\mathcal{S}$ becomes the pseudo asymptotic BDE of $\mathcal{S}_P$). Following the setting of Corollary 4.4, the pseudo asymptotic directions of $\mathcal{S}_P$ at p are given by $dxdy = 0$. Therefore, by Theorem 4.1 its associated pseudo principal directions at p are given by $dy^2 - dx^2 = 0$. The solutions of this equation are obtained by rotating the pseudo principal directions of $\mathcal{S}$ by $\pi/4$. The pseudo characteristic directions of $\mathcal{S}_P$ at p are given by $dy^2 + dx^2 = 0$ (Theorem 4.2). This is not surprising as the pseudo characteristic curves live in the pseudo elliptic region of $\mathcal{S}_P$ which is empty (all non pseudo umbilic points of $\mathcal{S}_P$ are pseudo hyperbolic points). $\square$

Remark 4.7 *It follows from the proof of Proposition 4.6 that if the pseudo asymptotic curves of a pseudo shape operator $\mathcal{S}$ are defined everywhere on M and are orthogonal away from pseudo umbilic points, then there are no pseudo characteristic curves on M and the pseudo principal curves are obtained by rotating the pseudo asymptotic curves of $\mathcal{S}$ by $\pi/4$.*

5 The configurations of the solution curves

The BDEs in §4 determine a pair of transverse foliations away from the discriminant. The pair of foliations together with the discriminant are called the *configuration* of the solutions of the BDE. We analyse here the configurations of the BDEs in §4 at points on their discriminants. One approach for investigating BDEs with coefficients not all vanishing at a given point consists of lifting the bi-valued direction field defined in the plane to a single direction field ξ on some surface $N \subset \mathbb{R}^3$ (see for example [12, 13, 14, 25, 36]).

It is shown in [11] and [12] (see [1] and [24] for alternative proofs) that if ξ does not vanish at the point in consideration then the BDE can locally be reduced, by smooth changes of coordinates in the plane, to $dy^2 - xdx^2 = 0$. The integral curves in this case is a family of cusps transverse to the discriminant.

If ξ has an elementary singularity (saddle/node/focus), then the corresponding point in the plane is called a *folded singularity* of the BDE. At folded singularities, the equation is locally smoothly equivalent to $dy^2 + (-y + \lambda x^2)dx^2 = 0$, with $\lambda \neq 0, \frac{1}{16}$, provided that ξ is linearizable at the singular point; see [13, 14]. (For normal forms at folded resonant saddles and nodes see [15].) There are three topological models (see [14] for references): a folded saddle if $\lambda < 0$, a folded node if $0 < \lambda < \frac{1}{16}$ and a folded focus if $\frac{1}{16} < \lambda$; Figure 1.

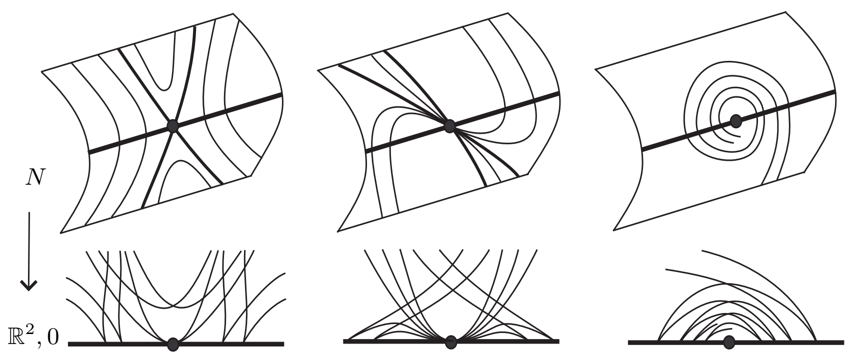

Figure 1: Folded saddle (left), node (centre) and focus (right).

The family of cusps and the folded singularities are the only locally structurally stable configurations of singular BDEs (1).

General BDEs with vanishing coefficients at a given point are studied for example in [3, 9, 22, 26]. One can lift the bi-valued field in the plane to a single field ξ on a surface $N \subset \mathbb{R}^2 \times \mathbb{R}P^1$. The whole exceptional fibre $(0,0) \times \mathbb{R}P^1$ is an integral curve of ξ. It turns out that when the discriminant has a Morse singularity and the field ξ has only elementary singularities, the topological models of the integral curves of the BDE are completely determined by the singularity type of the discriminant (an isolated point or a crossing), the number (1 or 3) and the type (saddle or node) of the singularities of ξ on the exceptional fibre (see for example [9]). If $j^1(a,b,c) = (a_1x + a_2y, b_1x + b_2y, c_1x + c_2y)$, then the singularities of ξ on the exceptional fibre are given by the roots of the cubic

$$\phi(p) = a_2p^3 + (2b_2 + a_1)p^2 + (2b_1 + c_2)p + c_1.$$

The eigenvalues of the linear part of ξ at a singularity are $-\phi'(p)$ and $\alpha_1(p)$, where

$$\alpha_1(p) = 2(a_2p^2 + (b_2 + a_1)p + b_1).$$

So the cubic ϕ and the quadratic α_1 determine the number and the type of the singularities of ξ.

It is shown in [7] that for generic surfaces in $\mathbb{R}^3$ the asymptotic and characteristic BDEs have folded singularities with opposite indices at cusps of Gauss, that is, on one side of the parabolic curve we have a folded saddle and on the other a folded node or focus (Figure 1). It is also shown in [7] that the type of an umbilic point of the characteristic and principal curves on surfaces in $\mathbb{R}^3$ are related. There are three topological configurations of the principal and characteristic curves at a generic umbilic point (so the discriminant is an isolated point): D_1 (lemon) where ξ has 1 saddle on the exceptional fibre, D_2 (monstar) where ξ has 1 node and 2 saddles, and D_3 (star) where ξ has 3 saddles, see Figure 2. We show below that the above relations do not hold in general for the pseudo asymptotic, characteristic and principal BDEs.

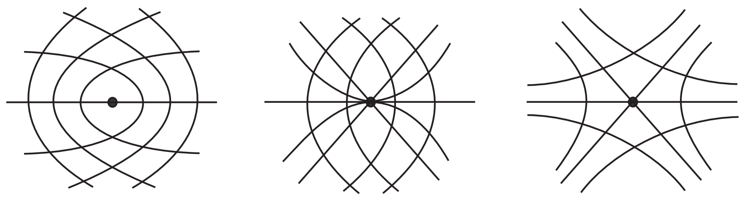

Figure 2: Darbouxian singularities: D_1 (lemon) left, D_2 (monstar) centre, D_3 (star) right.

Proposition 5.1 (1) *The pseudo asymptotic and characteristic BDEs have common singularities on the pseudo parabolic set. However, the types of these singularities are not related in general.*

(2) *The types of pseudo umbilics of the pseudo characteristic and principal BDEs are not related in general.*

Proof (1) We take the point in consideration to be the origin and suppose that the surface is given locally in Monge form $(x, y, f_3(x, y), \cdots, f_n(x, y))$, where the f_i's have zero 1-jets. So the 1-jets of E, F, G at the origin are equal to $1, 0, 1$ respectively. We suppose, without loss of generality, that the unique solution, at the origin, of the pseudo asymptotic BDE is parallel to $(0, 1)$. Given that the origin is a singularity of the pseudo asymptotic BDE, we can write the 2-jet of that equation in the form

$$n_0p^2 + 2(m_1x + m_2y)p + l_1x + l_2y + l_3x^2 + l_4xy + l_5y^2.$$

This singularity is a folded singularity if and only if

$$\lambda = \frac{1}{2l_2}(8n_0l_2l_3 - 8m_1^2l_2 - m_1) \neq 0, \frac{1}{16}$$

(see [8]). We have a folded saddle if $\lambda < 0$, a folded node if $0 < \lambda < \frac{1}{16}$ and a folded focus if $\lambda > \frac{1}{16}$; see Figure 1.

The 2-jet of the pseudo characteristic BDE is given by

$$n_0^2p^2 + 2n_0(m_1x + m_2y)p + l_1n_0x - l_2n_0y + (l_1(l_1 - n_1) - n_0l_3 + 2m_1^2)x^2 + (l_2(l_1 - n_1) + l_1(l_2 - n_2) + 4m_2m_1)xy + (l_2(l_2 - n_2) + 2m_2^2)y^2$$

The origin is also a singularity of this BDE. The singularity is a folded singularity if and only if

$$\mu = \frac{1}{2l_2}(8n_0^2 l_2 l_1^2 - 8n_0^2 l_2 l_1 n_1 - 8n_0^3 l_2 l_3 + 8m_1^2 n_0^2 l_2 + m_1) \neq 0, \frac{1}{16}.$$

It is clear that λ and μ are distinct in general.

(2) We write $j^1 l = l_0 + l_1 x + l_2 y$, $j^1 m = m_0 + m_1 x + m_2 y$, $j^1 n = n_0 + n_1 x + n_2 y$. The origin is a pseudo umbilic point of the pseudo principal (and of the pseudo characteristic) BDE if and only if the coefficients of the BDE all vanish at the origin, if and only if $m_0 = 0$ and $l_0 = n_0$. We shall assume that the pseudo umbilic point is not a pseudo parabolic point, that is $n_0 \neq 0$.

The 1-jet of the coefficients the pseudo principal BDE is given by

$$(-m_1 x - m_2 y, \frac{1}{2}((n_1 - l_1)x + (n_2 - l_2)y), m_1 x + m_2 y).$$

The number of the singularities of the lifted field and their type are determined by

$$\begin{array}{l} \phi_1 = -m_2 p^3 + (n_2 - m_1 - l_2)p^2 + (n_1 - l_1 + m_2)p + m_1, \\ \alpha_1^1 = -2m_2 p^2 + (n_2 - 2m_1 - l_2)p + n_1 - l_1. \end{array}$$

The 1-jet of the coefficients of the pseudo characteristic BDE is given by

$$((n_1 - l_1)x + (n_2 - l_2)y, 2m_1 x + 2m_2 y, -(n_1 - l_1)x - (n_2 - l_2)y).$$

The number of the singularities of the lifted field and their type are determined by

$$\begin{array}{l} \phi_2 = (n_2 - l_2)p^3 + (n_1 - l_1 + 4m_2)p^2 + (l_2 - n_2 + 4m_1)p + l_1 - n_1, \\ \alpha_1^2 = n_0((n_2 - l_2)p^2 + (2m_2 + n_1 - l_1)p + 2m_1). \end{array}$$

It is clear that the topological type of the pseudo umbilic of the pseudo characteristic and principal BDEs are not related in general. $\square$

Remark 5.2 *We can identify the set of pseudo shape operators with the set $\mathcal{C}(M, \mathbb{R}^3)$ of smooth maps $M \to \mathbb{R}^3$, and give this set the Whitney topology. It follows by Thom's transversality theorem and an analysis of the various conditions in the proof of Proposition 5.1 that the set of shape operators with the properties (i)-(iii) below form an open and dense subset of $\mathcal{C}(M, \mathbb{R}^3)$.*

(i) The pseudo parabolic set, when not empty, is a smooth curve.

(ii) The singularities of the pseudo asymptotic and characteristic BDEs are folded-saddles, nodes or foci.

(iii) The pseudo umbilic points are isolated and are Darbouxian singularities of the pseudo principal and characteristic BDEs.

6 Pseudo conjugate and reflective congruence

In [17] is constructed a natural 1-parameter family of BDEs, called *conjugate curve congruence*, that links the asymptotic curves BDE and the principal curves BDE on a smooth surface in $\mathbb{R}^3$. In [7] is constructed a natural 1-parameter family of BDEs, called *reflected conjugate congruence*, linking the characteristic curves BDE and that of the principal curves.

We define here analogous families for pseudo shape operators. We shall assume that the pseudo umbilic points are isolated and follow the notation in [7]. Consider the projective space PT_pM of all tangent directions through a point p of M which is neither a pseudo umbilic nor a pseudo parabolic point. Recall that $v \in T_pM$ is a conjugate direction to $u \in T_pM$ if $\langle \mathcal{S}(u), v\rangle = 0$. Conjugation gives an involution on PT_pM, $v \mapsto \overline{v} = C(v)$. There is another involution on PT_pM which is the reflection in either of the principal directions, $v \mapsto R(v)$.

Definition 6.1 *(1) Let $\Theta : PTM \to [-\pi/2, \pi/2]$ be given by $\Theta(p, v) = \alpha$, where α denotes the oriented angle between a direction v and the corresponding conjugate direction $\overline{v} = C(v)$. The* pseudo conjugate curve congruence*, for a fixed α, is defined to be $\Theta^{-1}(\alpha)$ which we denote $\mathcal{C}_\alpha$.*

(2) Let $\Phi : PTM \to [-\pi/2, \pi/2]$ be given by $\Phi(p, v) = \alpha$, where α is the signed angle between v and $R(\overline{v})(= R \circ C(v))$. Then the pseudo reflected conjugate curve congruence*, for a fixed α, is defined to be $\Phi^{-1}(\alpha)$, which we denote $\mathcal{R}_\alpha$.*

Note that Θ is not well defined at points corresponding to pseudo asymptotic directions at pseudo parabolic points, and Φ is not well defined at pseudo umbilic points.

We have a result below similar to that in [7], namely that $\mathcal{C}_\alpha$ and $\mathcal{R}_\alpha$ are given by families of BDEs.

Proposition 6.2 *1. The pseudo conjugate curve congruence $\mathcal{C}_\alpha$ of a parametrised surface is given by the BDE*

$$(\sin\alpha(mG - nF) - n\cos\alpha\sqrt{EG - F^2})dy^2 + (\sin\alpha(lG - nE) - 2m\cos\alpha\sqrt{EG - F^2})dydx + (\sin\alpha(lF - mE) - l\cos\alpha\sqrt{EG - F^2})dx^2 = 0,$$

and the pseudo reflected conjugate congruence $\mathcal{R}_\alpha$ is given by the BDE

$$\begin{aligned}&\{(2m(mG - nF) - n(Gl - En))\cos\alpha + (nF - mG)\frac{2mF - lG - nE}{\sqrt{(EG - F^2)}}\sin\alpha\}dy^2 +\\ &\{2(m(lG + nE) - 2lnF)\cos\alpha + (nE - lG)\frac{2mF - lG - nE}{\sqrt{(EG - F^2)}}\sin\alpha\}dydx +\\ &\{(l(lG - nE) - 2m(lF - mE))\cos\alpha + (mE - lF)\frac{2mF - lG - nE}{\sqrt{(EG - F^2)}}\sin\alpha\}dx^2 = 0.\end{aligned}$$

We indicate here some of the properties of the families $\mathcal{C}_\alpha$ and $\mathcal{R}_\alpha$ (see [7] for more details, adding the word pseudo to all the concepts there).

- $\mathcal{C}_0$ (resp. $\mathcal{R}_0$) is the pseudo asymptotic (resp. characteristic) BDE and $\mathcal{C}_{\pm\pi/2}$ (resp. $\mathcal{R}_{\pm\pi/2}$) is the pseudo principal BDE.

– In the setting of Corollary 4.4, the directions $\mathcal{C}_\alpha$ at p are given by

$$\kappa_2 \cos\alpha dy^2 + (\kappa_2 - \kappa_1)\sin\alpha dydx + \kappa_1 \cos\alpha dx^2 = 0,$$

and the directions $\mathcal{R}_\alpha$ are given by

$$\kappa_2 \cos\alpha dy^2 - (\kappa_1 + \kappa_2)\sin\alpha dydx - \kappa_1 \cos\alpha dx^2 = 0.$$

– The discriminant of $\mathcal{C}_\alpha$ is given by $H^2(x,y)\sin^2\alpha - K(x,y) = 0$ and that of $\mathcal{R}_\alpha$ by $H^2(x,y)\sin^2\alpha + K(x,y)\cos^2\alpha = 0$.

– The discriminants of $\mathcal{C}_\alpha$ and $\mathcal{C}_{-\alpha}$ (resp. $\mathcal{R}_\alpha$ and $\mathcal{R}_{-\alpha}$) coincide.

– The discriminants of the family $\mathcal{C}_\alpha$ (resp. $\mathcal{R}_\alpha$) foliate the pseudo elliptic (resp. hyperbolic) region and are given by $\kappa_1/\kappa_2 = constant$. On these discriminants the BDE $\mathcal{C}_\alpha$ (resp. $\mathcal{R}_\alpha$) determines a unique direction which is a pseudo characteristic (resp. asymptotic) direction and the other pseudo characteristic (resp. asymptotic) direction is the unique direction determined by $\mathcal{C}_{-\alpha}$ (resp. $\mathcal{R}_{-\alpha}$) there.

– As α varies in $[-\pi/2, \pi/2]$, the folded singularities of the members of $\mathcal{C}_\alpha$ (resp. $\mathcal{R}_\alpha$) trace a curve in the pseudo elliptic (resp. hyperbolic) region, that we label the *pseudo zero curve*. This curve has the same properties as that in [4].

7 Principal curves on surfaces in $\mathbb{R}^4$

Let M be a smooth compact oriented 2-dimensional surface in $\mathbb{R}^4$. Given a point $p \in M$, consider the unit circle in T_pM parametrised by $\theta \in [0, 2\pi]$. The set of the curvature vectors $\eta(\theta)$ of the normal sections of M by the hyperplane $N_pM \oplus \langle\theta\rangle$ form an ellipse in the normal plane N_pM (see for example [27]). This ellipse is called the curvature ellipse. (When θ varies in $[0, 2\pi]$, the vector $\eta(\theta)$ traces the ellipse twice.)

Asymptotic directions, labelled conjugate directions in [27], are defined as the directions along θ such that $\eta(\theta)$ is tangent to the curvature ellipse (see also [21, 29]). So there are 2/1/0 of these depending on whether p is outside/on/inside the curvature ellipse. An alternative description of the asymptotic directions is given in [5] via the singularities of the projections of M to hyperplanes. Following [5], a direction $u \in T_pM$ is said to be asymptotic if the projection of M along u to a transverse hyperplane has an $\mathcal{A}$-singularity more degenerate than a cross-cap at p (where $\mathcal{A}$ denotes the Mather group of smooth changes of coordinates in the source and target).

One important result for our investigation here is that the asymptotic curves on a surface in $\mathbb{R}^4$ are given by a BDE ([6, 21, 27, 29]). This BDE has the form

$$(b_3c_4 - b_4c_3)dy^2 + (a_3c_4 - a_4c_3)dxdy + (a_3b_4 - a_4b_3)dx^2 = 0$$

where a_i, b_i, c_i, $i = 1, 2$, are the coefficients of the second fundamental form at (x, y). This equation can also be written in a determinant form

$$\begin{vmatrix} dy^2 & -dxdy & dx^2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0.$$

The discriminant Δ of the above equation is labelled the parabolic set in [29], and separates the surface into two regions labelled hyperbolic and elliptic, with the asymptotic curves lying on the hyperbolic part of the surface ([29]). Points where Δ is singular (generically a Morse singularity $A_1^{\pm}$ modelled by $x^2 \pm y^2$) are labelled *inflection points*. The configurations of the asymptotic curves at inflection points where Δ has an A_1^+-singularity are given in [21], and the configurations at A_1^--singularities of Δ and at other points on the discriminant are given in [6].

As pointed out in the introduction, it appears that there is no previous study that produces in a natural way orthogonal pairs of tangent (principal) directions at most points on the surface. A definition of a principal direction is given in [27]. A unit direction θ in T_pM is called a principal direction if the curvature vector $\eta(\theta)$ is an extremity of one of the principal axes of the curvature ellipse. At a generic point on the surface pass four lines of curvatures ([27]; see also [23]). In [34], the authors define the v-principal curves which are, in the terminology of this paper, the pseudo principal curves of the shape operator S_v. A structural stability theorem, similar to the one in [35], is proved in [34] (see also [19] for some related results). Note that the v-principal curves depend on the choice of the normal vector field v. (Another pair of foliations on surfaces in $\mathbb{R}^4$ defined in terms of the curvature ellipse is studied in [18].)

We propose in this paper a definition of the principal curves based on the results in the previous sections (see Definition 2.4). We define a principal (resp. characteristic) curve on M as a pseudo principal (resp. characteristic) curve of the pseudo shape operator associated to the asymptotic BDE on M. With these definitions, the equations of the principal and characteristic curves are given by BDEs, and the triple asymptotic, characteristic and principal BDEs behave in the same way as their counterpart on surfaces in $\mathbb{R}^3$ (Proposition 4.5).

Corollary 7.1 *The principal curves on M are given by the BDE in Theorem 4.1(3) and the characteristic curves by the BDE in Theorem 4.2(2), where $l = a_3b_4 - a_4b_3$, $m = (a_3c_4 - a_4c_3)/2$ and $n = b_3c_4 - b_4c_3$.*

Remarks 7.2 (1) *At a hyperbolic point, the principal directions bisect the angle formed by the asymptotic directions (Corollary 4.4). So the curvature vector $\eta(\alpha_i)$ $(i = 1, 2)$ along a unit principal direction $\alpha_i \in T_pM$ is an extremity of the segment bisecting the internal and external angles formed by the curvature vectors $\eta(\theta_i)$, $i = 1, 2$, associated to the asymptotic directions; see Figure 7 (recall that $\eta(\theta_i)$, $i = 1, 2$, are tangent to the curvature ellipse). A similar observation can be made for points in the elliptic region. At such points, the curvature vector along a unit principal direction is an extremity of the segment bisecting the internal and external angles formed by the curvature vectors associated to the characteristic directions.*

(2) *The pseudo parabolic set of the pseudo shape operator associated to the asymptotic BDE on M is the parabolic set Δ (as defined in* [29]*) and the pseudo hyperbolic and elliptic regions are the hyperbolic and elliptic regions defined in* [29].

(3) *Umbilics are isolated points on generic surfaces and occur in the elliptic region of the surface. Also, the inflection points are umbilic points but the converse is not true in general.*

(4) *The results in Proposition 5.1 still hold here. That is, although the asymptotic curves and the characteristic curves are singular at the same points on* Δ *(see* [6] *for their location), their types are not related in general. The type of umbilic points of the characteristic and principal curves are also not related in general. In fact, the 1-jets of all the coefficients of the BDE of the characteristic curves vanish at inflection points.*

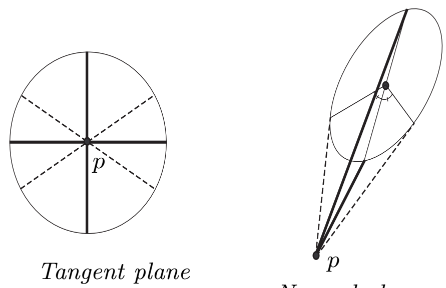

Figure 3: Unit principal directions (thick lines) and asymptotic directions (dotted lines) in the tangent space left, and their associated normal vectors on the curvature ellipse at a hyperbolic point.

We relate in Theorem 2.5 the principal curves defined here to the v-principal curves defined in [34]. Below is the proof of that result.

Proof of Theorem 2.5

(1) It follows from Proposition 3.2 that a pseudo shape operator with coefficients A, B, C is a shape operator if and only if

$$(b_3c_4 - c_3b_4)A - (a_3c_4 - c_3a_4)B + (a_3b_4 - b_3a_4)C = 0.$$

But this is exactly the condition for the point $(A : 2B : C)$ to be on the polar line of $(a_3b_4 - b_3a_4 : a_3c_4 - c_3a_4 : b_3c_4 - c_3b_4)$, that is, on the polar line of the asymptotic BDE of M.

(2) The principal curves are v-principal if and only if they are the pseudo principal curves of the shape operator S_v. So by (1) above the BDE of S_v must lie on the polar line of the BDE of the asymptotic curves. (Note that as v varies on the unit circle in N_pM, the BDE of S_v traces the polar line of the asymptotic BDE of M.) However, the BDE of S_v must also lie on the polar line of the BDE of the principal curves. Therefore the BDE of S_v is precisely that of the characteristic curves (which is the intersection point of the polar lines of the asymptotic and principal BDEs).

One can compute v as in the proof of Proposition 3.2, where l, m, n refer now to the coefficients of the characteristic BDE of M. □

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