Two parameter families of binary differential equations

Farid Tari

Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
E-mail: farid.tari@durham.ac.uk

Abstract

We obtain in this paper topological models of binary differential equation at local codimension 2 singularities where all the coefficients of the equation vanish at the singular points. We also study the bifurcations of these singularities when the equation is deformed in a generic 2-parameter families of equations.

1 Introduction

A binary/quadratic differential equation (BDE) is an implicit differential equation written locally, in some open subset U of $\mathbb{R}^2$, in the form

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0, \tag{1}$$

where the coefficients a, b, c are smooth (i.e. C^∞) functions in $(x,y) \in U$. BDEs are studied extensively and have applications to differential geometry, partial differential equations and control theory (see for example [13, 17, 29] for references and [30]). For example, lines of curvature, asymptotic and characteristic curves on a smooth surface in $\mathbb{R}^3$ are given by BDEs (see for example [13]) and the characteristic lines of a general linear second-order differential equation are also given by a BDE (see for example [29]).

A BDE defines two directions at each point in the region where $\delta = (b^2 - ac)(x,y) > 0$ and no directions in the region where $\delta < 0$. On the *discriminant* set $\Delta = \{(x,y) \in U : \delta(x,y) = 0\}$, the equation defines a double direction at points where not all the

2000 Mathematics Subject classification 34A09, 37G10, 37C10, 58K05, 53A05.

Key Words and Phrases. Binary differential equations, bifurcations, singularities, topological models.

coefficients vanish. When they all do, then every direction is a solution of (1). Away from the discriminant, the BDE (1) has no solutions or determines locally a pair of transverse 1-dimensional foliations. For generic BDEs and at generic points on the discriminant, the solutions of (1) form a family of cusps, with the cusp points tracing the discriminant. However, the configuration of the solutions can be more complicated at isolated points on the discriminant. We call such points the singularities (or zeros) of the BDE.

We are interested in the configurations of the solutions of the BDE around a given singularity. We can choose, without loss of generality, the point of interest to be the origin. If the coefficients of the BDE (1) do not all vanish at the origin, we can change coordinates and write the equation in the form $dy^2 + f(x,y)dx^2 = 0$ ([11]). The stable and codimension ≤ 2 local singularities are known in this case: see [2, 14, 15, 16, 28, 29, 38, 39] for the stable cases, [6, 18, 36] for the codimension 1 singularities and their bifurcations and [35] for the codimension 2 singularities and their bifurcations.

Our interest in this paper is when all the coefficients of the BDE vanish at the origin. This makes the origin a singularity of codimension ≥ 1 (see §2 for details). The local topological models of codimension 1 singularities are given in [4, 9, 10, 23, 29]. More degenerate BDEs whose discriminants are isolated points are also studied in [22, 24, 25, 26, 27, 37]. We study in this paper local codimension 2 singularities in BDEs (1) when all the coefficients vanish at the origin. Our study is split into two parts. In the first part, we obtain the topological models of the integral curves in a neighbourhood of the singularity (§3). This is done by extending Guíñez's blowing up technique ([24, 25, 26]) to cover the cases where the discriminant is not an isolated point. The results in §3 are summarised in Theorem 1.1 below.

Theorem 1.1 *A local codimension 2 singularity of a BDE (1) with vanishing coefficients at the singular point is locally topologically equivalent to one of the normal forms in Table 1.*

Table 1: Normal forms of local codimension 2 singularities.

Section	*Name*	*Normal form*	*Figure*
§*3.1*	*MT2 criminant saddle-node*	$(y, 2x+3y+x^2, y)$ $(y, -\frac{3}{4}x+\frac{1}{4}y+x^2, y)$ $(x+y, x^2, y)$	*Figure 5*
§*3.2*	*MT2 saddle-node*	$(y, x+y+y^2, -y)$ $(y, 4x+3y+y^2, y)$ $(y, -\frac{3}{8}x+\frac{1}{2}y+y^2, y)$	*Figure 7*
§*3.3*	*Cusp singularity*	$(y, -x+y, y^2)$ $(y, x+2y, y^2)$ $(y, x+y, y^2)$	*Figure 8*

In the second part (§4) we consider the way the local codimension 2 singularities bifurcate in generic two parameter families of BDEs. Bifurcations of vector or direction fields are usually very hard to deal with. The work in this paper and in [35] show that one can make good progress in the study of the *local* bifurcations of degenerate singularities of BDEs when following a strategy that uses a combination of tools from singularity theory and the qualitative theory of vector fields. One key observation is that for BDEs all the complications occur on the discriminant. The discriminant is (a germ) of a plane curve but has special singularities. These singularities are those of the determinant of a family of symmetric matrices associated to the BDE and can be studied using Bruce's framework in [3]. One can determine, in particular, the way the discriminant curve bifurcates in a generic family of equations. The next step is to locate on the deformed discriminant the singularities of the BDE and determine their type. This can be done in a fairly straightforward way. One can then draw the configuration around these singularities. Joining theses configurations together is not difficult in most cases as BDEs define pairs of transverse foliations away from the discriminant. However, one must consider the occurrence of non-local phenomena that emerge form the local once. The results in §4 are summarised in Theorem 1.2 below.

Theorem 1.2 *A generic two parameter family of a singularity in Theorem 1.1 is locally (fibre) topologically equivalent to one of the normal forms in Table 2.*

Table 2: Normal forms of generic two parameter families of BDEs.

Section	*Name*	*Generic family*	*Figure*
§4.1	*MT2 criminant saddle-node*	$(y, 2x+3y+x^2, y+ux+v)$	*Figure 10*
		$(y, -\frac{3}{4}x+\frac{1}{4}y+x^2, y+ux+v)$	*Figure 11*
		$(x+y, x^2+ux+v, y)^*$	*Figure 12*
§4.2	*MT2 saddle-node*	$(y, x+y+y^2, -y+ux+v)$	*Figure 13*
		$(y, 4x+3y+y^2, y+ux+v)$	*Figure 14*
		$(y, -\frac{3}{8}x+\frac{1}{2}y+y^2, y+ux+v)$	*Figure 15*
§4.3	*Cusp singularity*	$(y, -x+y, y^2+uy+v)$	*Figure 16*
		$(y, x+2y, y^2+uy+v)$	*Figure 17*
		$(y, x+y, y^2+uy+v)$	*Figure 18*

($*$) *For this case, due to the emergence of a non-local phenomena, we impose on the family to be a "good family" (see Definition 4.1 and the Appendix for details).*

The work in this paper is motivated by applications to differential geometry and has applications to other area where BDEs appear. There is an interest in finding out how the geometry of a surface S in the Euclidean space $\mathbb{R}^3$ changes as the surface is deformed in 1-parameter family of surfaces (see for example [8, 39]). It is shown in [21] (see also [5, 13] for details) that there is a natural 1-parameter family of BDEs on a smooth surface in $\mathbb{R}^3$, called the family of conjugate curve congruences and denoted by $\mathcal{C}_\alpha$, that links the BDE of the lines of curvature to that of the asymptotic curves. Geometric properties of the surface can be derived form this family ([5, 13]). When the surface is deformed in 1-parameter families of surfaces, we obtain a 2-parameter families of conjugate curve congruences $\mathcal{C}_{\alpha,t}$. The results in this paper are applied in [32] to obtain some of the bifurcations in $\mathcal{C}_{\alpha,t}$ at umbilic and flat umbilics of an evolving surface. There is also another natural 1-parameter family of BDEs on S that links the BDE of the lines of curvature to that of the characteristic curves, called the family of reflected curve congruences and denoted by $\mathcal{R}_\alpha$ ([13]). The bifurcations in the family $\mathcal{R}_{\alpha,t}$ on an evolving surface will be dealt with in a forthcoming paper.

2 Preliminaries

As pointed out in the introduction, we are interested in the local qualitative behaviour of the solution curves of BDEs (1) where $a,\ b,\ c$ all vanish at the origin. For this reason, we consider $a,\ b,\ c$ as germs of smooth functions $\mathbb{R}^2,0 \to \mathbb{R},0$, that is, equivalent classes of functions under the equivalence of being identical in some neighbourhood of the origin. (This means that we exclude in our study semi-local or global phenomena such as connections involving separatrices.) As usual, properties on germs are checked on representatives of the germs, and in all the paper we use the same notation for a germ and its representative. We shall denote (the germ of) the BDE (1) by $w=(a,b,c)$ and write

$$\begin{array}{l} j^2a = a_0 + a_1x + a_2y + a_{20}x^2 + a_{21}xy + a_{22}y^2,\\ j^2b = b_0 + b_1x + b_2y + b_{20}x^2 + b_{21}xy + b_{22}y^2,\\ j^2c = c_0 + c_1x + c_2y + c_{20}x^2 + c_{21}xy + c_{22}y^2, \end{array}$$

where j^kf (the k-jet of f at the origin) denotes the Taylor polynomial of order k of f at the origin. Our interest is when $a_0=b_0=c_0=0$.

We review now two methods used for obtaining local topological models of BDEs with zero coefficients at the singular point and comment on their advantages and limitations in Remark 2.2. We start with the method that lifts the bi-valued direction field determined by the BDE to a single direction field on a surface (see for example [4] for the case $c=-a$ and in [9] for the general case). Consider the surface M associated to the BDE given by

$$M=\{(x,y,[\alpha:\beta])\in\mathbb{R}^2,0\times\mathbb{R}P^1 : a\beta^2+2b\alpha\beta+c\alpha^2=0\}.$$

As the coefficients of the BDE all vanish at the origin, $0\times\mathbb{R}P^1\subset M$ (this set will be refered to as the *exceptional fibre*). The discriminant function $\delta=b^2-ac$ plays a key role. When δ has a Morse singularity the surface M is smooth and the projection $\pi: M\to\mathbb{R}^2,0$ is a double cover of the set $\{(x,y):\delta(x,y)>0\}$ ([9]). We label these BDEs *Morse Type 2* (*MT*2 for short). The bi-valued direction field defined by the BDE in the plane lifts to a single direction field ξ on M and extends smoothly to $\pi^{-1}(0)$. Note that the exceptional fibre $0\times\mathbb{R}P^1\subset\pi^{-1}(\Delta)$ is an integral curve of ξ. The closure of the set $\pi^{-1}(\Delta)-(0\times\mathbb{R}P^1)$ is the *criminant* of the equation.

There is an involution σ on $M-(0\times\mathbb{R}P^1)$ that interchanges points with the same image under the projection to $\mathbb{R}^2,0$. It is shown in [9] that σ extends to M when the coefficients a,b,c are analytic. (In fact the result is true when the coefficients are smooth functions; see Remark 3.3.) Points on M are identified with their images by σ. A bi-valued field on the quotient space $M'=M/\sigma$ is then studied and models of the configurations of the integral curves of the BDE are obtained by blowing-down. It turns out that when the discriminant has a Morse singularity and the field ξ has only elementary singularities on the exceptional fibre, the topological models of the integral curves of the BDE are completely determined by the singularity type of the discriminant (an isolated point (A_1^+) or a crossing (A_1^-)), and the number (1 or 3) and type (saddle (S) or node (N)) of the singularities of ξ on the exceptional fibre ([9], Figures 3, 4, top figures).

The above construction also works for BDEs with coefficients not all vanishing at the origin. Then M can be considered a surface in $\mathbb{R}^3$. When M is smooth and ξ has an elementary singularity (saddle/node/focus), the corresponding point in the plane is labelled *folded saddle/node/focus* (see [17]). We shall call these points the *zeros* of the BDE. We observe that a separatrix of a folded saddle is the projection of a separatrix of ξ at the saddle, and a separatrix of a folded node is the projection of a strong separatrix of the node singularity of ξ. (See for example Figure 4 first (resp. second) bottom left for the configurations at a folded saddle (resp. node).)

Back to the case when all the coefficients vanish at the origin and consider the affine chart $p=\frac{\beta}{\alpha}$ (we also need to consider the chart $q=\frac{\alpha}{\beta}$). Let

$$F(x,y,p)=a(x,y)p^2+2b(x,y)p+c(x,y).$$

Then the lifted direction field is parallel to the vector field

$$\xi=F_p\frac{\partial}{\partial x}+pF_p\frac{\partial}{\partial y}-(F_x+pF_y)\frac{\partial}{\partial p}.$$

The singularities of ξ on the exceptional fibre are given by the roots of the cubic

$$\begin{aligned}\phi(p) &= (F_x+pF_y)(0,0,p)\\ &= a_2p^3+(2b_2+a_1)p^2+(2b_1+c_2)p+c_1.\end{aligned}$$

The eigenvalues of the linear part of ξ at a singularity are $-\phi'(p)$ and $\alpha_1(p)$, where

$$\alpha_1(p)=2(a_2p^2+(b_2+a_1)p+b_1).$$

(See for example [9].) So the cubic ϕ and the quadratic α_1 determine the number and type of the singularities of ξ.

The calculations simplify considerably when the 1-jet of the BDE is taken as follows.

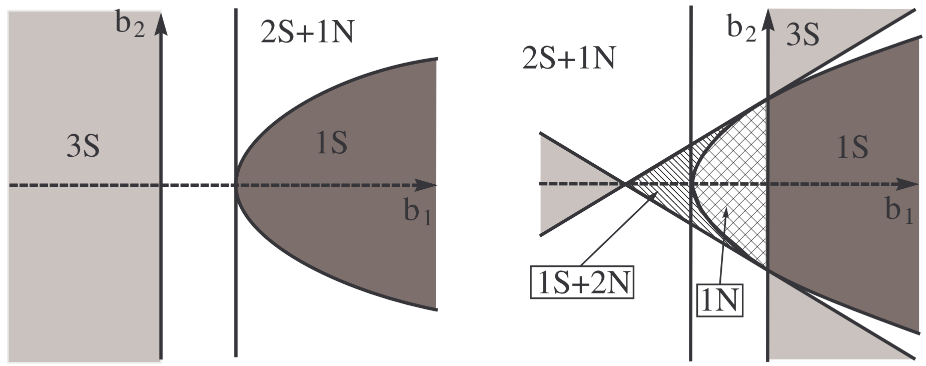

Figure 1: Partition of the (b_1, b_2)-plane, $\epsilon = -1$ left and $\epsilon = +1$ right.

Proposition 2.1 *The 1-jet of the coefficients of a BDE without constant terms can be reduced by linear changes of coordinates and multiplication by non-zero constants to one of the following normal forms.*
(1) When M is regular:
(i) If α_1 and ϕ have no common roots or if ϕ has more that one root ([9, 23])*:*

$$j^1(a, b, c) = (y, b_1 x + b_2 y, \epsilon y),\ \epsilon = \pm 1.$$

(ii) If α_1 and ϕ have a common root and ϕ has only one root:

$$j^1(a, b, c) = (x + a_2 y, 0, y), a_2 > \tfrac{1}{4}.$$

(2) When M is singular, the least degenerate case is ([11])*:*

$$j^1(a, b, c) = (y, \epsilon x + b_2 y, 0),\ \epsilon = \pm 1.$$

Proof The case (1)(ii) follows by calculations similar to those in [9] and [23]. □

In the case (1)(i) of Proposition 2.1, the number and type of the singularities of ξ are determined by the pair (b_1, b_2). There are special curves in the (b_1, b_2)-plane that bound open regions where the topological type of the BDE is constant ([9]). These curves are as follows.

– The discriminant has a degenerate singularity (worse than Morse): $b_1 = 0$.
– The cubic ϕ has a double root: $2b_1 + \epsilon = 0$ or $b_1 = \frac{1}{2}(b_2^2 - \epsilon)$.
– The quadratic α_1 and the cubic ϕ have a common root: $\epsilon = 1$, $b_1 = \pm b_2 - 1$.

See Figure 1, left for the case $\epsilon = -1$ and right for $\epsilon = +1$. Observe that the change of co-ordinates $(x, y) \to (x, -y)$ produces an equivalent BDE given by a pair $(b_1, -b_2)$, there is a symmetry in the (b_1, b_2)-plane with respect to the b_1-axis. Therefore it is enough to restrict to the case $b_2 \geq 0$.

The topological type of the BDE is constant in the complement of the above curves (Figures 3 and 4, top figures); see [9]. These singularities are of codimension 1 and their bifurcations in generic families are studied in [10] (see also [29] for the case $\epsilon = -1$). The figures on both sides of the bifurcations are equivalent, so only one of them is shown in each case in Figures 3 and 4. The figures are labelled according to the number and type of the singularities of ξ on the exceptional fibre. Our study here pointed out at a missing case in [10]: Figure 4, (2S+1N) case (b). In fact, for the (2S+1N) $MT2$-singularity there are two possible bifurcations. The two cases are distinguished by the way a separatrix of a saddle of the lifted field accumulates at the node singularity; see Figure 2 where the separatrix is drawn in thick colour and Figure 3 for the bifurcations in the plane. (Examples of both bifurcations can be plotted using [31].)

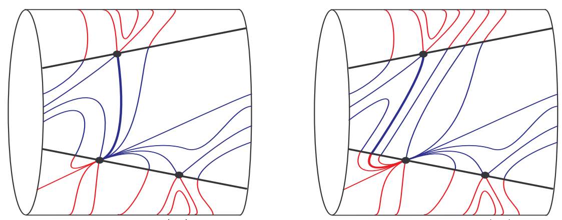

Figure 2: The two bifurcations of the lifted field at a (2S+1N) $MT2$-singularity.

The second method used for studying BDEs (1) consists of blowing-up the singularity. This is first done in [34] where topological models of the lines of curvature at umbilic points on a smooth surface in $\mathbb{R}^3$ are sought. Guíñez [24] used this technique on BDEs whose discriminants are isolated points, which he labels *positive quadratic differential equations*. However, we show here and in [37], that Guíñez's technique can be extended to deal with general BDEs. This extended technique allows us to obtain the local topological models of the integral curves at codimension 2 singularities.

We highlight Guíñez's method below for the case when $j^1w = (y, b_1x + b_2y, \epsilon y)$, with $\epsilon = -1$. The necessary adjustment when $\epsilon = +1$ (i.e. when the discriminant is not an isolated point) are made in §3.

Following the notation in [24], let $f_i(w)$, $i = 1, 2$ denote the foliation associated to the BDE (1), $\omega = (a, b, c)$, which is tangent to the vector field $a\frac{\partial}{\partial u} + (-b + (-1)^i\sqrt{b^2 - ac}\,)\frac{\partial}{\partial v}$. If ψ is a diffeomorphism and $\lambda(x, y)$ is a non-vanishing real valued function, then ([24]) for $k = 1, 2$

1. $\psi(f_k(w)) = f_k(\psi^*(\omega))$, if ψ is orientation preserving;
2. $\psi(f_k(w)) = f_{3-k}(\psi^*(\omega))$, if ψ is orientation reversing;
3. $f_k(\lambda w) = f_k(\omega)$, if $\lambda(x, y)$ is positive;
4. $f_k(\lambda w) = f_{3-k}(\omega)$, if $\lambda(x, y)$ is negative.

We write $w = (y + M_1(x, y), b_1x + b_2y + M_2(x, y), \epsilon y + M_3(x, y))$ and consider the directional blowing-up $x = u, y = uv$. (We also consider the blowing-up $x = uv, y = v$,

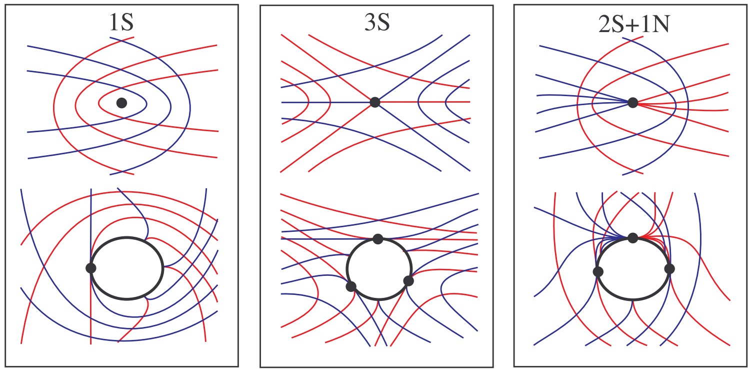

Figure 3: Bifurcations of $MT2$-singularities: the discriminant has an A_1^+-singularity.

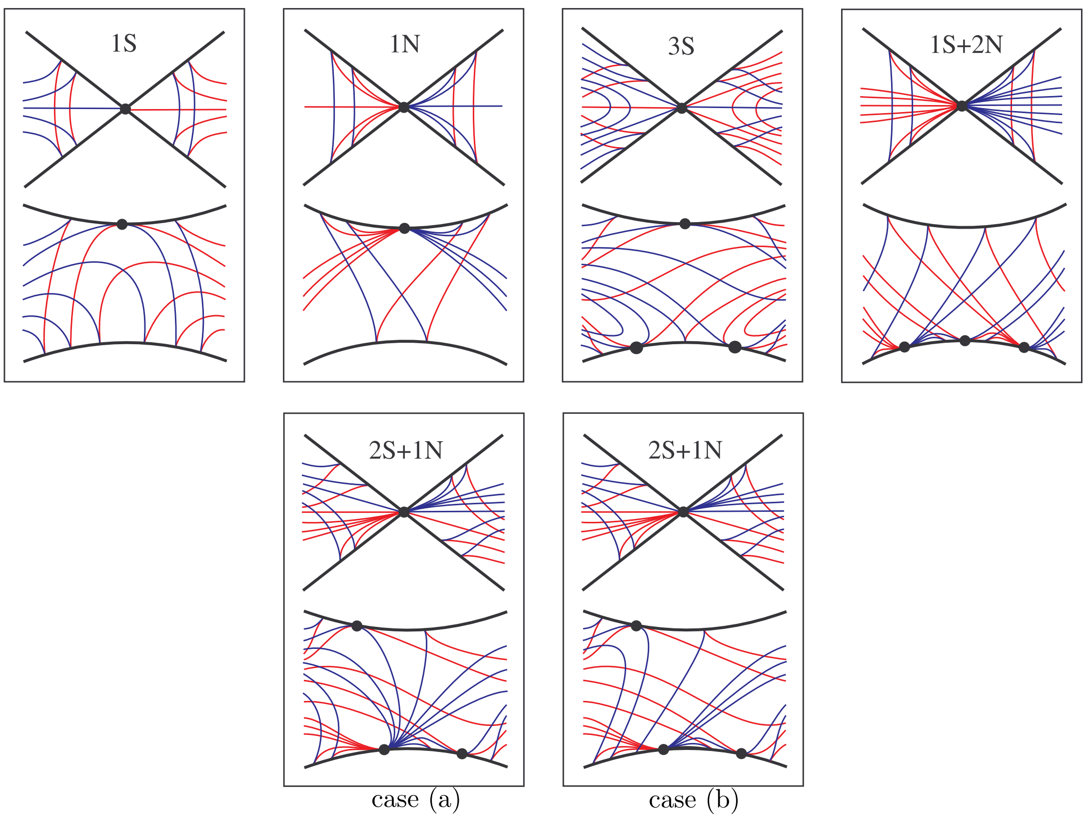

Figure 4: Bifurcations of $MT2$-singularities: the discriminant has an A_1^--singularity.

but this does not yield extra information in general.) Then the new BDE $\omega_0 = (u,v)^*\omega$ has coefficients

$$\begin{aligned}\bar a =&\ u^2(uv + M_1(u,uv)),\\ \bar b =&\ uv(uv + M_1(u,uv)) + u(b_1u + b_2uv + M_2(u,uv)),\\ \bar c =&\ v^2(uv + M_1(u,uv)) + 2v(b_1u + b_2uv + M_2(u,uv)) + \epsilon uv + M_3(u,uv).\end{aligned}$$

We can write $(\bar a, \bar b, \bar c) = u(u^2A_1, uB_1, C_1)$ with

$$\begin{aligned}A_1 =&\ v + uN_1(u,v),\\ B_1 =&\ v^2 + b_2v + b_1 + u(N_2(u,v) + vN_1(u,v)),\\ C_1 =&\ v(v^2 + 2b_2v + 2b_1 + \epsilon) + u(v^2N_1(u,v) + 2vN_2(u,v) + N_3(u,v)),\end{aligned}$$

and $M_i(u,uv) = u^2N_i(u,v)$, $i = 1,2,3$.

The quadratic form $\omega_1 = (u^2A_1, uB_1, C_1)$ can be decomposed into two 1-forms, and to these 1-forms are associated the vector fields

$$X_i = (u^2A_1, -uB_1 + (-1)^i\sqrt{u^2(B_1^2 - A_1C_1)}\,), \quad i = 1,2.$$

These vector fields are tangent to the foliations defined by ω_1. It is clear that we can factor out the term u in X_i, with an appropriate sign change when $u < 0$. The vector fields

$$Y_i = (uA_1, -B_1 + (-1)^i\sqrt{B_1^2 - A_1C_1}\,), \quad i = 1,2$$

are then considered. Since the blowing up is orientation preserving if $u > 0$ and orientation reversing if $u < 0$, and we factored out u twice, it follows from the observation above (see [24]) that Y_1 corresponds to the foliation $\mathcal{F}_1$ of w if $u > 0$ and to $\mathcal{F}_2$ if $u < 0$; while Y_2 corresponds to $\mathcal{F}_2$ if $u > 0$ and to $\mathcal{F}_1$ if $u < 0$.

We study the vector fields Y_i in a neighbourhood of the exceptional fibre $u = 0$, and blow down to obtain the configuration of the integral curves of the original BDE. We then proceed as in [6, 35, 36] to show that any two such configurations are homeomorphic.

Remark 2.2 *The first method (lifting the bi-valued field to a single field on M) is geometrical and works well when the surface M is smooth. However, when M is singular, dealing with the involution σ presents some obstacles. One needs to show first that σ extends to the exceptional fibre and this is not trivial. Then one needs to consider the bi-valued field on the quotient space $M' = M/\sigma$ which has a complicated structure. The second method (blowing-up) is computational and the calculations are sometimes long and winding. It is not very hard to keep track of which foliation projects to a given one in the plane when only one blow-up is needed. However, this could become harder for more degenerate singularities when several blow-ups may be required. In this paper, we use the second method to obtain the topological models at the codimension 2 singularities. But we also make extensive use of the surface M and of the field ξ as these provide useful information about the BDE, especially when dealing with the bifurcations in generic families. In fact, the results here are announced in terms of M and ξ.*

In all the paper one foliation is drawn in blue and the other in red. The discriminant is drawn in thick black and the singularities are represented by thick dots. The figures in this paper are also checked using a computer programme written by A. Montesinos ([31]).

3 Codimension 2 singularities

We consider here topological equivalence between (germs of) BDEs and say that two BDEs ω and τ are equivalent if there is a germ of a homeomorphism $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ that takes the integral curves of ω to those of τ.

When the 1-jet of the BDE with coefficients all vanishing at the origin is written in the form $(y, b_1x + b_2y, \epsilon y)$, local degeneracies of codimension 2 occur when (b_1, b_2) is a generic point on one of the exceptional curves in Figure 1. Degeneracies of codimension 2 also occur if the 1-jet of the BDE is as in Proposition 2.1 (1)-(ii) and (2). Both cases are also treated here and are included in one of the following categories.

1. *$MT2$ criminant saddle-node singularity*: occurs when the quadratic α_1 and the cubic ϕ have a common root. Then the singularity of the discriminant is of type A_1^-, so the criminant is the union of two smooth curves C_1 and C_2 crossing transversally the exceptional fibre at two distinct points. It is not difficult to prove that α_1 and ϕ have a common root if and only if a root of ϕ is at the point of intersection of one of the curves C_i $(i = 1, 2)$ with the exceptional fibre. One can also show that the lifted field ξ has a saddle-node singularity at that point.

2. *$MT2$ saddle-node singularity*: occurs when the discriminant has a Morse singularity and ξ has a saddle-node singularity on the exceptional fibre (away from the criminant). This is equivalent to the cubic ϕ having a double root.

3. *Cusp singularity*: this occurs when the discriminant has a cusp singularity.

We give in §4 a definition of the codimension of a singularity of a BDE and show there that the above singularities are indeed of codimension 2. We treat each of the above cases separately and determine in this section their topological models.

We observe that the notation $MT1$ (Morse Type 1) is reserved for the case when the discriminant has a Morse singularity but the coefficients of the BDE do not all vanish at the origin.

We now proceed to obtain the topological models at the singularities. We give full details of the calculations for the $MT2$ criminant saddle-node singularity and suppress some of these for the other cases.

3.1 $MT2$ criminant saddle-node singularity

We have two cases to consider depending on the number of roots of ϕ (see Proposition 2.1). The surface M is smooth in both cases.

Suppose that ϕ has three roots so we can set $j^1 w = (y, b_1 x + b_2 y, y)$. Then α_1 and ϕ have a common root if $b_1 = \pm b_2 - 1$. Note that the change of coordinates $(x, y) \mapsto (-x, y)$ yields a pair $(b_1, -b_2)$, so we only need to consider the case $b_1 = b_2 - 1$. Then ϕ and α_1 are given by

$$\begin{aligned} \phi(p) &= (p+1)(p+2b_2-1)p, \\ \alpha_1(p) &= (p+1)(p+b_2-1). \end{aligned}$$

We assume here that $(b_1, b_2) \neq (0, 1)$ (the discriminant has a Morse singularity), and $(b_1, b_2) \neq (-1, 0)$ (α_1 and ϕ have only 1 common root), otherwise the resulting BDE is of codimension > 2. (The case when α_1 and ϕ have 2 common roots is treated in [37].)

The common root of ϕ and α_1 is at $p_1 = -1$. The lifted field ξ has generically a singularity of type saddle-node at this point. The remaining two roots of ϕ, $p_2 = 0$ and $p_3 = -2b_2 + 1$, are simple singularities of ξ (saddle or node). More precisely, both singularities are saddles if $b_2 < 0$ or $b_2 > 1$, and one is a saddle and the other is a node if $0 < b_2 < 1$.

If ϕ has one root, we set $j^1 w = (x + a_2 y, 0, y)$, with $a_2 > \frac{1}{4}$ (see Proposition 2.1). Then

$$\begin{aligned} \phi(p) &= p(a_2 p^2 + p + 1), \\ \alpha_1(p) &= p(a_2 p + 1). \end{aligned}$$

So the common root is at $p = 0$. The lifted field ξ has generically a singularity of type saddle-node at this point.

Theorem 3.1 *Suppose that discriminant has a Morse A_1^--singularity and that the vector field ξ has a singularity at the point of intersection of the criminant with the exceptional fibre (i.e. the cubic ϕ and the quadratic α_1 have one common root). Then the BDE is generically locally topologically equivalent to one of the following normal forms.*

If ϕ has three roots:

(i) $(y, 2x + 3y + x^2, y)$, *Figure 5, bottom left, or*

(ii) $(y, -\frac{3}{4}x + \frac{1}{4}y + x^2, y)$, *Figure 5, bottom centre.*

If ϕ has one root:

(iii) $(x + y, x^2, y)$, *Figure 5, bottom right.*

The topological type is completely determined by the 2-jets of the coefficients of the BDE. In all the cases the lifted field ξ has one singularity of type saddle-node on the exceptional fibre. In cases (i) (resp. (ii)) it has two additional saddles (resp. one saddle and one node).

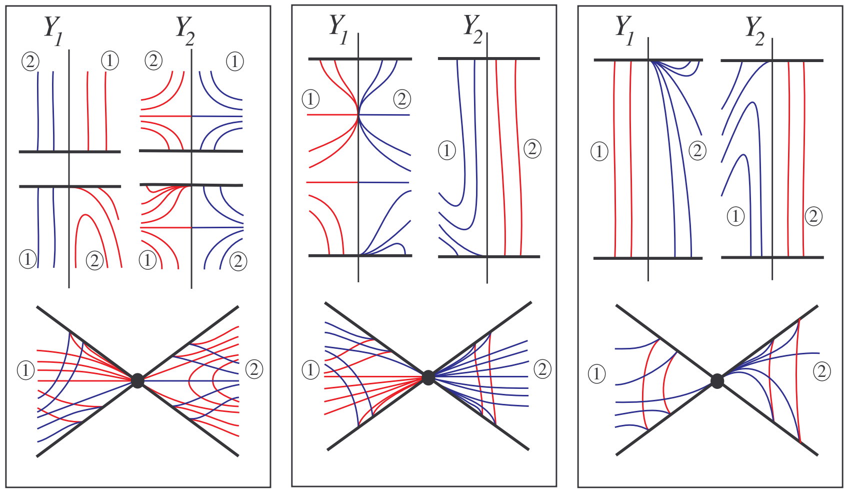

Figure 5: Topological models of the $MT2$ criminant saddle-node singularities (bottom figures) and their associated directional blowing-up models (top figures).

Proof (i)-(ii). We consider the blowing-up $x = u, y = uv$ and analyse the vector fields

$$Y_i = (uA_1, -B_1 + (-1)^i\sqrt{B_1^2 - A_1C_1}\,), \quad i = 1, 2$$

(see §2) along the exceptional fibre $u = 0$. Here, unlike the cases treated by Guíñez, the fields Y_i are only defined in the regions where $B_1^2 - A_1C_1 \geq 0$ and not in a neighbourhood of the whole exceptional fibre. On $u = 0$, this means that

$$(b_2^2 - 1)(v + 1)(v + \frac{b_2 - 1}{b_2 + 1}) \geq 0.$$

So we need to analyse the vector fields Y_i, $i = 1, 2$, at their singular points along the exceptional fibre as well as at the boundary points $v = -1$ and $v = -(b_2 - 1)/(b_2 + 1)$.

The singularities of Y_1 on $u = 0$ occur when $-B_1 - \sqrt{B_1^2 - A_1C_1} = 0$, that is when $-B_1 = \sqrt{B_1^2 - A_1C_1}$. This is equivalent to

$$A_1C_1 = v^2(v + 1)(v + 2b_2 - 1) = 0 \quad \text{and} \quad B_1 = (v + 1)(v + b_2 - 1) \leq 0.$$

So the possible singular points on the exceptional fibre are

(a) $v = -2b_2 + 1$,
(b) $v = 0$,
(c) $v = -1$.

(a) The point $v=-2b_2+1$ is a singularity if $0<b_2<1$. The 1-jet of Y_1 at $(0,-2b_2+1)$ is given by

$$j^1Y_1=(-2b_2+1)u\frac{\partial}{\partial u}+(\kappa u+\frac{(b_2-1)^2}{b_2}(v+2b_2-1))\frac{\partial}{\partial v},$$

for some scalar κ. So the singularity is a node if $0<b_2<\frac{1}{2}$ and a saddle if $\frac{1}{2}<b_2<1$. (We exclude the value $b_2=\frac{1}{2}$ as this yields a singularity of higher codimension.)

(b) The point $v=0$ is a singularity if $b_2<1$. At this point $B_1(0,0)=b_2-1<0$, so

$$-B_1-\sqrt{B_1^2-A_1C_1}=-B_1+B_1\sqrt{1-\frac{A_1C_1}{B_1^2}}=-\frac{A_1C_1}{2B_1}+A_1^2g(u,v)$$

for some germ of a smooth function g with a zero 1-jet at the origin. Therefore Y_1 is singular along the curve $A_1(u,v)=0$. We consider the vector field $\tilde{Y}_1=Y_1/A_1$. The 1-jet of $\tilde{Y}_1$ at $(0,0)$ is given by

$$j^1\tilde{Y}_1=u\frac{\partial}{\partial u}+(\kappa u-\frac{2b_2-1}{2(b_2-1)}v)\frac{\partial}{\partial v},$$

for some scalar κ. So $\tilde{Y}_1$ has a singularity at the origin of type saddle if $b_2<\frac{1}{2}$ and node if $\frac{1}{2}<b_2<1$.

(c) The point $v=-1$ is a singularity of Y_1 for any value of b_2. The singularity occurs at the point of intersection of the exceptional fibre with one branch of the criminant. We consider here the case $b_2<1$ (the remaining cases follow in an analogous way).

We can make smooth changes of coordinates and set $j^ka=y$, for any $k\geq 2$. We write j^2b and j^2c as in §2. Then

$$B_1^2-A_1C_1=2(1-b_2)(1+v+\lambda u+O_2(u,v)),$$

with

$$\lambda=\frac{c_{20}-c_{21}+c_{22}-2(b_{20}-b_{21}+b_{22})}{2(1-b_2)}.$$

We change variables and set $s=u, t^2=1+v+\lambda u+O_2(u,v)$, with $t\geq 0$. (We can consider instead $t\leq 0$ but then we have to adjust the sign when taking the square root.) Then

$$(s,t)^*Y_1=(2st+h.o.t)\frac{\partial}{\partial s}+(\Lambda s+\sqrt{2(1-b_2)}\,t+h.o.t)\frac{\partial}{\partial t},$$

where

$$\Lambda=\frac{(3-b_2)(c_{20}-c_{21}+c_{22})-4(b_{20}-b_{21}+b_{22})}{2(1-b_2)}.$$

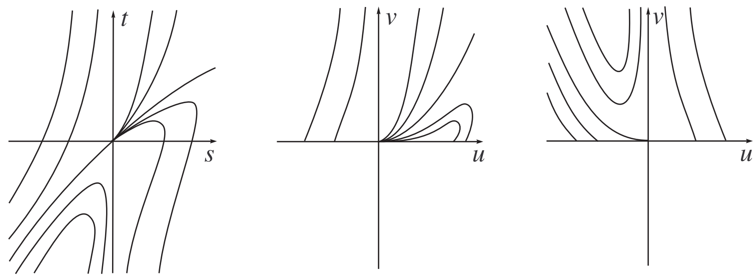

Figure 6: Integral curves of Y_1 (centre) and Y_2 (right) at $v=-1$.

So the origin is a saddle-node singularity of $(s,t)^*Y_1$ provided $\Lambda \neq 0$. The integral curves, up to a reflection with respect to the vertical axis, are as shown in Figure 6 left, and therefore those of Y_1 are as in Figure 6 centre. (Observe that the centre manifold is transverse to the exceptional fibre.) These configurations depend only on the 2-jets of the coefficients of the BDE, more precisely on $\Lambda \neq 0$.

At the remaining boundary point $v=-(b_2-1)/(b_2+1)$, the integral curves of Y_1 are smooth segments ending on the criminant.

We consider now the vector field Y_2. It is singular on the exceptional fibre when $-B_1+\sqrt{B_1^2-A_1C_1}=0$, that is when $B_1=\sqrt{B_1^2-A_1C_1}$. Equivalently, when

$$A_1C_1=v^2(v+1)(v+2b_2-1)=0 \quad \text{and} \quad B_1=(v+1)(v+b_2-1)\geq 0.$$

We have, as for Y_1, three possible singular points at (a) $v=-2b_2+1$, (b) $v=0$, (c) $v=-1$.

(a) The point $v=-2b_2+1$ is a singularity if $b_2<0$ or $b_2>1$. The 1-jet of Y_2 at $(0,-2b_2+1)$ is given

$$j^1Y_2=(-2b_2+1)u\frac{\partial}{\partial u}+(\kappa u+\frac{(2b_2-1)^2}{b_2}(v+2b_2-1))\frac{\partial}{\partial v},$$

for some scalar κ. So the singularity is always a saddle.

(b) The point $v=0$ is a singularity if $b_2>1$. A calculation similar to that for Y_1 shows that Y_2 is singular along the curve $A_1(u,v)=0$. The 1-jet of the vector field $\tilde{Y}_2=Y_2/A_1$ at $(0,0)$ is given

$$j^1\tilde{Y}_2=u\frac{\partial}{\partial u}+(\kappa u-\frac{2b_2-1}{2(b_2-1)}v)\frac{\partial}{\partial v},$$

so $\tilde{Y}_2$ has a saddle singularity at the origin.

(c) The point $v=-1$ is always a singularity of Y_2. Similar calculations to those for Y_1 at $v=-1$ yields a vector field $(s,t)^*Y_2$ given by

$$(s,t)^*Y_2=(2st+h.o.t)\frac{\partial}{\partial s}+(\Lambda s-\sqrt{2(1-b_2)}\,t+h.o.t)\frac{\partial}{\partial t},$$

with Λ as above. So when $\Lambda \neq 0$, the integral curves of the field Y_2 are as in Figure 6 right.

At the remaining boundary point $v = -(b_2-1)/(b_2+1)$, the integral curves of Y_2 are smooth segments ending on the criminant.

We can now draw the integral curves of the fields Y_1 and Y_2, as illustrated in Figure 5, top figures, and blow down to obtain the configurations of the integral curves of the associated BDE (Figure 5, bottom figures).

Recall that the integral curves of Y_1 project to one of the foliations defined by the BDE, say $\mathcal{F}_1$ if $u>0$ and to the other foliation, say $\mathcal{F}_2$, if $u<0$; while those of Y_2 project the foliation $\mathcal{F}_2$ if $u>0$ and to $\mathcal{F}_1$ if $u<0$ (see §2).

Part of the integral curves of the fields Y_i, $i=1,2$ that project to the foliation $\mathcal{F}_i$, $i=1,2$, are drawn with the same colour. (In Figure 5, when the discriminant is a node, we highlight which integral curves project to a given region by labeling these ① and ②.)

To show that two BDEs with the same configuration are topologically equivalent, we follow the method in [6, 35, 36] and construct a homeomorphism taking the integral curves of one BDE to the other. The homeomorphism is constructed by choosing an appropriate neighbourhood of the singularity and by sliding along integral curves. It is completely determined by its restriction to a subset of the boundary of the neighbourhood.

The configurations depend only on the type of the singularities of ξ (and on $\Lambda \neq 0$), so we can give (b_1,b_2) any value on the line $b_1 = b_2 - 1$, provided $b_2 \neq 0, \frac{1}{2}, 1$. We take (b_1,b_2) as in the statement of the theorem and choose values for the coefficients of the second degree terms so that $\Lambda \neq 0$.

(iii). Suppose that ϕ has one root and write $w = (x + a_2 y + M_1(x,y), M_2(x,y), y + M_3(x,y))$, with $a_2 > \frac{1}{4}$. We can changes coordinates and set $j^k M_3(x,y) = 0$ and $j^k M_2(x,y) = B_2(x)$, for any $k \geq 2$. We consider the blowing up $x = u, y = uv$ and study the vector fields $Y_i = (uA_1, -B_1 + (-1)^i \sqrt{B_1^2 - A_1 C_1})$, $i=1,2$ where

$$\begin{aligned} A_1 =&\ \ 1 + a_2 v + uN_1(u,v) \\ B_1 =&\ \ v(1+a_2 v) + u(vN_1(u,v) + N_2(u,v)) \\ C_1 =&\ \ v(1+v+a_2v^2) + u(v^2N_1(u,v) + 2vN_2(u,v) + N_3(u,v)) \end{aligned}$$

and $M_i(u,uv) = u^2 N_i(u,v)$. Calculations similar to those above show that the only singularities of Y_1 and Y_2 are at the point of intersection of the exceptional fibre with one branch of the criminant, and the configurations of the integral curves of these fields are as in Figure 5, top right. So blowing down yields the configuration in Figure 5, bottom right. The configuration depends only on the 2-jets of the coefficients of the BDE, more precisely, on $b_{20} \neq 0$. We can take $(a,b,c) = (x+y, x^2, y)$ as a topological model. $\square$

3.2 $MT2$ saddle-node singularity

We take $j^1 w = (y, b_1 x + b_2 y, \epsilon y)$, $\epsilon = \pm 1$. Then the cubic ϕ has a double root if and only if $b_1 = \frac{1}{2}(b_2^2 - \epsilon)$ or $b_1 = -\frac{1}{2}\epsilon$. The case $\epsilon = -1$ is studied in [24]. It is shown there that when $b_1 = \frac{1}{2}$, one can change coordinates and reduce to the case where $b_1 = \frac{1}{2}(b_2^2 + 1)$. A similar calculation shows that this is also true when $\epsilon = 1$. So we shall only deal with the case $\epsilon = +1$ and $b_1 = \frac{1}{2}(b_2^2 - 1)$. We also take $b_2 > 0$ as the change of coordinates $(x, y) \mapsto (-x, y)$ yields a pair $(b_1, -b_2)$.

In this case, the surface M is smooth and $\phi(p) = p(p + b_2)^2$. The field ξ has a saddle-node singularity at $p = -b_2$ and a saddle (resp. node) at $p = 0$ if $b_2 > 1$ (resp. $0 < b_2 < 1$).

Theorem 3.2 *Suppose that the discriminant has a Morse singularity and that ξ has a saddle-node singularity away from the criminant (i.e. the cubic ϕ has a repeated root). Then the BDE is generically locally topologically equivalent to one of the following normal forms.*

(i) $(y, x + y + y^2, -y)$ [24], *Figure 7, bottom left,*
(ii) $(y, 4x + 3y + y^2, y)$ *Figure 7, bottom centre,*
(iii) $(y, -\frac{3}{8}x + \frac{1}{2}y + y^2, y)$ *Figure 7, bottom right.*

The topological type is completely determined by the 1-jets of the coefficients of the BDE. (The degree 2 term is added so that the generic families of these singularities are given in simple forms, see §4.2.)

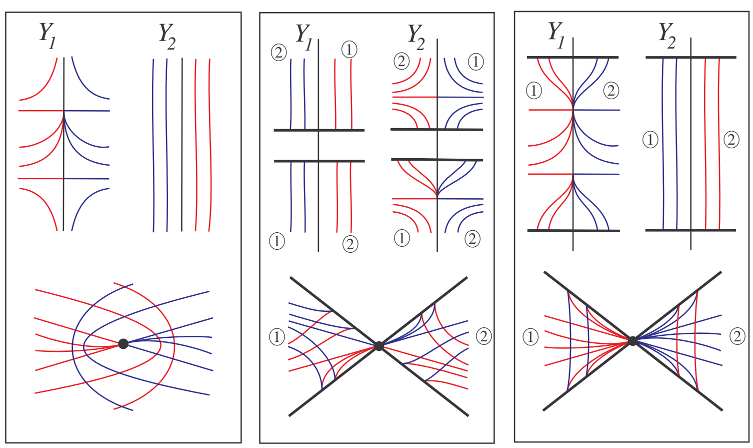

Figure 7: Topological models of the $MT2$ saddle-node singularities (bottom figures) and their associated directional blowing-up models (top figures).

Proof For (i) see [24]. For (ii) and (iii) the proof is similar to the case in §3.1, so we highlight only the differences and suppress most of the details. We have $(B_1^2 - A_1C_1)(0,v) = (b_2^2-1)(v+\frac{b_2+1}{2})(v+\frac{b_2-1}{2})$.

If $0 < b_2 < 1$ then Y_1 has a node singularity at $v=0$ and a saddle-node at $v=-b_2$, whereas Y_2 has no singularities. Both fields end transversally on the criminant; see Figure 7 right. The centre manifold of the saddle-node singularity is along the exceptional fibre.

If $b_2 > 1$ then Y_2 has a saddle singularity at $v=0$ and a saddle-node at $v=-b_2$, whereas Y_1 has no singularities. Both fields end transversally on the criminant; see Figure 7 centre. The centre manifold of the saddle-node singularity is along the exceptional fibre.

The configurations depend only on the type of the singularities of ξ and of the discriminant δ. So to get a model we can give (b_1,b_2) any value on one of the connected component (determined by the exceptional points) of the exceptional curve in consideration. We take (b_1,b_2) as in the statement of the theorem. $\square$

Remark 3.3 *The results in Theorems 3.1 and 3.2 can also be proved using the method in* [4, 9] *(see §2). In both cases the surface M is smooth and there is a well defined involution σ on $M - 0\times\mathbb{R}P^1$ (see §2). One can show that Proposition 4.4 in* [9] *also holds when the coefficients are smooth functions (instead of analytic), that is, σ extends to a smooth function on M. (I would like to thank Carlos Biasi for his help with the proof of this statement.) We then proceed as in* [9]*. The difference here with the cases treated in* [4, 9] *is that one singularity of the lifted field ξ on the exceptional fibre is a saddle-node away from the criminant in the "MT2 saddle-node" case (with the centre manifold along the exceptional fibre) and on the criminant in the "MT2 criminant saddle-node" case (with the centre manifold transverse to the exceptional fibre).*

3.3 The cusp singularity

We set $j^1w = (y, \epsilon x + b_2y, 0)$, $\epsilon = \pm 1$ (see [11]). It is shown in [11] that, for almost all values of b_2, any k-jet of the coefficients of the BDE can be reduced to $(y, \epsilon x + b_2y, g(y))$, where g is a polynomial with zero 1-jet. In fact, when the discriminant has a cusp singularity, the same calculation in [11] shows that we can reduce any k-jet of the coefficients to $(y, \epsilon x + b_2y + b(x,y), y^2 + g(y))$, where b and g have zero 2-jets, for any value of b_2. So we shall take

$$(a,b,c) = (y + M_1(x,y), \epsilon x + b_2y + M_2(x,y), y^2 + M_3(x,y)),$$

with M_1 having a zero k-jet and $j^kM_3 = g(y)$ (for the calculation here, it is enough to take $k=2$).

The surface M has a Morse singularity at $p=0$ and $\phi(p) = p(p^2+2b_2p+2\epsilon)$. When $b_2^2 - 2\epsilon > 0$ the vector field ξ has two singularities at regular points on the exceptional fibre. When $\epsilon = -1$ these singularities are both saddles. When $\epsilon = +1$ and $b_2^2 - 2 > 0$ one is a saddle and the other is a node. There are no singularities of ξ at regular points on the exceptional fibre when $\epsilon = +1$ and $b_2^2 - 2 < 0$.

Theorem 3.4 *Suppose that the discriminant has a cusp singularity. Then the BDE is generically locally topologically equivalent to one of the following normal forms.*

(i) $(y, -x + y, y^2)$ *Figure 8, bottom left,*
(ii) $(y, x + 2y, y^2)$ *Figure 8, bottom centre,*
(iii) $(y, x + y, y^2)$ *Figure 8, bottom right.*

The topological type is completely determined by the 2-jets of the coefficients of the BDE.

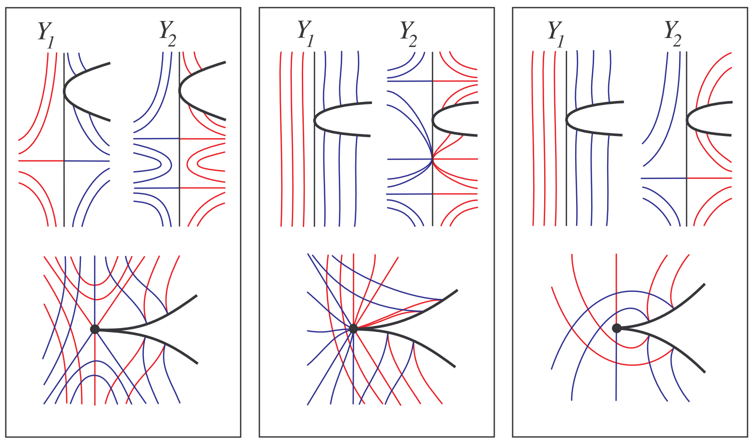

Figure 8: Topological models of the cusp singularity (bottom figures) and their associated directional blowing-up models (top figures).

Proof We consider, as in §3.1, the blowing-up $x = u, y = uv$ and study the vector fields $Y_i = (uA_1, -B_1 + (-1)^i\sqrt{B_1^2 - A_1C_1}\,)$, $\quad i = 1, 2$ with

$$
\begin{aligned}
A_1 = &\ v + uN_1(u, v),\\
B_1 = &\ v^2 + b_2v + \epsilon + u(vN_1(u, v) + N_2(u, v)),\\
C_1 = &\ v(v^2 + 2b_2v + 2\epsilon) + u(v^2 + v^2N_1(u, v) + 2vN_2(u, v) + N_3(u, v)).
\end{aligned}
$$

The fields Y_i, $i = 1, 2$ are defined in the region where $B_1^2 - A_1C_1 \geq 0$. The set $B_1^2 - A_1C_1 = 0$ is a smooth curve tangent to the exceptional fibre at $v = -\epsilon/b_2$ and we have $(B_1^2 - A_1C_1)(0, v) = (b_2v + \epsilon)^2$, so the whole exceptional fibre is an integral curve for both Y_1 and Y_2.

At $v=-\epsilon/b_2$, we have $C_1(0,-\frac{\epsilon}{b_2})\neq 0$. The solutions of $u^2A_1dv^2+2uB_1dudv+C_1du^2=0$ are also tangent to the vector fields $(u(-B_1+(-1)^i\sqrt{B_1^2-A_1C_1}\,),C_1)$. It is not hard to show that the integral curves of these fields (which are the same as those of Y_i) are as in Figure 8, top figures, at $v=-\epsilon/b_2$.

The singularities of Y_i on $u=0$ occur when

$$-(v^2+b_2v+\epsilon)+(-1)^i|b_2v+\epsilon|=0.$$

When $\epsilon=-1$, Y_1 has a line of singularities passing through $v=0$. We can write $Y_1=A_1(u,v)\tilde{Y}_1$, and show that $\tilde{Y}_1$ has a saddle singularity at $(0,0)$. The vector field Y_2 has two singularities at the solutions of $v^2+2b_2v-2=0$. These are both saddles. Therefore the integral curves of Y_1 and Y_2 are in as Figure 8, top left. Blowing down yields the configuration in Figure 8, bottom left.

When $\epsilon=+1$ and $b_2^2-2>0$, Y_2 has two singularities at the solutions of $v^2+2b_2v+2=0$. One singularity is a saddle and the other a node (the node is located between the saddle and the point of contact of $B_1^2-A_1C_1=0$ with $u=0$). At $v=0$ we can write $Y_2=A_1(u,v)\tilde{Y}_2$, with $\tilde{Y}_2$ having a singularity of type saddle at $v=0$. The vector field Y_1 has no singularities on $u=0$. Therefore the integral curves of Y_1 and Y_2 are in as Figure 8, top centre. Blowing down yields the configuration in Figure 8, bottom centre.

When $\epsilon=+1$ and $b_2^2-2<0$ the situation is as above except that Y_2 has only one singularity at $v=0$. The curves of Y_1 and Y_2 are as in Figure 8, top right. Blowing down yields the configuration in Figure 8, bottom right. $\square$

4 Generic two parameter families of BDEs

We adopt here, following [20] (see also [33]), the notion of fibre topological equivalence for families of BDEs. Two germs of families of BDEs ω_t and τ_s, depending smoothly on the parameters t and s respectively, are said to be *locally fibre topologically equivalent* if, for any of their representatives, there exist neighbourhoods U and W of 0 in respectively the phase space (x,y) and the parameter space t, and a family of homeomorphisms h_t, for $t\in W$, all defined on U such that h_t is a topological equivalence between ω_t and $\tau_{\psi(t)}$, where ψ is a homeomorphism defined on W. (We exclude from our study the semi-local/global phenomena in ω_0 but these can of course appear in ω_t, for $t\neq 0$.)

We associate to a germ of an r-parameter family of BDEs $\tilde{\omega}=(\tilde{a},\tilde{b},\tilde{c})$ the jet-extension map

$$\begin{array}{rccc}\Phi: & \mathbb{R}^2\times\mathbb{R}^r,(0,0) & \to & J^k(2,3)\\ & ((x,y),t) & \mapsto & j^k(\tilde{a},\tilde{b},\tilde{c})_t|_{(x,y)}\end{array}$$

where $J^k(2,3)$ denotes the vector space of polynomial maps of degree k from $\mathbb{R}^2$ to $\mathbb{R}^3$, and $j^k(\tilde{a},\tilde{b},\tilde{c})_t|_{(x,y)}$ is the k-jet of $(\tilde{a},\tilde{b},\tilde{c})$ at (x,y) with t fixed. (This is simply the Taylor expansion of order k of $(\tilde{a},\tilde{b},\tilde{c})_t$ at (x,y).) A singularity in the family is of *codimension* m if the conditions that define it yield a semi-algebraic set V of codimension $m+2$ in $J^k(2,3)$, for any $k\geq k_0$. The set V is supposed to be invariant under the natural action of the $k+1$-jets of diffeomorphisms in (x,y) and multiplication by non-zero functions in (x,y).

The family $\tilde{\omega}$ is said to be *generic* if the map Φ is transverse to V in $J^k(2,3)$. Observe that a necessary condition for genericity is that $r\geq m$. It follows from Thom's Transversality Theorem that the set of generic families is residual in the set of smooth map germs $\mathbb{R}^2\times\mathbb{R}^r,0\to\mathbb{R}^3,0$.

Our interest here is when the codimension $m=2$. The *bifurcation set* of a generic 2-parameter family consists of the set of parameters t where the associated BDE has a singularity of codimension ≥ 1.

We take, as in §2, an affine chart $p=\frac{dy}{dx}$ (we also consider the chart $q=\frac{dx}{dy}$) and set $F(x,y,t,p)=\tilde{a}(x,y,t)p^2+2\tilde{b}(x,y,t)p+\tilde{c}(x,y,t)$. (We still denote by F the restriction of this function to $t=$ *constant*.) A zero of the BDE w_t, for t fixed, at a smooth point on the discriminant is given by $F=F_p=F_x+pF_y=0$. Suppose, without loss of generality, that $F_y\neq 0$ at the point in consideration, say $q_0=(x_0,y_0,p_0)$, so that the surface of the equation is given by $y=h(x,p)$ for some germ of a smooth function h at (x_0,p_0). So $F(x,h(x,p),p)=0$ and $F_x+h_xF_y=F_yh_p+F_p=0$. Therefore, the linear part of the projection of the lifted field ξ_t to the (x,p)-plane is given by

$$A=\left(\begin{array}{cc} F_{xp}-\frac{F_x}{F_y}F_{yp} & F_{pp}-\frac{F_p}{F_y}F_{yp} \\ \\ -F_{xx}+\frac{F_x}{F_y}F_{xy}-p(F_{xy}-\frac{F_x}{F_y}F_{yy}) & -F_y-F_{xp}+\frac{F_p}{F_y}F_{xy}-p(F_{yp}-\frac{F_p}{F_y}F_{yy}) \end{array}\right),$$

where the entries are evaluated at q_0.

The origin is a zero-dimensional stratum of the bifurcation set. The components of the bifurcation set corresponding to local codimension 1 singularities are as follows.

$(MT1)$-*stratum*: is given by the set of parameters t for which w_t has a Morse Type 1 singularity at some point in the plane (i.e. the coefficients of w_t do not all vanish at this point and the discriminant has a Morse singular). This is given by

$$F=F_p=F_x=F_y=0.$$

$(MT2)$-*stratum*: is given by the set of parameters t for which w_t has a Morse Type 2 singularity at some point in the plane (i.e. the coefficients of w_t all vanish at this point). This is given by

$$a=b=c=0.$$

(FSN)-*stratum*: is given by the set of parameters t for which w_t has a folded saddle-node singularity at some point in the plane (i.e. the discriminant of w_t is smooth and the lifted field has a saddle-node singularity). This is given by

$$F = F_p = F_x + pF_y = \det(A) = 0.$$

The node-focus change component of the bifurcation set ([35]) is empty for the singularities studied here. This can be verified by direct calculations but is not surprising as the lifted field ξ has no singularities of type focus on the exceptional fibre at a Morse Type 2 singularity. We have also shown in [35] that Poincaré-Andronov (Hopf) bifurcations do not occur on a smooth discriminant.

We need to consider codimension 1 non-local phenomena as well. There are the ones that correspond to non-local singularities of ξ_t on M_t (see for example [1], p.91 for the list of the generic possibilities). Geometric arguments, such as the location of the zeros, the fact that the pair of foliations are transverse away from the discriminant and the sector decomposition at the codimension 2 singularities, exclude the emergence of a homoclinic trajectory of a saddle-node and of a separatrix loop at a saddle in all the cases treated here. (Recall that we are only dealing with the bifurcations of the local singularities of BDEs.) We need to consider the case below which is topologically stable for vector fields but not for BDEs due to the presence of the involution σ_t on M_t (see §4.1).

> $(FSNC)$-*stratum*: is given by the set of parameters t for which a separatrix at a folded saddle and a separatrix at a folded node end at the same point on another component of the discriminant. (We label this stratum *folded saddle node connection.*)

We observe here that at a saddle-node bifurcation of a planar vector field, there is an invariant curve that links the saddle and the node and is a stable/unstable manifold at these points (see for example [1], Corollary in p. 69). This is a local phenomena and is distinct from the above.

We obtain a stratification $\mathcal{S}$ of a neighbourhood U of the origin in the parameter space $\mathbb{R}^2$, given by the origin, the above strata and the complement of the union of these sets.

Our aim is to show that any two generic 2-parameter families of a codimension 2 singularity are fibre topologically equivalent and obtain thus a model for such families. For this we take the following steps (compare with [20], [35]).

- Obtain a model for the BDE at $t = 0$ (this is done in §3).
- Reduce the N-jet of the family to a normal form.
- Obtain a condition for the family to be generic.
- Show that the bifurcation sets of generic families are homeomorphic.
- Obtain the configuration of the discriminant in each stratum of $\mathcal{S}$.
- Show that the number of singularities, their type and their position on the discriminant are constant in each stratum of $\mathcal{S}$.
- Show that the configurations of the integral curves have a constant topological type in each stratum of $\mathcal{S}$.

The above strategy is justified by the following observations. A key feature of the BDE $w_{(u,v)}$ is that all the zeros occur on the discriminant, and away from it the pair of foliations defined by the equation are transverse. Therefore, if we know the type of the zeros at smooth points on the discriminant we can draw locally the integral curves of the BDE at such points. It is then a matter of putting together the local figures to get the configuration of the integral curves of $w_{(u,v)}$ in a neighbourhood of the origin in the plane. This is done by looking at what happens on the codimension 1 strata.

We determine the differential structure of the $(MT1)$ and $(MT2)$ strata by considering families of symmetric matrices. To a BDE $w = (a, b, c)$ is associated the family of symmetric matrices

$$S(x, y) = \begin{pmatrix} a(x, y) & b(x, y) \\ b(x, y) & c(x, y) \end{pmatrix}.$$

We write $S = (a, b, c)$. The discriminant of the BDE is precisely the determinant of S. Bruce classified in [3] families of symmetric matrices up to an equivalence relation that preserves the singularities of the determinant. Let $S(n, \mathbb{K})$ denotes the space of $n \times n$-symmetric matrices with coefficients in the field $\mathbb{K}$ of real or complex numbers. A family of symmetric matrices is a smooth map germ $\mathbb{K}^r, 0 \to S(n, \mathbb{K})$. Denote by $\mathcal{G}$ the group of smooth changes of parameters in the source and parametrised conjugation in the target. A list of all the $\mathcal{G}$-simple singularities of families of symmetric matrices is obtained in [3].

The cases of interests to our investigation are $n = r = 2$ and when the $\mathcal{G}$-codimension of S is ≤ 2. The cases that we need from [3] are those in the series $(x, y, \pm x^k)$, $k > 1$, with $\mathcal{G}$-codimension k. A $\mathcal{G}$-versal unfolding is given by $(x, y, \pm x^k + \sum_{i=0}^{k-1} v_i x^i)$. The bifurcation set of this family has two components. One component is given by the set of parameters where the determinant is singular at some point q but the coefficients of the matrix do not all vanish at q. This corresponds to the $(MT1)$-stratum and is the discriminant of $\pm x^k + \sum_{i=0}^{k-1} v_i x^i$, so is diffeomorphic to the product of the discriminant of an A_{k-1}-singularity with a line. The second component is given by the vanishing of all the coefficients of the matrix at some point q and corresponds to the $(MT2)$-stratum. It is given by $v_0 = 0$.

We show that a generic 2-parameter family of BDEs w_t, with w_0 having a codimension 2 singularity at the origin, yields a $\mathcal{G}$-versal deformation of the matrix of w_0. This allows us to deduce the differential structure of the union of the $(MT1)$ and $(MT2)$ strata. This approach also allows us to obtain the configuration of the discriminant in each stratum of $\mathcal{S}$.

4.1 Bifurcations of the $MT2$ criminant saddle-node

We consider generic 2-parameter families of BDEs $w_{(u,v)} = (\tilde{a}, \tilde{b}, \tilde{c})$ with w_0 as in Theorem 3.1. There are two cases to consider depending on the number of roots of ϕ_0;

see Theorem 3.1. (We identify a polynomial map in $J^2(2,3)$ with its coefficients with respect to the generating monomials.)

The case ϕ_0 has 3 roots

We take $j^1 w_0 = (y, b_1 x + b_2 y, y)$ with $b_1 = \pm b_2 - 1$. As pointed out in §3.1, it is enough to consider the case $b_1 = b_2 - 1$. We can change coordinates and set $j^k \tilde{a} = y$ (for any $k \geq 2$) and $\tilde{b}(0,0,u,v) = 0$. We write $j^2(a,b,c)$ in the form

$$\begin{aligned} j^2\tilde{a} &= y, \\ j^2\tilde{b} &= b_1 x + b_2 y + (\beta_{10}u + \beta_{11}v)x + (\beta_{20}u + \beta_{21}v)y + B_2(x,y), \\ j^2\tilde{c} &= y + \gamma_1 u + \gamma_2 v + (\gamma_{10}u + \gamma_{11}v)x + (\gamma_{20}u + \gamma_{21}v)y + C_2(x,y) + \Gamma_2(u,v) \end{aligned} \tag{2}$$

where B_2, C_2 and Γ_2 have zero 1-jets, and $b_1 = b_2 - 1$. The variety V_1 of BDEs having an $MT2$ criminant saddle-node singularity is given by

$$V_1 = \{a_0 = b_0 = c_0 = 0,\ R = \text{Resultant}(\phi, \alpha_1) = 0\},$$

where $R = -3a_2b_2b_1c_1 - b_2^2c_2b_1 - a_2b_2c_2c_1 + 2b_2^2a_1c_1 - a_1b_1^2b_2 - a_2a_1b_1c_1 - a_2a_1c_2c_1 + b_2a_1^2c_1 + 2a_2b_1^2c_2 + a_2^2c_1^2 + b_2^3c_1 + a_2b_1c_2^2 + a_2b_1^3 - a_1c_2b_2b_1$. At w_0, we have

$$\left(\frac{\partial R}{\partial a_1}, \frac{\partial R}{\partial a_2}, \frac{\partial R}{\partial b_1}, \frac{\partial R}{\partial b_2}, \frac{\partial R}{\partial c_2}, \frac{\partial R}{\partial c_1}\right) = b_2(b_2 - 1)\left(-b_2, b_2, 2, -2, b_2 - 2, 2 - b_2\right).$$

Restricting to any k-jet space $J^k(2,3)$, with $k \geq 2$, and working with appropriate jets, V_1 is smooth at w_0 provided $b_2 \neq 0, 1$. The tangent space to V_1 at w_0 is given by the intersection of the kernels of the 1-forms α_i, $i = 1,2,3,4$ with

$$\begin{gathered} \alpha_1 = \mathrm{d}a_0,\ \alpha_2 = \mathrm{d}b_0,\ \alpha_3 = \mathrm{d}c_0, \\ \alpha_4 = \mathrm{d}R|_{w_0} = b_2\mathrm{d}a_1 - b_2\mathrm{d}a_2 - 2\mathrm{d}b_1 + 2\mathrm{d}b_2 - (b_2 - 2)\mathrm{d}c_1 + (b_2 - 2)b_2\mathrm{d}c_2. \end{gathered}$$

The map Φ is transverse to V_1 at w_0 if and only if there is no non-zero vector $Z = \lambda_1\Phi_x + \lambda_2\Phi_y + \lambda_3\Phi_u + \lambda_4\Phi_v$ that belongs to the intersection of the kernels of the 1-forms α_i, $i = 1,2,3,4$. This gives a linear system $\alpha_i(Z) = 0$, $i = 1,2,3,4$ in $(\lambda_1, ..., \lambda_4)$, and the transversality condition is equivalent to the matrix of this linear system having a non zero determinant. The determinant is equal to $-(b_2 - 1)$ times

$$(\gamma_1(\gamma_{11} - \gamma_{21}) + \gamma_2(\gamma_{20} - \gamma_{10}))b_2 - 2\gamma_1(\gamma_{11} - \gamma_{21} + \beta_{21} - \beta_{11}) + 2\gamma_2(\beta_{20} - \beta_{10} + \gamma_{10} - \gamma_{20}).$$

As $b_2 \neq 1$, a necessary condition for the determinant to be non zero is $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$. When this is the case we can make a change of variable and set $v = \tilde{c}(0,0,u,v)$, so $\gamma_2 = 1$, $\gamma_1 = 0$ and $\Gamma_2 = 0$ in (2) above. Then the condition for transversality becomes

$$(b_2 - 2)(\gamma_{20} - \gamma_{10}) + 2(\beta_{20} - \beta_{10}) \neq 0.$$

This condition is satisfied if we take $j^2w = (y, (b_2 - 1)x + b_2y + x^2, y + ux + v)$, with $b_2 \neq 2$. When $b_2 = 2$, we need to take $j^2w = (y, (b_2 - 1 + u)x + b_2y + x^2, y + v)$, but this is similar to the previous case and will not be considered here.

We determine now the bifurcation set of a generic family with $j^2(\tilde{a}, \tilde{b}, \tilde{c})$ in the simplified form above.

The symmetric matrix associated to w_0 is $\mathcal{G}$-equivalent to (y, x, y) which has $\mathcal{G}$-codimension 1. A generic 2-parameter family of BDEs yields a $\mathcal{G}$-versal deformation of the singularity of the symmetric matrix of w_0. Therefore the $(MT2)$-stratum is a smooth curve and the $(MT1)$-stratum is empty. It follows by the implicit functions theorem that the $(MT2)$-stratum is given by $v = h(u)$ for some germ of a smooth function h with a zero 1-jet at the origin.

The (FSN)-stratum is given by $F = F_p = F_x + pF_y = \det(A) = 0$, where p is close to $p_1 = -1$ and A is the matrix at the beginning of §4. If we consider the map-germ $G = (F, F_p, F_x + pF_y, \det(A))(x, y, p, u, v)$, then $D_{(x,y,p,v)}G$ has maximal rank at $(0, 0, -1, 0, 0)$ if and only if $b_2 \neq 1$ (which we assume is the case, see §3.1) and

$$(c_{20} - c_{21} + c_{22})(b_2 - 3) + 4(b_{20} - b_{21} + b_{22}) \neq 0.$$

This is precisely the condition $\Lambda \neq 0$ in §3.1, so is satisfied at a $MT2$ criminant saddle-node singularity. The by the implicit function theorem, the (FSN)-stratum is a smooth curve provided the above condition is satisfied. One can then compute the initial term of a parametrisation of this stratum. It is given by

$$v = \frac{(2(\beta_{10} - \beta_{20} - \gamma_{10} + \gamma_{20}) + (\gamma_{10} - \gamma_{20})b_2)^2}{4(b_2 - 1)((c_{20} - c_{21} + c_{22})(b_2 - 3) + 4(b_{20} - b_{21} + b_{22}))}u^2 + h.o.t.$$

It follows that for generic families $w_{(u,v)}$, the (FSN)-stratum has an ordinary tangency at the origin with the $(MT2)$-stratum (bifurcation sets in boxes in Figures 10, 11, 12).

The multiplicity of w_0 (see [11] for definition and related results) is generically 4, that is, we expect a maximum of 4 folded singularities of $w_{(u,v)}$ in an open stratum of $\mathcal{S}$. (The genericity condition depends on j^2w_0 and is satisfied for the normal form taken here.)

We analyse now the topological configurations of the integral curves of the members of the family with (u, v) on the $(MT2)$-stratum. For (u, v) on this stratum, the lifted field $\xi_{(u,v)}$ has three simple singularities on the exceptional fibre. Two of them p_2, p_3 have the same type as those of ξ_0. The third singularity has eigenvalues $\alpha_{1_{(u,v)}}(p_1)$ and $-\phi'_{(u,v)}(p_1)$. As $\phi'_0(p_1) \neq 0$ and $\alpha_{1_{(u,v)}}(p_1)$ changes sign as we move from one component of the $MT2$-stratum to the other, p_1 is a saddle (resp. node) when (u, v) is on one component (resp. the other) of the $MT2$-stratum. Therefore we have the following Morse Type 2 singularities on the discriminant. (Recall that these singularities are classified by the type of the Morse singularity of the discriminant (A_1^- here) and the number and the type of the singularities of ξ; see Figure 1).

(i) $b_2 < 0$ or $b_2 > 1$: 2S+1N on one component of the $(MT2)$-stratum and 3S on the other (Figure 10, ② and ⑧ respectively).

(ii) $0 < b_2 < 1$: 1S+2N on one component of the $(MT2)$-stratum and 2S+1N on the other (Figure 11, ② and ⑧ respectively).

It is not difficult to show that as (u, v) varies on the $(MT2)$-stratum, the induced deformation of the saddle-node singularity of $\xi_{(0,0)}$ at $p_1 = -1$ is not versal. In fact, we have a saddle and a node on both sides of the transition. One of these singularities remain on the exceptional fibre while the other lies on the criminant. So as (u, v) moves along the $(MT2)$-stratum across the origin, a singularity of type folded saddle appears on one side of a branch of the discriminant and a folded node on the other side of the same branch of the discriminant (see Figures 10, 11 ② and ⑧). With these information and the way Morse Type 2 singularities bifurcate ([10]; Figure 4; Figures 10, 11 around the $(MT2)$-stratum), we are able to determine the number of singularities on the discriminants of the BDEs in each stratum of the stratification $\mathcal{S}$, as well as identify their type and position on the discriminant. We can also determine the configurations in the open strata delimited by the $(MT2)$-stratum (see Figures 10, 11 ③, ⑦ and ⑨) by using the results in [10]. Observe that in Figure 10 ⑧ the bifurcation of the 2S+1N $MT2$-singularity is modeled by case (a) in Figure 3 and that in Figure 11 ⑧ by case (b). This is determined by the separatrix of the folded node (resp. saddle) for (u, v) on the $MT2$-stratum.

One can keep track of which saddle and node will disappear as we cross the (FSN)-stratum. (These can be identified by looking at the lifted field $\xi_{(u,v)}$. The saddle and node of $\xi_{(u,v)}$ of interest are those linked to the saddle-node singularity of $\xi_{(0,0)}$). The configuration at a folded saddle-node singularity and the way it bifurcates are given in [36] (Figures 10, 10 around the (FSN)-stratum).

It remains to show that the configuration of the integral curves is constant in each stratum. This is done by choosing an appropriate neighbourhood of the discriminant and constructing the required homeomorphism by sliding along integral curves (see for example [6, 35, 36] for details).

The case ϕ_0 has 1 root

We take $w_0 = (x + a_2 y, 0, y)$, with $a_2 > \frac{1}{4}$. Let $w = (\tilde{a}, \tilde{b}, \tilde{c})$ be a 2-parameter deformation of w_0. We can change coordinates and set $j^k\tilde{c} = y$ (for any $k \geq 2$), $\tilde{a}(0, 0, u, v) = 0$ and write

$$
\begin{aligned}
&j^2\tilde{a} = x + a_2 y + (\gamma_{10}u + \gamma_{11}v)x + (\gamma_{20}u + \gamma_{21}v)y + A_2(x, y),\\
&j^2\tilde{b} = b_{20}x^2 + \beta_1 u + \beta_2 v + (\beta_{10}u + \beta_{11}v)x + (\beta_{20}u + \beta_{21}v)y + \Gamma_2(u, v),\\
&j^2\tilde{c} = y,
\end{aligned}
$$

where A_2 and Γ_2 have zero 1-jets. Following the calculations for the previous case we have, at w_0,

$$
\left(\frac{\partial R}{\partial a_1}, \frac{\partial R}{\partial a_2}, \frac{\partial R}{\partial b_1}, \frac{\partial R}{\partial b_2}, \frac{\partial R}{\partial c_2}, \frac{\partial R}{\partial c_1}\right) = (0, 0, a_2, 0, -a_2, 0)\,.
$$

where $R = \text{Resultant}(\phi, \alpha_1)$. Working in any k-jet space, with $k \geq 2$, and restricting to appropriate jets, V_1 is smooth at w_0 if and only if $a_2 \neq 0$, which is the case as $a_2 > \frac{1}{4}$.

The tangent space at w_0 is given by the intersection of the kernel of the 1-forms α_i, $i = 1, 2, 3$, in the previous case and of $\alpha_4 = \mathrm{d}R|_{w_0} = \mathrm{d}b_1 - \mathrm{d}c_1$. Following the same argument in the previous case, we can determine the condition for the jet extension map to be transverse to V_1 at w_0 (a determinant of a matrix being non zero). This is given by

$$-a_2(\beta_1\beta_{11} - \beta_2\beta_{10}) \neq 0.$$

A necessary condition for this to happen is $\beta_1 \neq 0$ or $\beta_2 \neq 0$. We can then set $v = \tilde{b}(0, 0, u, v)$, and the condition for transversality becomes $\beta_{10} \neq 0$ (as $a_2 \neq 0$). In this case we rescale and set $\beta_{10} = 1$.

The $(MT2)$-stratum is a smooth curve and the $(MT1)$-stratum is empty. A calculation similar to the one for the previous case shows that the (FSN)-stratum is smooth provided $b_{20} \neq 0$ and has ordinary tangency, at the origin, with the $(MT2)$-stratum (Figure 12). The analysis around the $(MT2)$ and (FSN) strata is similar to the previous case.

In Figure 12 ② on the $(MT2)$-stratum (the situation is similar on the other branch of the $(MT2)$-stratum), one separatrix of the folded saddle ends at the singularity of the discriminant and the other, say C_1, ends at a smooth point on the discriminant (Figure 12 ②, thick curve). This can be checked by looking at the changes on the lifted field on the surface of the equation, Figure 9. (A stable/unstable manifold at the saddle and node at their birth in a saddle-node bifurcation are close to the stable/unstable manifold of the saddle-node singularity, which is the exceptional fibre in this case.)

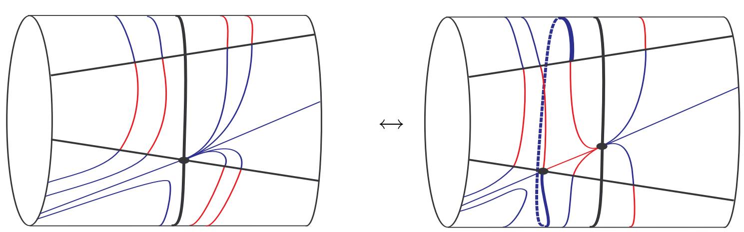

Figure 9: The lifted field associated to a point on the $(MT2)$-stratum.

As we cross the $(MT2)$-stratum (Figure 12 ③), and for (u, v) close to the $(MT2)$-stratum, the separatrix C_2 of the folded node (the thick curve which is on the same side as the folded saddle) and the curve C_1 do not intersect. However, for (u, v) close to the (FSN)-stratum (Figure 12 ⑤), these curves do intersect (this is how folded saddle-node singularities bifurcate, see Lemma 5.1 in the Appendix). Therefore, there are values of (u,v) in the region delimited by the strata ② and ⑥ in the bifurcation set Figure 12 (box), where the two curves end at the same point on the discriminant. So the $(FSNC)$-stratum is not empty. Ideally, this stratum should consist of only one curve, that is, the above separatrices ended up at the same point for just one value of (u,v) on each line u =constant. If this is the case, we can draw the configurations on each stratum as shown in Figure 12. For this reason, we introduce the following concept.

Definition 4.1 *A generic 2-parameter family of a BDE w_0 with an $MT2$-criminant saddle-node singularity and with ϕ_0 having one root is called a "good family" if the $(FSNC)$-stratum consists of two branches, one in each region delimited by the $(MT2)$ and (FSN)-strata.*

We show in Theorem 5.2 in the Appendix that the family $(x+y, x^2+ux+v, y)$ is a good family. We have thus the following result.

Theorem 4.2 *Any generic 2-parameter family of BDEs w with w_0 as in Theorem 3.1 is generically locally fibre topologically equivalent to one of the following normal forms.*
If ϕ_0 has three roots:
(i) $(y, 2x+3y+x^2, y+ux+v)$, *Figure 10,*
(ii) $(y, -\frac{3}{4}x+\frac{1}{4}y+x^2, y+ux+v)$, *Figure 11.*
If ϕ_0 has one root and when the family is a "good family":
(iii) $(x+y, x^2+ux+v, y)$, *Figure 12.*

4.2 Bifurcations of the $MT2$ saddle-node singularity

We consider generic 2-parameter families of BDEs $w_{(u,v)} = (\tilde{a}, \tilde{b}, \tilde{c})$ with w_0 as in Theorem 3.2. We take $j^1w_0 = (y, b_1x+b_2y, \epsilon y)$ with $b_1 = \frac{1}{2}(b_2^2-\epsilon)$, $\epsilon = \pm 1$ (the case $b_1 = -\frac{\epsilon}{2}$ is equivalent to this one; see §3.2). We make changes of variables and write $j^2(a,b,c)$ in the form (2) in §4.1, with (b_1,b_2) as above.

The variety V_2 of BDEs with ξ having a saddle-node singularity on the exceptional fibre and away from the criminant (i.e. the cubic ϕ has a repeated root) is given by

$$V_2 = \{a_0 = b_0 = c_0 = 0,\ R = \mathrm{Resultant}(\phi, \phi') = 0\},$$

with $R = 48a_2b_1^2c_2 - 72a_2c_1b_2b_1 - 36a_2c_1b_2c_2 - 18a_2c_1a_1c_2 - 36a_2c_1a_1b_1 - 16b_1b_2^2c_2 - 16b_1^2b_2a_1 - 4c_2^2b_2a_1 - 4c_2a_1^2b_1 + 48b_2^2a_1c_1 + 24b_2a_1^2c_1 - 16b_1^2b_2^2 - 4c_2^2b_2^2 - c_2^2a_1^2 - 4b_1^2a_1^2 + 32b_2^3c_1 + 4a_1^3c_1 + 24a_2b_1c_2^2 + 27a_2^2c_1^2 + 32a_2b_1^3 + 4a_2c_2^3 - 16b_1b_2a_1c_2$. At w_0, we have

$$\left(\frac{\partial R}{\partial a_1}, \frac{\partial R}{\partial a_2}, \frac{\partial R}{\partial b_1}, \frac{\partial R}{\partial b_2}, \frac{\partial R}{\partial c_2}, \frac{\partial R}{\partial c_1}\right) = 4b_2^3\left(-b_2^2, b_2^3, 2b_2, -2b_2^2, -1, b_2\right).$$

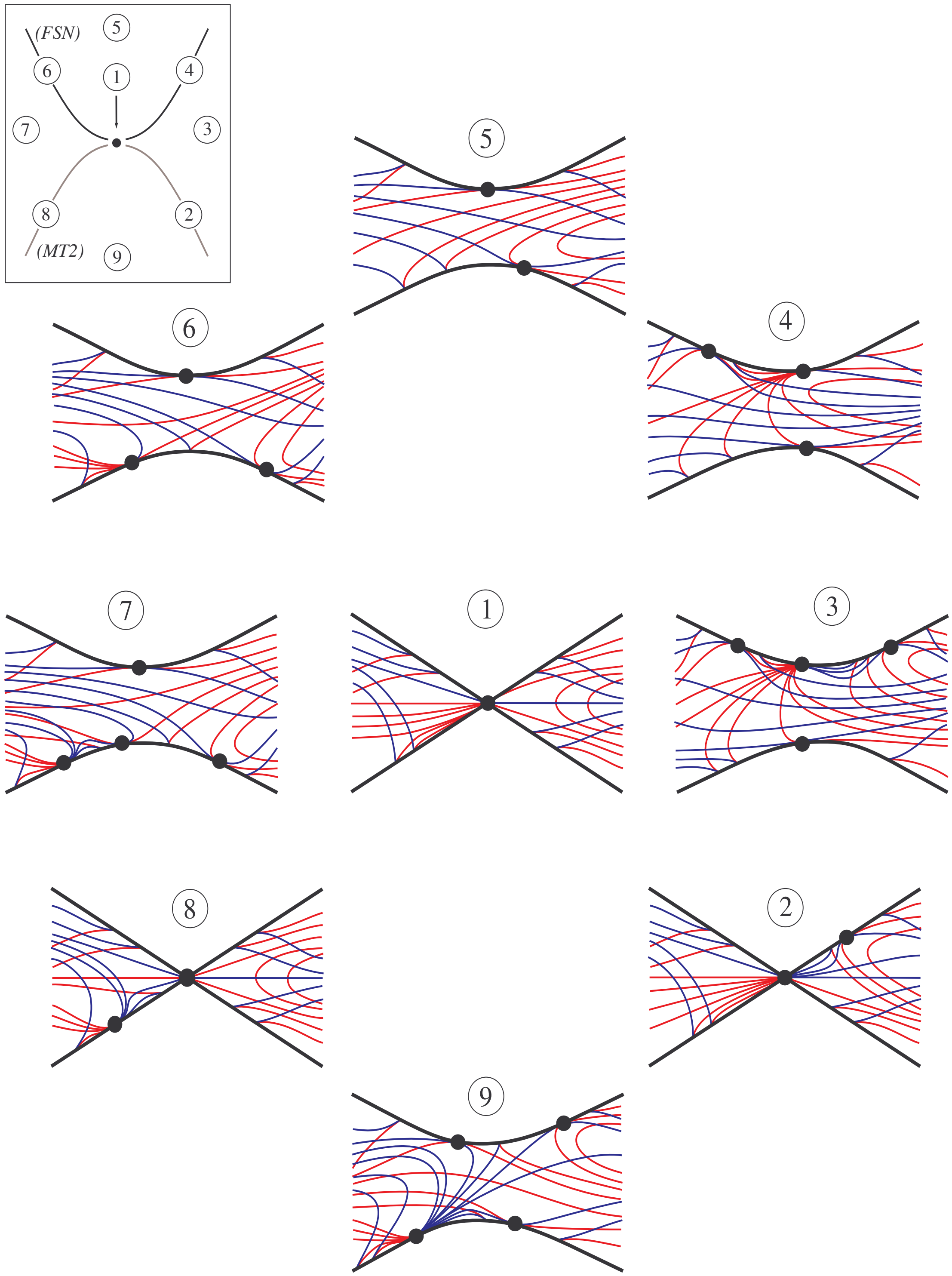

Figure 10: Stratification of the parameters space (box) and generic bifurcations of the $MT2$ criminant saddle-node singularities: case (i) in Theorem 4.2.

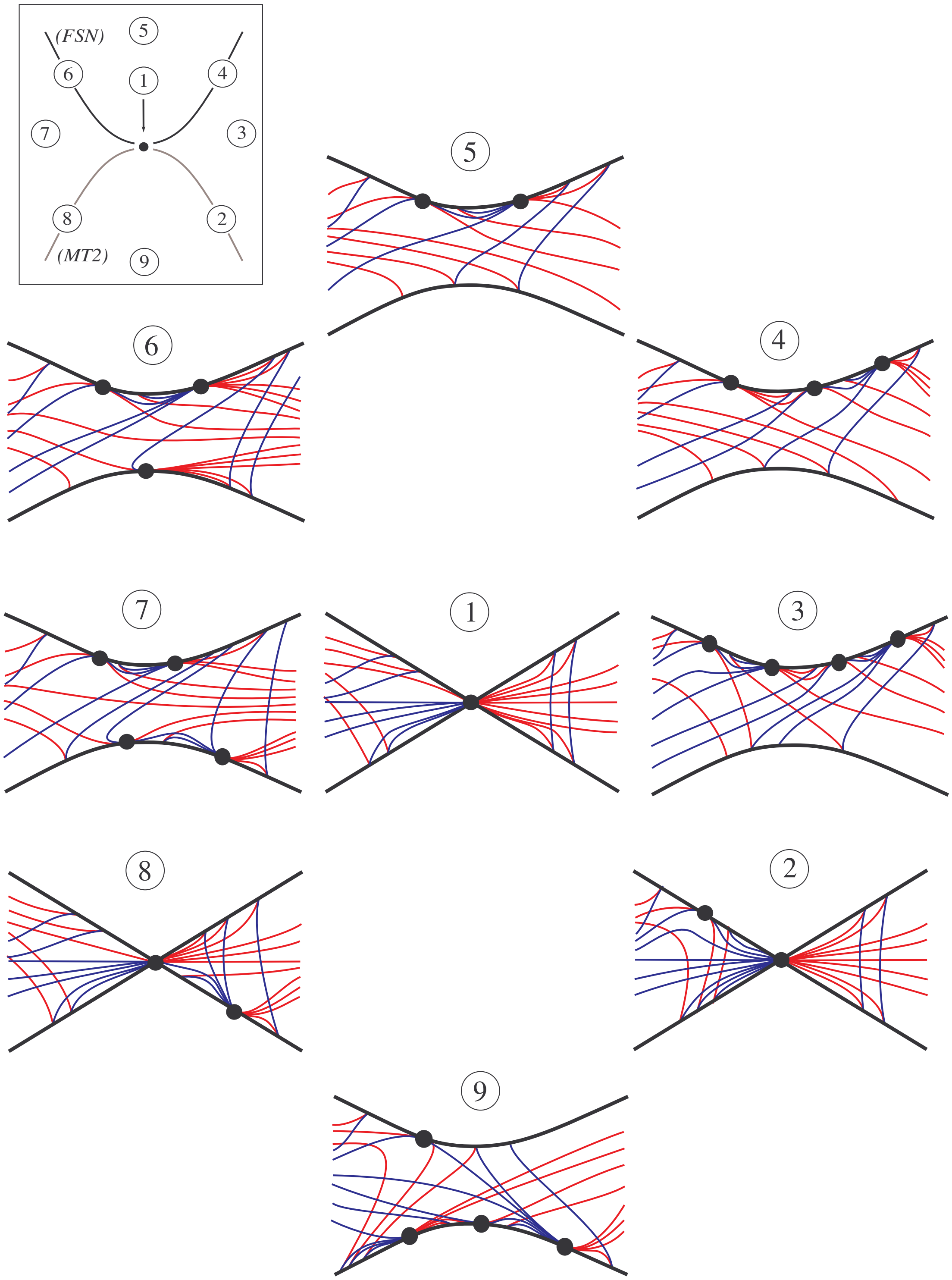

Figure 11: Stratification of the parameters space (box) and generic bifurcations of the $MT2$ criminant saddle-node singularities: case (ii) in Theorem 4.2.

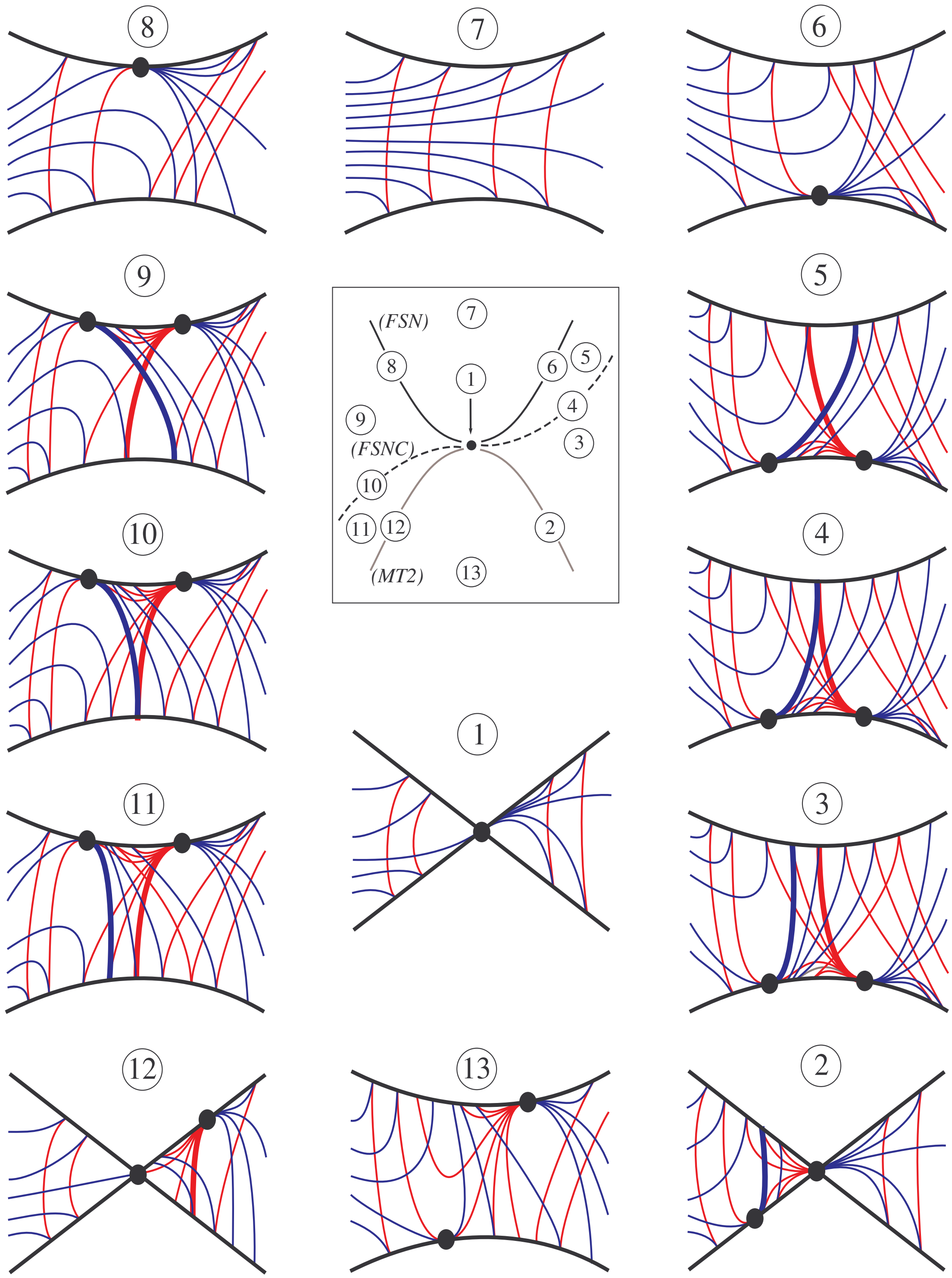

Figure 12: Stratification of the parameters space (box) and generic bifurcations of the $MT2$ criminant saddle-node singularities: case (iii) in Theorem 4.2.

So the variety V_2 is smooth at w_0 provided $b_2 \neq 0$ (here we work in any k-jet, space with $k \geq 2$). The tangent space to V_2 at w_0 is given by the intersection of the kernels of the 1-forms α_i, $i = 1, 2, 3$ in § 4.1 and

$$\alpha_4 = \mathrm{d}R|_{w_0} = b_2^2 \mathrm{d}a_1 - b_2^3 \mathrm{d}a_2 - 2b_2 \mathrm{d}b_1 + 2b_2^2 \mathrm{d}b_2 + \mathrm{d}c_1 - b_2 \mathrm{d}c_2.$$

Calculations similar to those in §4.1 show that the map Φ is transverse to V_2 at w_0 if and only if

$$b_2(b_2^2 - \epsilon)\left((\gamma_{11} - (2\beta_{11} + 2\beta_{21} + b_2\gamma_{21})b_2)\gamma_1 + ((2\beta_{10} + 2\beta_{20} + \gamma_{20})b_2 - \gamma_{10})\gamma_2\right) \neq 0.$$

A necessary condition for this to hold is $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$. When this is the case, we can change variables and set $v = \tilde{c}(0, 0, u, v)$. The conditions for transversality become then $b_2(b_2^2 - \epsilon) \neq 0$ and

$$(2\beta_{10} + 2\beta_{20} + \gamma_{20})b_2 - \gamma_{10} \neq 0.$$

We determine the bifurcation set of a generic family with j^2w as above. The symmetric matrix associated to w_0 is $\mathcal{G}$-equivalent to (y, x, y) which has $\mathcal{G}$-codimension 1. Similar argument to that in §4.1 shows that the $(MT2)$-stratum is a smooth curve and $(MT1)$-stratum is empty. The $(MT2)$-stratum is given by $v = h(u)$ for some germ of a smooth function h with a zero 1-jet.

The (FSN)-stratum is given by $F = F_p = F_x + pF_y = \det(A) = 0$, where p is close to $-b_2$. Following the same steps in §4.1, one can show that the (FSN)-stratum is a a smooth curve provided

$$b_2(b_2^2 - \epsilon)(2\beta_{20}b_2^2 - (\gamma_{20} + 2\beta_{10})b_2 + \gamma_{10}) \neq 0.$$

This curve is transverse to the $(MT2)$-stratum at the origin (bifurcations sets in boxes in Figures 13, 14, 15) when $-2\beta_{21}b_2^4 + (\gamma_{21} + 2\beta_{11})b_2^3 + (2\beta_{21}\epsilon - \gamma_{11} - 4b_{22})b_2^2 + (-2\beta_{11}\epsilon + 2b_{21} - \epsilon\gamma_{21} + 2c_{22})b_2 + \epsilon\gamma_{11} - c_{21} \neq 0$.

The above conditions and that for transversality of the jet-extension map to the variety V_2 are satisfied if we take $j^2w = (y, b_1x + b_2y + y^2, \epsilon y + ux + v)$, with $b_1 = \frac{1}{2}(b_2^2 - \epsilon)$ and $b_2(b_2^2 - \epsilon) \neq 0$. (A second degree term is added to the normal form at $u = v = 0$ in order to have a generic family in a simple form.)

The multiplicity of w_0 is 3, so we expect a maximum of 3 folded singularities of $w_{(u,v)}$ in the open strata of $\mathcal{S}$.

We analyse the members of the family with (u, v) on the $(MT2)$-stratum. As (u, v) varies on the $(MT2)$-stratum, the induced deformation of the saddle-node singularity of ξ_0 is versal. So for (u, v) on this stratum, the lifted field $\xi_{(u,v)}$ has simple singularities (1 or 3) on the exceptional fibre. These are as follows (and depend only on ϵ and b_2):

(i) $\epsilon = -1$: 2S+1N on one component of the $(MT2)$-stratum and 1S on the other (Figure 13, 2 and 6 respectively);

(ii) $\epsilon = +1$, $b_2^2 - 1 > 0$: 2S+1N on one component of the $(MT2)$-stratum and 1S on the other (Figure 14, ② and ⑥ respectively). The 2S+1N singularity bifurcates as in Figure 4 bottom centre;

(iii) $\epsilon = +1$, $b_2^2 - 1 < 0$: 1S+2N on one component of the $(MT2)$-stratum and 1N on the other (Figure 15, ② and ⑥ respectively).

The family of BDEs induces a versal deformation of the Morse Type 2 singularities on the $(MT2)$-stratum. We can use the results in [10] to determine the way these singularities bifurcate (Figures 13, 14, 15 around the $MT2$-stratum). It is not difficult now to determine the configuration of the integral curves in the remaining strata. In Figure 14 ②, the bifurcation of the 2S+1N $MT2$-singularity is that modeled by case (b) in Figure 3. One can show by analysing the sectors in Figure 14 ⑦ and ⑨ that case (a) in Figure 3 cannot occur here.

One can show, using the arguments in §4.1, that the topological type of configuration of the integral curves is constant in each stratum. We have thus the following result.

Theorem 4.3 *Any generic 2-parameter family of a BDEs w with w_0 as in Theorem 3.2 is generically fibre topologically equivalent to one of the following normal forms.*

(i) $(y, x + y + y^2, -y + ux + v)$, *Figure 13,*

(ii) $(y, 4x + 3y + y^2, y + ux + v)$, *Figure 14,*

(iii) $(y, -\frac{3}{8}x + \frac{1}{2}y + y^2, y + ux + v)$, *Figure 15.*

4.3 Bifurcations of the cusp singularity

Let $w_{(u,v)} = (\tilde{a}, \tilde{b}, \tilde{c})$ be a 2-parameter family of BDEs with $j^2 w_0 = (y, \epsilon x + b_2 y, y^2)$, $\epsilon = \pm 1$ (see §3.3). The variety V_3 of BDEs with discriminants having a cusp singularity at the origin is given by

$$V_3 = \{a_0 = b_0 = c_0 = 0, (2b_1 b_2 - a_1 c_2 - a_2 c_1)^2 - 4(b_2^2 - a_2 c_2)(b_1^2 - a_1 c_1) = 0\}.$$

Working in any k-jet space, with $k \geq 2$, V_3 is smooth at w_0. The tangent space to V_3 at w_0 is given by the intersection of the kernels of the 1-forms α_i, $i = 1, 2, 3$ (as in § 4.1) and of $\alpha_4 = -\epsilon b_2 \mathrm{d}c_1 + \mathrm{d}c_2$. When Φ is transverse to V_4, we can make changes of variables and set

$$j^1\tilde{a} = y, \quad j^1\tilde{b} = \epsilon x + b_2 y, \quad j^2\tilde{c} = y^2 + uy + v.$$

The symmetric matrix of w_0 is $\mathcal{G}$-equivalent to (y, x, y^2). It has $\mathcal{G}$-codimension 2 and a $\mathcal{G}$-versal unfolding is given by $(y, x, y^2 + sy + t)$. Matrices with coefficients vanishing at a given point occur when $t = 0$. The determinant has an A_1^+ singularity if $s < 0$ and an A_1^- singularity if $s > 0$. Matrices with singular determinant and with coefficients

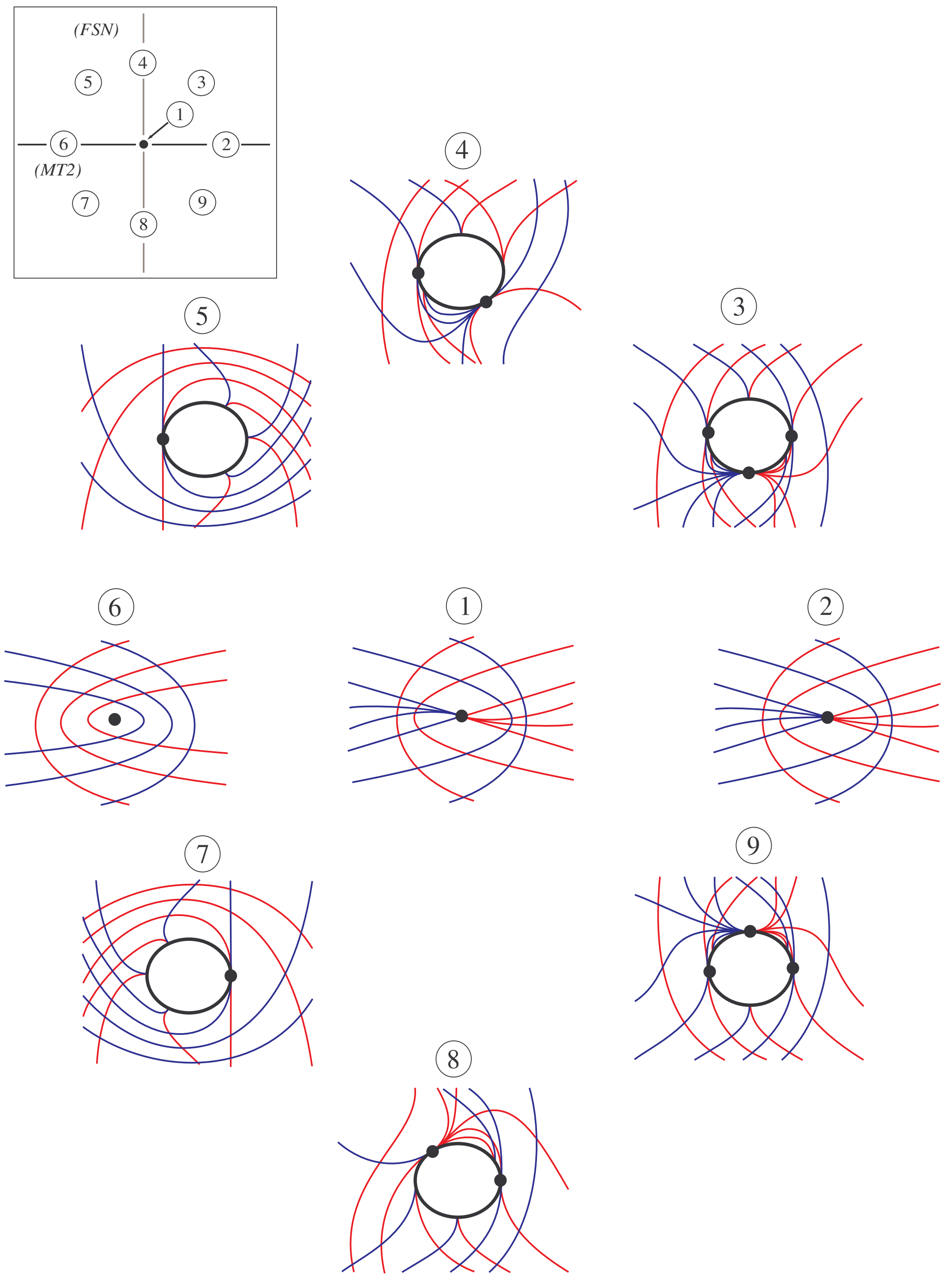

Figure 13: Stratification of the parameters space (box) and generic bifurcations of the $MT2$ saddle-node singularities: case (i) in Theorem 4.3.

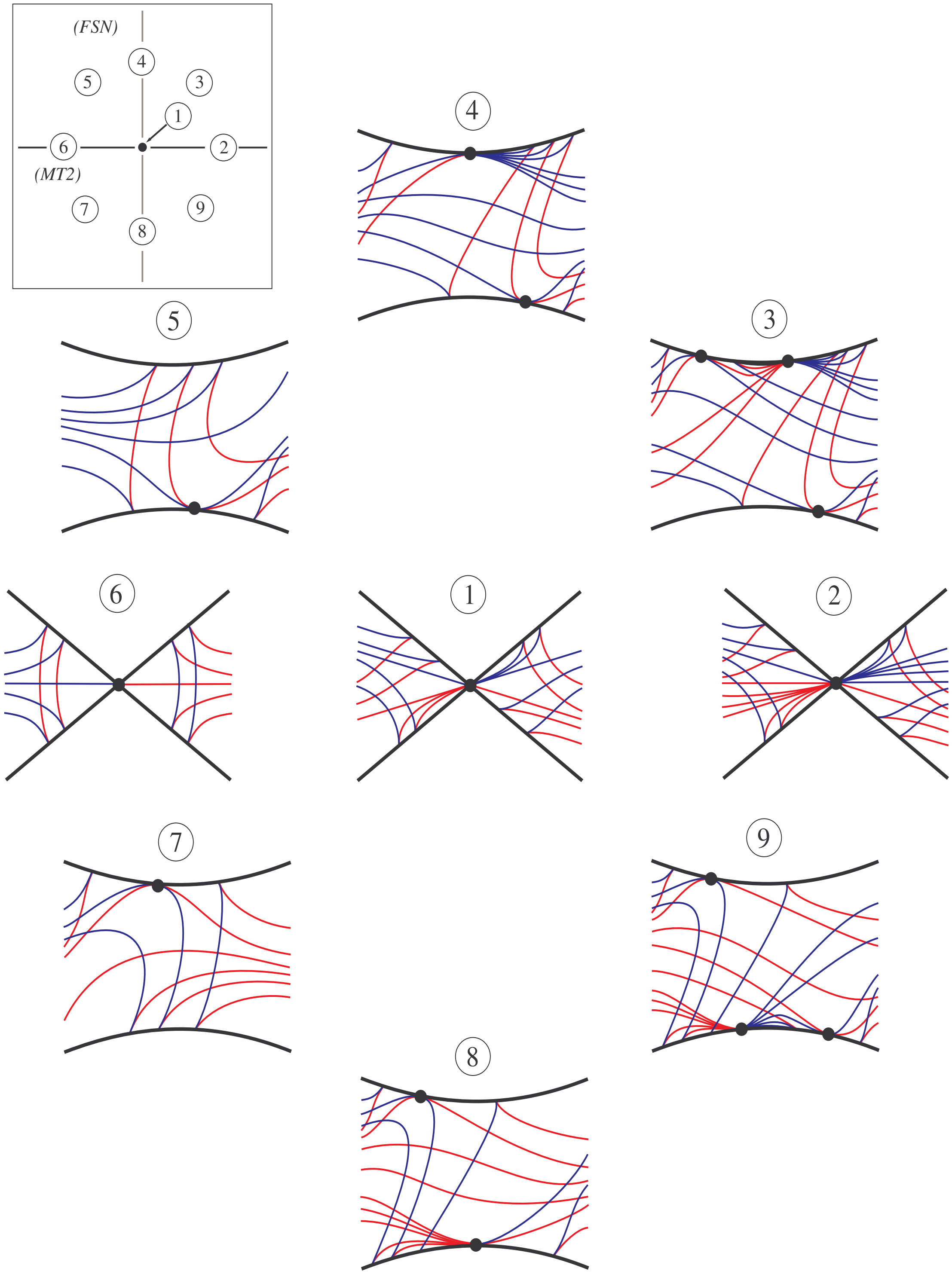

Figure 14: Stratification of the parameters space (box) and generic bifurcations of the $MT2$ saddle-node singularities: case (ii) in Theorem 4.3.

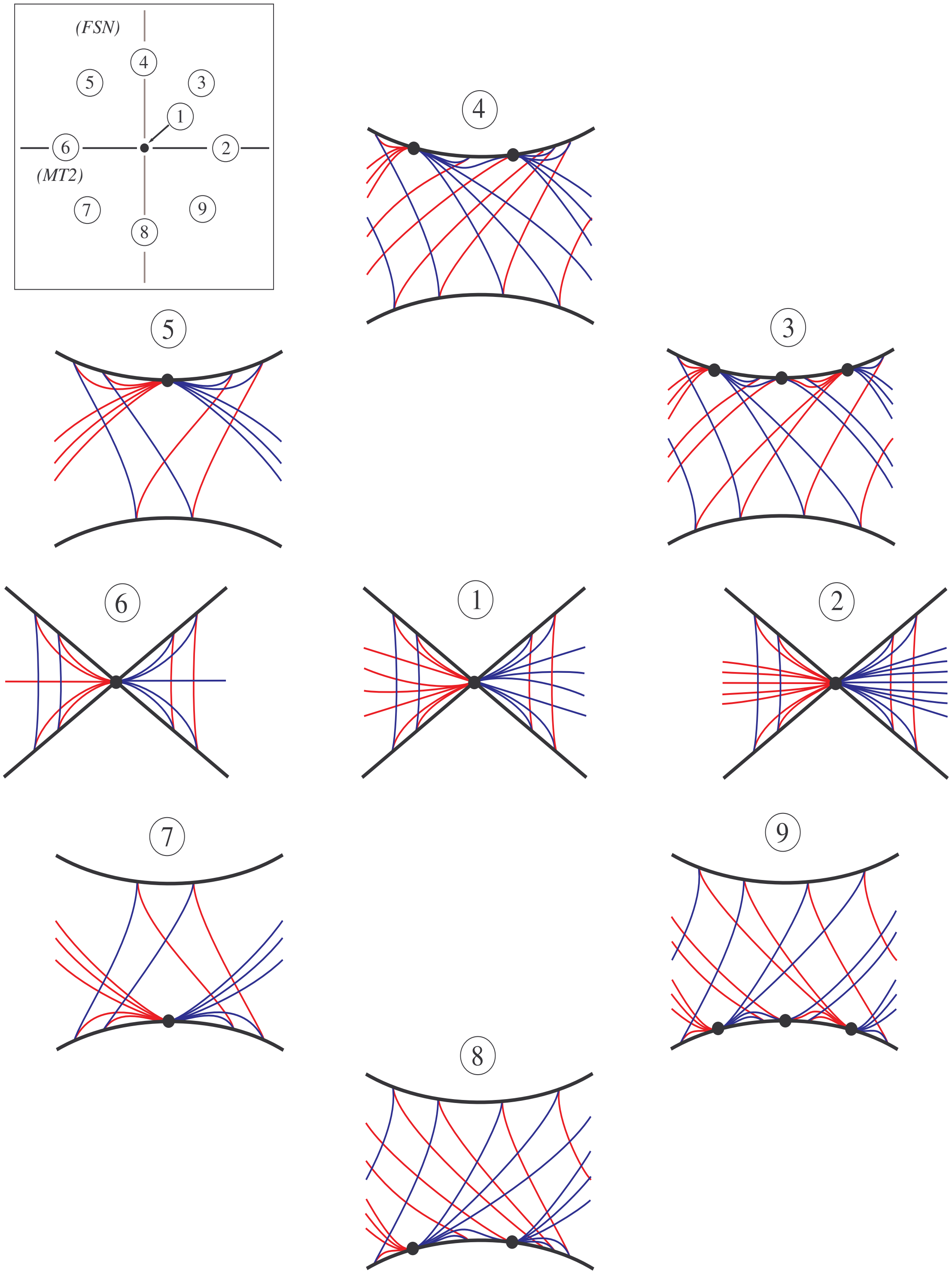

Figure 15: Stratification of the parameters space (box) and generic bifurcations of the $MT2$ saddle-node singularities: case (iii) in Theorem 4.3.

not all vanishing at the singularity of the determinant occur when $t = \frac{1}{4}s^2$. The discriminant has an A_1^+ singularity if $s > 0$ and an A_1^- singularity if $s < 0$.

It is clear that a generic family of w_0 induces a $\mathcal{G}$-versal unfolding of the matrix of w_0. Therefore the $(MT1)$ and $(MT2)$ strata are smooth curves meeting tangentially at the origin (bifurcation sets in boxes in Figures 16, 17, 18). A calculation shows that the (FSN)-stratum is empty (the determinant of the matrix A in §4 does not vanish at the origin).

The multiplicity of w_0 is 4, so we expect a maximum of 4 folded singularities of $w_{(u,v)}$ in the open strata of $\mathcal{S}$.

We analyse now the members of the family with (u,v) on the $(MT2)$-stratum. The cubic $\phi_{u,v}$ depends continuously on (u,v), so for (u,v) on the $(MT2)$-stratum, $\xi_{(u,v)}$ has three singularities when $\epsilon = -1$ or $\epsilon = +1$ and $b_2^2 - 2 > 0$ and one singularity when $\epsilon = +1$ and $b_2^2 - 2 < 0$ (recall from §3.3 that $\phi_0 = \phi = p(p^2 + 2b_2 p + 2\epsilon)$). When there are three singularities, two of them p_2 and p_3 have the same type as those of ξ_0 (both saddles if $\epsilon = -1$ and one is a saddle and the other a node if $\epsilon = +1$, see §3.3). At the third root of $\phi_{(u,v)}$ (or its unique root when ϕ has a single root) $j^1\xi_{(u,v)}$ has eigenvalues $\alpha_{1_{(u,v)}}(p_1)$ and $-\phi'_{(u,v)}(p_1)$, with p_1 close to zero. As $-\phi'_0(0)(\alpha_1)_0(0) = -1$, the singularity at p_1 is a saddle. From the discussion above about families of symmetric matrices, the discriminant of the Morse Type 2 singularity is of type A_1^+ on one component of the $(MT2)$-stratum and of type A_1^- on the other. Therefore we have the following Morse Type 2 singularities on the $(MT2)$-stratum (see Figure 1):

(i) $\epsilon = -1$: (A_1^-)-3S on one component and an (A_1^+)-3S on the other (Figure 16, 2 and 8 respectively);

(ii) $\epsilon = +1$ and $b_2^2 - 2 > 0$: (A_1^-)-2S+1N on one component and an (A_1^+) 2S+1N on the other (Figure 17, 2 and 8 respectively);

(iii) $\epsilon = +1$ and $b_2^2 - 2 < 0$: (A_1^-)-1S on one component and an (A_1^+)-1S on the other (Figure 18, 2 and 8 respectively).

One can also show that for (u,v) on the $(MT2)$-stratum, there is always an elementary singularity at a smooth point of the discriminant of type folded saddle and its location is as shown in Figures 16, 17, 18 2 and 8. The bifurcations of the 2S+1N $MT2$-singularity in Figure 17 (3 and 9) are those modeled by case (a) in Figure 3. This can be checked by analysing the behaviour of the separatrices in the other strata.

On the $(MT1)$-stratum, the Morse singularity of the discriminant changes type as we cross the origin (see discussion above about families of symmetric matrices). On one component we have an isolated point (A_1^+) an the other a crossing (A_1^-), Figures 16, 17, 18 4 and 6.

Morse Type 1 singularities are either of type saddle or focus ([6], that is two saddles/foci appear on one side of the transition and none on the other). For BDEs with $j^2F = p^2 + 2(l_1x + l_2y)p + m_1x^2 + m_2xy + m_3x^2$, the origin is a saddle Morse Type 1 singularity if $l_1^2 - m_2 > 0$ and of focus type if $l_1^2 - m_2 < 0$ ([12]). We can make changes of coordinates and write the 2-jet of the BDE as above when (u, v) is on the $(MT1)$-stratum. We find that $l_1^2 - m_2 > 0$, so the Morse Type 1 singularity is of type saddle.

With these information and the way Morse Type 1 and 2 singularities bifurcate ([6, 10]), we are able to determine the number of singularities of $w_{(u,v)}$ in each stratum of $\mathcal{S}$ as well as identify their type and position on the discriminant. We can then draw the integral curves of the BDEs in each stratum and show that any two configurations in the same stratum are homeomorphic. We thus have the following result.

Theorem 4.4 *Any generic 2-parameter family of BDEs w with w_0 as in Theorem 3.4 is locally fibre topologically equivalent to one of the following normal forms.*

(i) $(y, -x + y, y^2 + uy + v)$, *Figure 16,*
(ii) $(y, x + 2y, y^2 + uy + v)$, *Figure 17,*
(iii) $(y, x + y, y^2 + uy + v)$, *Figure 18.*

5 Appendix

We give in this appendix details of a result used in the proof of Theorem 4.2(iii) to show the existence of a $(FSNC)$-stratum. We also prove that the family $(x+y, x^2+ux+v, y)$ is a good family.

Lemma 5.1 *A generic 1-parameter deformation w_t of a folded saddle-node singularity w_0 results in a birth of a folded saddle and a folded node singularities on one side of the deformation, say $t > 0$, and no singularities on the other side of the deformation ($t < 0$ in this case). For t small enough, a separatrix at the folded saddle of one of the direction fields determined by w_t always intersects a separatrix at the folded node of the other direction field determined by w_t.*

Proof As the coefficients of the BDE w_0 are not all zero at a folded saddle-node singularity, those of w_t also do not vanish simultaneously at any point near the singularity of w_t for t small enough. So we can make smooth changes of coordinates and multiply by a non-zero function and set the family w_t in the form $dy^2 - f(x, y, t)dx^2 = 0$ (see [11]). We can also take $f(x, y, 0) = y + x^3 + \lambda x^4$, where λ is a parameter modulus (it can be set to zero for the topological model), see [18].

The surface $M = F_0^{-1}(0)$ of the equation w_0, where $F_0(x, y, p) = p^2 - (y + x^3 + \lambda x^4)$, is the graph of the function $y = p^2 - x^3 - \lambda x^4$. So for simplicity, we work with the projection of the lifted field ξ_0 to the (x, p)-plane (the tangent space of M_0 at the origin). This field is given by $\tilde{\xi}_0 = 2p\partial/\partial x + (3x^2 + 4\lambda x^3 + p)\partial/\partial p$. The centre manifold is a smooth curve tangent to the criminant, which is the p-axis. The other stable/unstable manifold is a smooth curve C transverse to the criminant. The curve C is tangent to the eigenvector associated to the non-zero eigenvalue of $\tilde{\xi}_0$ at the origin

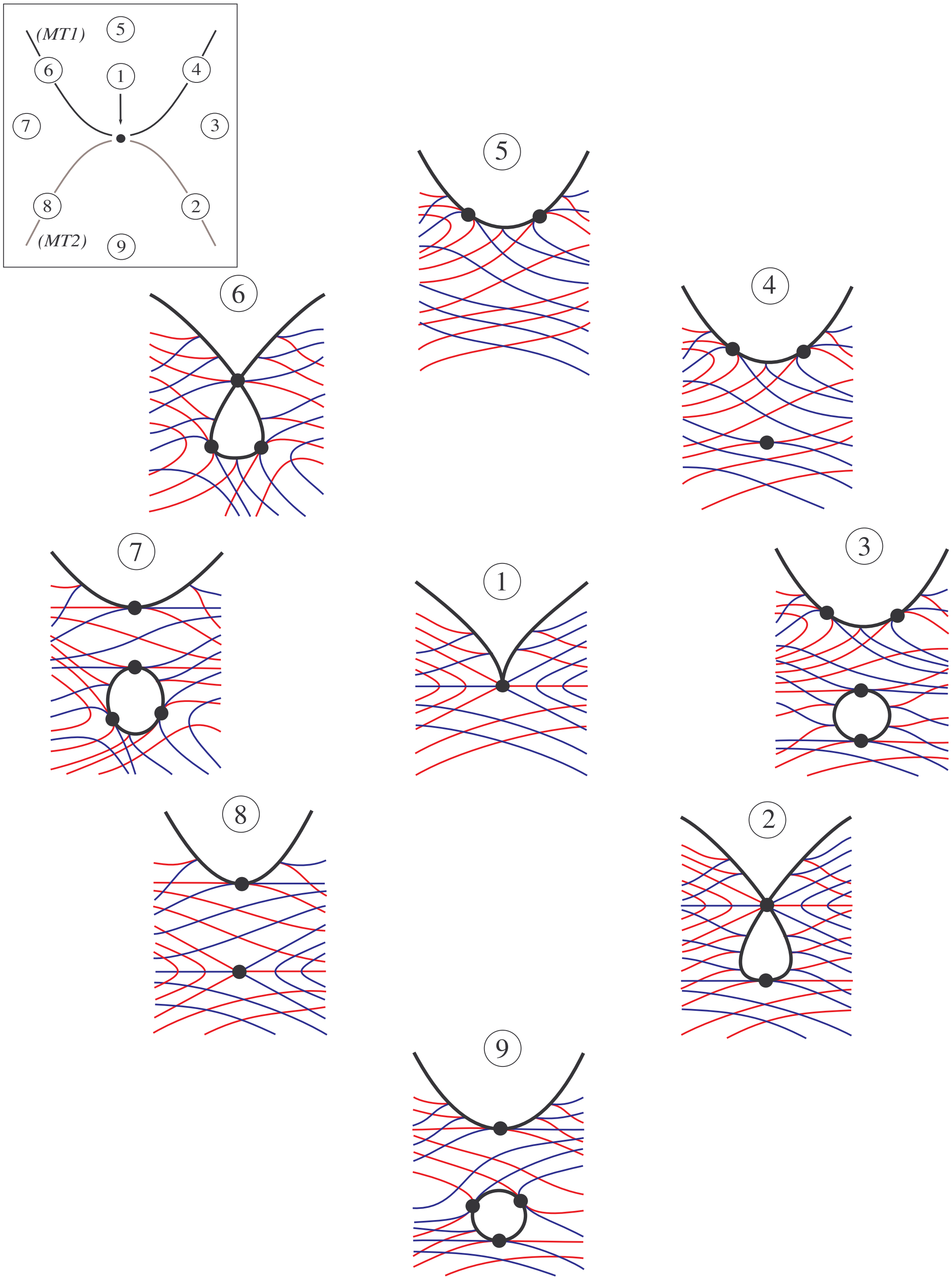

Figure 16: Stratification of the parameters space (box) and generic bifurcations of the cusp singularity: case (i) in Theorem 4.4.

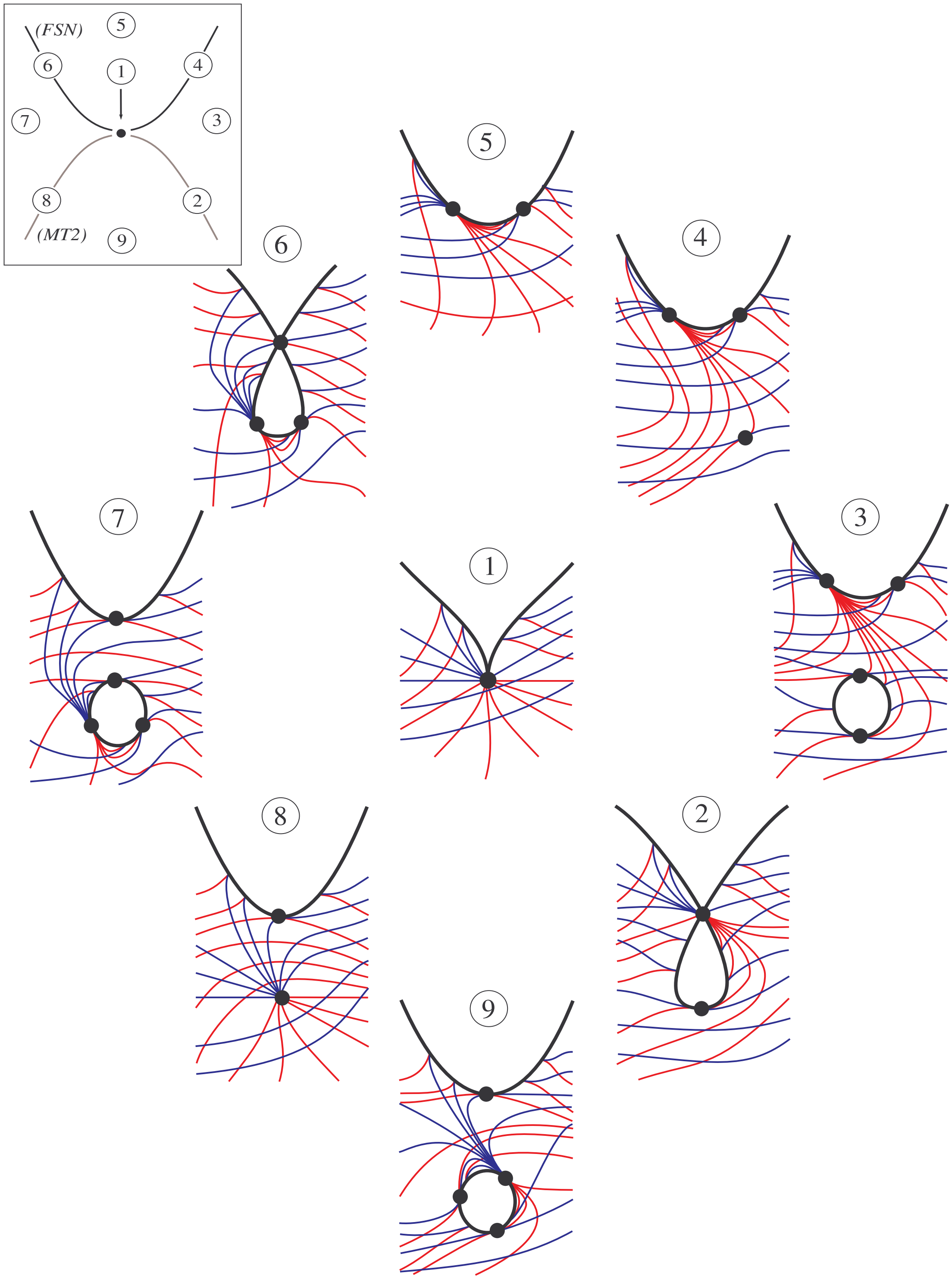

Figure 17: Stratification of the parameters space (box) and generic bifurcations of the cusp singularity: case (ii) in Theorem 4.4.

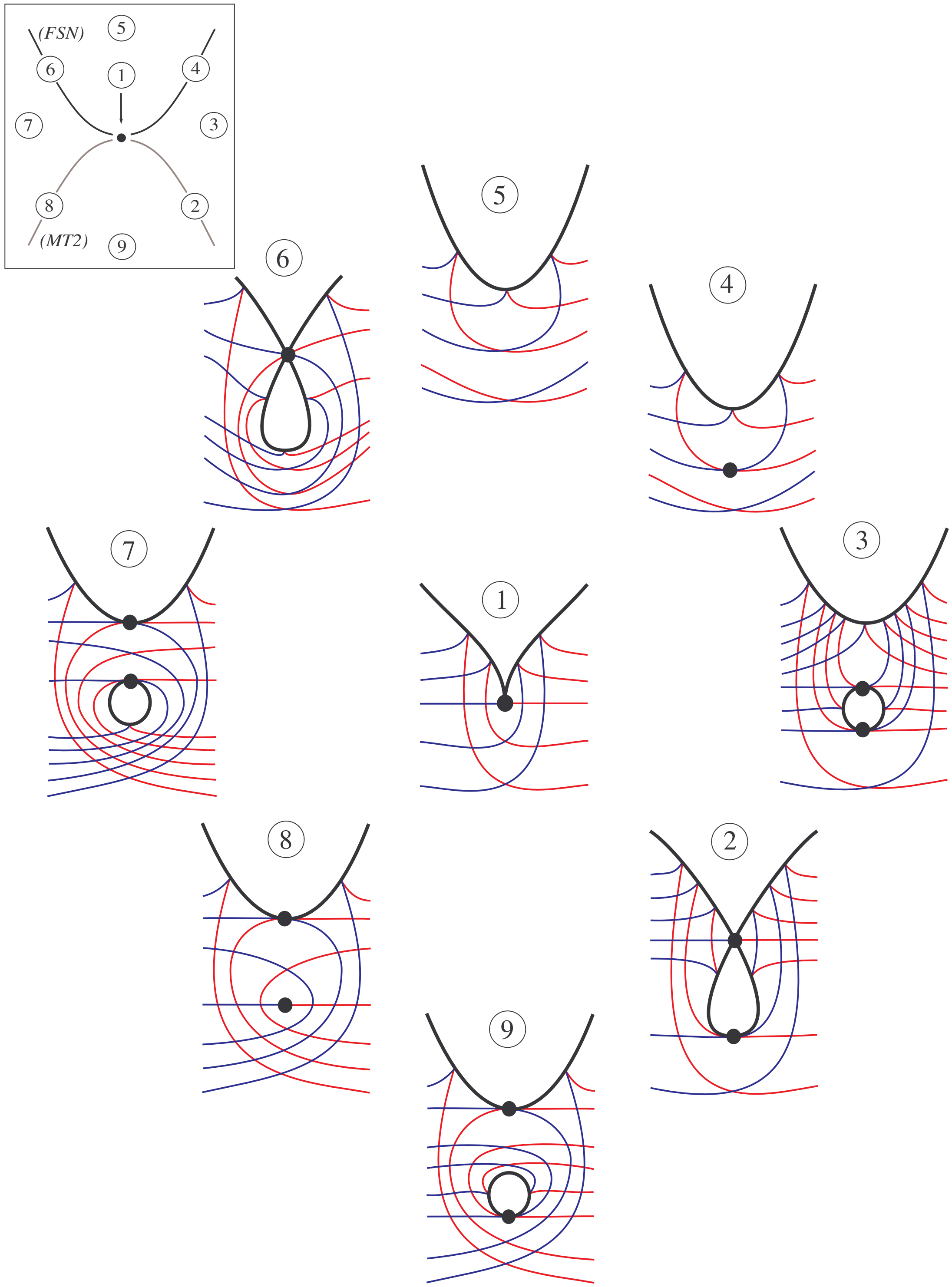

Figure 18: Stratification of the parameters space (box) and generic bifurcations of the cusp singularity: case (iii) in Theorem 4.4.

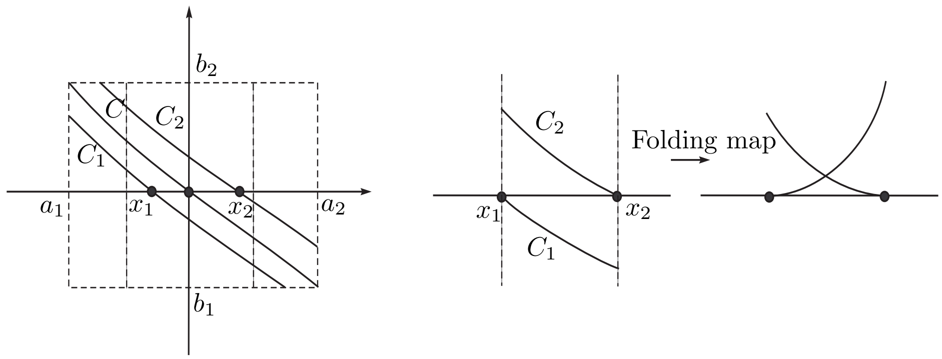

Figure 19: Intersection of separatrices at a deformation of a folded saddle-node singularity.

(see [36]). We choose a small rectangular neighbourhood of the origin $(x, p) \in V = [a_1, a_2] \times [b_1, b_2]$ where C is given as a graph of a function $p = h(x)$ (Figure 19 left).

The singularities x_i, $i = 1, 2$, of $\tilde{\xi}_t$ depend continuously on t. Hence, for any $\epsilon > 0$, there exists $\eta > 0$, such that $|t| < \eta$ implies $|x_i| < \epsilon$, $i = 1, 2$. The stable/unstable manifolds C_i, $i = 1, 2$, of $\tilde{\xi}_t$ associated to the eigenvalues not close to zero also depend continuously on t (this follows, for instance, by considering the model generic family $(x^2+t)\partial/\partial x + p\partial/\partial p$ of a saddle-node singularity). So there exists η' such that $|t| < \eta'$ implies $d(C_i, C) < \epsilon$, $i = 1, 2$, where d denotes the distance between the two curves in the neighbourhood V.

Take $\epsilon < \min(a_1, a_2)/4$ so that the lines $x = a_i/2$, $i = 1, 2$ intersect both curves C_1 and C_2 (Figure 19 left). This implies that the line $x = x_1$ intersects C_2 and the line $x = x_2$ intersects C_1 (Figure 19 centre). Applying the folding map $(x, p) \to (x, p^2)$ to C_1 and C_2 yields a separatrices of w_t at the singular points x_1 and x_2 respectively. Let $(x, h_i(x))$, $i = 1, 2$, be parametrisations of these separatrices. Observe that, for $x \in]x_1, x_2[$, we have $h_i(x) > 0$, $i = 1, 2$, and $h_1(x_2) > 0$ and $h_2(x_1) > 0$. The result now follows by applying the intermediate theorem to $g = h_1 - h_2$. Indeed, we have $g(x_1) = h_1(x_1) - h_2(x_1) = 0 - h_2(x_1) < 0$ and $g(x_2) = h_1(x_2) - h_2(x_2) = h_2(x_2) - 0 > 0$, so there exists x_0, $x_1 < x_0 < x_2$ such that $g(x_0) = 0$, that is $h_1(x_0) = h_2(x_0)$ (Figure 19 right). $\square$

Theorem 5.2 *The family* $w_{(u,v)} = (a, b, c) = (x + y, x^2 + ux + v, y)$ *is a "good family".*

Proof We fix u, say with $u > 0$ ($u < 0$ follows in a similar way), and take v in the interval $]\epsilon_1, u^2/4 - \epsilon_2[$, with ϵ_i, $i = 1, 2$, small enough. Then (u, v) is between the $(MT2)$ and (FSN)-strata.

The zeros of $w_{(u,v)}$ occur when $y = 0$ and $x^2 + ux + v = 0$. Let $x_1 = (-u - \sqrt{u^2 - 4v})/2$ and $x_2 = (-u + \sqrt{u^2 - 4v})/2$ and denote by $z_1 = (x_1, 0)$ and $z_2 = (x_2, 0)$ the zeros of $w_{(u,v)}$. Observe that the zeros move away from each other with non-zero velocity as v-decreases. At the zeros, the BDE $w_{(u,v)}$ determines a unique direction parallel to $(1, 0)$. (In fact $y = 0$ is an integral curve of the BDE.) The separatrices of $w_{(u,v)}$ of interest (around the value of v for which they intersect) lie in the region (R): $x_1 < x < x_2$, $y > 0$ and $\delta > 0$ (Figure 20 left, where Δ_1 and Δ_2 are the two branches of the discriminant).

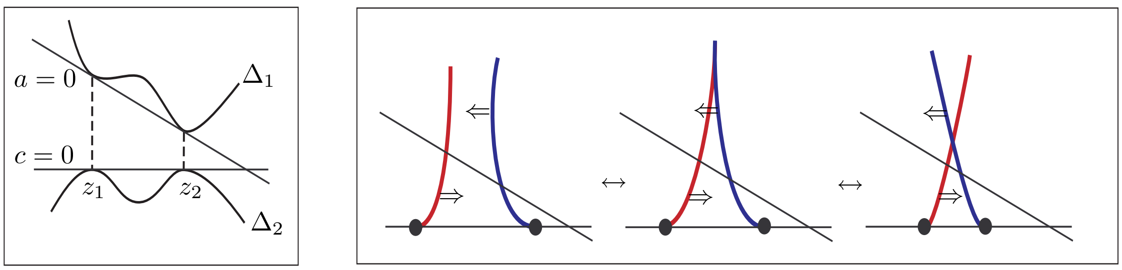

Figure 20: Direction of movement of the separatrices.

Consider the vector fields $Y_1 = (-b + \sqrt{\delta})\partial/\partial x + c\partial/\partial y$ determined by w_t. In the region (R) we have $b < 0$, so the slope of the direction determined by Y_1 is positive. Therefore the separatrix (of interest) of Y_1 at z_1 lies in (R). Also, Y_1 is neither singular nor vertical in (R), Figure 20, right (the red curves). We shall show that these separatrices of Y_1 do not intersect as v varies in the above interval. (Of course the region (R) depends on v, but this is not a problem as we can consider these sepratrices as curves in the plane. The important fact is that they do not intersect.)

In the region (R), the separatrices of Y_1 (with v as the parameter) are projections of the separatrices of the lifted field ξ. These are smooth curves and depend smoothly on v. So we can consider the family of separatrices of Y_1 as a family of plane curves given by $G(x, y, v) = 0$, for some smooth function G. To show that they do not intersect in the region (R), it is enough to show that the envelope of the family $G(x, y, v) = 0$ consists only of the line $y = 0$ (see for example [7] for definitions and results on envelopes). The envelope is given by the set of points (x, y) for which there exits v such that $G(x, y, v) = G_v(x, y, v) = 0$.

The surfaces of the family of equations are given by $y = -(xp^2 + 2(x^2 + ux + v)p)/(1 + p^2)$ and are smooth when v varies in the above interval. They have the x-axis in common and are disjoint elsewhere (as $\partial y/\partial v \neq 0$). Also one can show that the family of the separatrices of Y_1 can in fact be parametrised in the form $(x, y(x, v), v)$. We have $G(x, y(x, v), v) = 0$, so $G_y y_v + G_v = 0$. Then $G_v = 0$ if and only if $y_v = 0$. Now $(-b + \sqrt{\delta})dy - cdx = 0$, so $(-b + \sqrt{\delta})(x, y(x, v))y_x - c(x, y(x, v)) = 0$. Differentiating with respect to v we get, $((-b+\sqrt{\delta})_v y_x - c_y)y_v - (-b+\sqrt{\delta})y_{xv} = 0$. But

$$y_{xv} = \frac{\partial}{\partial v}\frac{\partial y}{\partial x} = \frac{\partial}{\partial v}\left(\frac{c}{-b+\sqrt{\delta}}\right) = \frac{c}{\sqrt{\delta}(-b+\sqrt{\delta})}.$$

which does not vanish in (R). This shows that the envelope of the separatrices is empty in (R), so these form a family of non-intersecting curves. Furthermore, as x_1 increases on the line $y=0$ as v increases in the interval $]\epsilon_1, u^2/4-\epsilon_2[$, the separatrix of Y_1 moves to the right as v increases in the above interval.

We consider now the other vector field $Y_2 = (-b-\sqrt{\delta})\partial/\partial x + c\partial/\partial y$ determined by w_t. The problem here is that Y_2 has a line singularity on $a=0$, so we consider instead a parallel field given by $Z_1 = a\partial/\partial x + (-b+\sqrt{\delta})\partial/\partial y$. This is vertical on $a=0$. We are interested in the behaviour of a separatrices of Z_1 when v varies in the above interval. The same arguments for Y_1 show that they form a family of non-intersecting curves away from the line $a=0$. Note that as x_2 decreases on the line $y=0$ as v increases in the interval $]\epsilon_1, u^2/4-\epsilon_2[$, the separatrix of Z_1 moves to the left in (R) with $a<0$ as v increases in the above interval. If the separatrices of Z_1 do not intersect on the line $a=0$, then they move to the left in (R) as v increases. As those of Y_1 move to the right, this shows that they end up on the same point on the discriminant for a unique value of v in $]\epsilon_1, u^2/4-\epsilon_2[$. (Recall that by Lemma 5.1, the sepratrices do intersect when v is close to $u^2/4$ and from the arguments in §4.1 they do not when v is close to zero. So there is a value of v for which they do end up at the same point on the discriminant.)

Suppose that two sepratrices of Z_1 associated to distinct values of v intersect on $a=0$. Then they must be tangential at such point as Z_1 is vertical on the line $a=0$. These two sepratrices have ordinary tangency at the point of intersection. Indeed, at such point, we can parametrise locally each separatrix in the form $x=x(y)$. These curves satisfy the equation

$$(x(y)+y) + 2(x(y)^2 + ux(y) + v)x'(y) + yx'(y)^2 = 0.$$

At points on the line $a=0$, $x'(y)=0$. Differentiating with respect to y and evaluation at points on $a=0$ we get

$$1 + 2(x(y)^2 + ux(y) + v)x''(y) = 0,$$

which shows that the separatrices have ordinary tangency if they meet on $a=0$, as $x''(y)$ has distinct (non-zero) values for distinct values of v. It follows from this that as the separatrix of Z_1 moves to the left in (R) when $a<0$ as v increases, it also moves to the left in (R) when $a>0$ as v increases. Therefore, in this case too we can conclude that the separatrix of Z_1 and that of Y_1 end up on the same point on the discriminant for a unique value of v in $]\epsilon_1, u^2/4-\epsilon_2[$. □

References

[1] V. I. Arnold (Ed.), *Dynamical systems. V. Bifurcation theory and catastrophe theory*. Encyclopaedia of Mathematical Sciences, 5. Springer-Verlag, Berlin, 1994.

[2] J. W. Bruce, A note on first order differential equations of degree greater than one and wavefront evolution. *Bull. London Math. Soc.* 16 (1984), 139–144.

[3] J. W. Bruce, On families of symmetric matrices. *Moscow Math. J.* 3 (2003), 335–360.

[4] J. W. Bruce and D. Fidal, On binary differential equations and umbilics. *Proc. Royal Soc. Edinburgh* 111A (1989), 147–168.

[5] J. W. Bruce, G. J. Fletcher and F. Tari, Zero curves of families of curve congruences. Real and complex singularities, 1–18, Contemp. Math., 354, Amer. Math. Soc., Providence, RI, 2004.

[6] J. W. Bruce, G. J. Fletcher and F. Tari, Bifurcations of implicit differential equations. *Proc. Royal Soc. Edinburgh* 130A (2000), 485–506.

[7] J. W. Bruce and P. J. Giblin, *Curves and Singularities*. Cambridge University Press, 1992.

[8] J. W. Bruce, P. J. Giblin and F. Tari, Families of surfaces: height functions, Gauss maps and duals. *Real and complex singularities (São Carlos, 1994). Pitman Res. Notes Math. Ser.* 333 (1995), 148–178.

[9] J. W. Bruce and F. Tari, On binary differential equations. *Nonlinearity* 8 (1995), 255–271.

[10] J. W. Bruce and F. Tari, Generic 1-parameter families of binary differential equations of Morse type. *Discrete Contin. Dynam. Systems* 3 (1997), 79–90.

[11] J. W. Bruce and F. Tari, On the multiplicity of implicit differential equations. *J. of Differential Equations* 148 (1998), 122–147.

[12] J. W. Bruce and F. Tari, Duality and implicit differential equations. *Nonlinearity* 13 (2000), 791–811.

[13] J. W. Bruce and F. Tari, Dupin indicatrices and families of curve congruences. *Trans. Amer. Math. Soc.* 357 (2005), 267–285.

[14] M. Cibrario, Sulla reduzione a forma delle equationi lineari alle derviate parziale di secondo ordine di tipo misto. *Accademia di Scienze e Lettere, Instituto Lombardo Redicconti* 65, 889-906 (1932).

[15] L. Dara, Singularités génériques des équations differentielles multiformes. *Bol. Soc. Brasil Math.* 6 (1975), 95–128.

[16] A. A. Davydov, Normal forms of differential equations unresolved with respect to derivatives in a neighbourhood of its singular point. *Functional Anal. Appl.* 19 (1985), 1–10.

[17] A. A. Davydov, *Qualitative control theory.* Translations of Mathematical Monographs 142, AMS, Providence, RI, 1994.

[18] A. A. Davydov and L. Ortiz-Bobadilla, Smooth normal forms of folded elementary singular points. *J. Dynam. Control Systems* 1 (1995), 463–482.

[19] A. A. Davydov and E. Rosales-González, Smooth normal forms of folded resonance saddles and nodes and complete classification of generic linear second order PDE's on the plane. International Conference on Differential Equations (Lisboa, 1995), 59–68, *World Sci. Publishing*, 1998.

[20] F. Dumortier, Techniques in the theory of local bifurcations: blow-up, normal forms, nilpotent bifurcations, singular perturbations. Bifurcations and periodic orbits of direction fields (Montreal, PQ, 1992), 19–73, *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, 408, Kluwer Acad. Publ., Dordrecht, 1993.

[21] G. J. Fletcher, *Geometrical problems in computer vision.* Ph.D thesis, Liverpool University, 1996.

[22] R. Garcia and J. Sotomayor, Codimension two umbilic points on surfaces immersed in $\mathbb{R}^3$. *Discrete Contin. Dynam. Systems* 17 (2007), 293–308.

[23] V. Guíñez, Positive quadratic differential forms and foliations with singularities on surfaces. *Trans. Amer. Math. Soc.* 309 (1988), 447–502.

[24] V. Guíñez, Locally stable singularities for positive quadratic differential forms. *J. Differential Equations* 110 (1994), 1–37.

[25] V. Guíñez, Rank two codimension 1 singularities of positive quadratic differential forms. *Nonlinearity* 10 (1997), 631–654.

[26] V. Guíñez and C. Gutierrez, Rank-1 codimension one singularities of positive quadratic differential forms. *J. Differential Equations* 206 (2004), 127–155.

[27] C. Gutiérrez, J. Sotomayor and R. Garcia, Bifurcations of umbilic points and related principal cycles. *J. Dynam. Differential Equations* 16 (2004), 321–346.

[28] A. Hayakawa, G. Ishikawa, S. Izumiya, K. Yamaguchi, Classification of generic integral diagrams and first order ordinary differential equations. *Internat. J. Math.* 5 (1994), 447–489.

[29] A.G. Kuz'min, Nonclassical equations of mixed type and their applications in gas dynamics. *International Series of Numerical Mathematics*, 109. Birkhauser Verlag, Basel, 1992.

[30] T. Miernowski, Formes normales d'une mètrique mixte analytique réelle générique. Preprint, 2006.

[31] A. Montesinos Amilibia, ODEinR2 v. 1.2.5. Computer program freely available at http://www.uv.es/˜montesin.

[32] A. C. Nabarro and F. Tari, Families of surfaces and conjugate curve congruences. In preparation.

[33] R. Roussarie, *Bifurcations of planar vector fields and Hilbert's 16th Problem*. Publications of the 20th Brazilian Mathematical Colloquium, IMPA, 1995.

[34] J. Sotomayor and C. Gutiérrez, *Structurally stable configurations of lines of principal curvature*. Bifurcation, ergodic theory and applications (Dijon, 1981), 195–215, Astérisque, 98-99, Soc. Math. France, Paris, 1982.

[35] F. Tari, Two-parameter families of implicit differential equations. *Discrete Contin. Dynam. Systems* 13 (2005), 139–162.

[36] F. Tari, Geometric properties of the solutions of implicit differential equations. *Discrete Contin. Dynam. Systems* 17 (2007), 349–364.

[37] F. Tari, On pairs of geometric foliations on a cross-cap. *Tohoku Math. J.* 59 (2007), 233–258.

[38] R. Thom, Sur les équations différentielles multiformes et leurs intégrales singulières. *Bol. Soc. Brasil. Mat.* 3 (1972), 1–11.

[39] R. Uribe-Vargas, Surface evolution, implicit differential equations and pairs of Lengendrian fibrations. Preprint.