On the geometry of the cross-cap in the Minkoswki 3-space

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Abstract

We initiate in this paper the study of the geometry of the cross-cap in the Minkowski 3-space $\mathbb{R}^3_1$. We distinguish between three types of cross caps, spacelike, timelike or lightlike according to the type of the unique tangent direction at the cross-cap point. We obtain special parametrisations for the three types of cross-caps and consider their affine properties. The induced metric changes signature along a curve and the singularities of this curve depend on the type of the cross-cap. We obtain the topological configurations of the lightlike curves and those of the lines of principal curvatures in the source of a parametrisation as well as on the cross-cap surface.

1 Introduction

Whitney showed that maps $\mathbb{R}^2 \to \mathbb{R}^3$ can have a stable local singularity under smooth changes of coordinates in the source and target. A model of this local singularity under these changes of coordinates is given by $(x, y) \mapsto (x, xy, y^2)$. The image of this map is a singular surface called a cross-cap (it is also called a surface with a pinch-point or a Withney umbrella).

Because the cross-cap is a stable singular surface, it is natural to seek to understand its geometry. The extrinsic differential geometry of the cross-cap in the Euclidean 3-space is investigated in [5, 7, 8, 9, 12, 18, 22, 23], and in [12] the authors considered its intrinsic properties. It is shown, for instance, in [5, 23] that there are generically

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two types of cross-caps, labelled *hyperbolic cross-cap* and *elliptic cross-cap* (Figure 1), and these are characterised by the singularity type of their parabolic set in the source (see also §4 for another characterisation and [17, 18] for applications to the geometry of surfaces in $\mathbb{R}^4$). The change from an elliptic to a hyperbolic cross-cap occurs at a parabolic cross-cap, Figure 1.

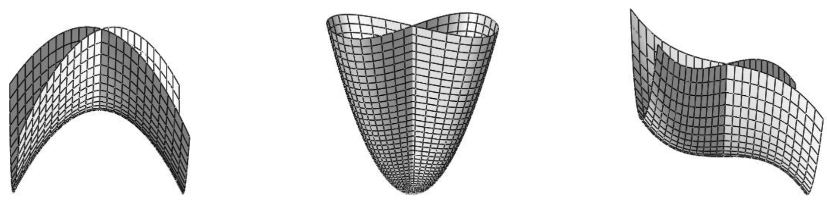


Figure 1: Hyperbolic, elliptic and parabolic cross-caps.

We initiate in this paper the study of the geometry of the cross-cap in the Minkowski 3-space $\mathbb{R}^3_1$. At the cross-cap point, the tangent plane to the surface degenerates to a line, which we label the tangent line of the cross-cap. We call the cross-cap spacelike, timelike or lightlike if its tangent line is respectively spacelike, timelike or lightlike. (Generically, cross-caps in $\mathbb{R}^3_1$ are either spacelike or timelike.) We obtain in §3 parametrisations of the cross-cap in simplified forms using smooth changes of coordinates in the source and Lorentzian motions in the target. From these parametrisations we get pairs of quadratic forms (Q_1, Q_2) in (x, y). We show in §4 that the $\mathcal{G} = GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$-class of (Q_1, Q_2) determines if the cross-cap is elliptic, hyperbolic or parabolic and obtain affine invariant properties of the cross-cap (such us its Dupin indicatrices and its focal conic).

In §5, we study the induced metric on the cross-cap and determine the generic topological configurations of the lightlike curves in the source and their images on the cross-cap. We study in §6 the lines of principal curvature on the cross-cap. Some of the configurations in §5 and in §6 are obtained in §7 using the blowing-up technique on general binary differential equations

The configurations of the solution curves of the binary differential equations (BDEs) in this papers have all been checked using Montesinos program [16]. The solutions of a BDE form a pair of foliations in $\mathbb{R}^2$ or on a surface. In all the figures in this paper, we draw one foliation in continuous line and the other in dashed line. The discriminant curve is drawn in thick black. The double point curve of the cross-cap is drawn in thick grey.

2 Preliminaries

The *Minkowski space* $(\mathbb{R}^3_1, \langle , \rangle)$ is the vector space $\mathbb{R}^3$ endowed with the metric given by the pseudo-scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -u_0v_0 + u_1v_1 + u_2v_2,$$

for any vectors $\boldsymbol{u} = (u_0, u_1, u_2)$ and $\boldsymbol{v} = (v_0, v_1, v_2)$ in $\mathbb{R}^3$ (see for example [19], p55). We say that a non-zero vector $\boldsymbol{u} \in \mathbb{R}^3_1$ is *spacelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$, *lightlike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and *timelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$. The norm of a vector $\boldsymbol{u} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$.

We have the following pseudo-spheres in $\mathbb{R}^3_1$ with centre $p \in \mathbb{R}^3_1$ and radius $r > 0$,

$$\begin{array}{rcl} H^2(p, -r) &=& \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = -r^2\}, \\ S^2_1(p, r) &=& \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = r^2\}, \\ LC^*(p) &=& \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = 0\}. \end{array}$$

We denote by $H^2(-r)$ and $S^2_1(r)$ the pseudo-spheres centred at the origin in $\mathbb{R}^3_1$.

We consider the set $\mathcal{C}$ of smooth map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^3_1$ with a cross-cap singularity at the origin and endowed with the Whitney C^∞-topology. We say that a property of the cross-cap is *generic* if it is satisfied by map-germs in a residual subset of $\mathcal{C}$.

Let $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$ be representative of a map-germ with a cross-cap singularity at the origin and denote its image by M. Let

$$E = \langle \phi_x, \phi_x \rangle, \quad F = \langle \phi_x, \phi_y \rangle, \quad G = \langle \phi_y, \phi_y \rangle$$

denote the coefficients of the first fundamental form of M (the subscripts denote partial derivatives).

We label the *Pre-Locus of Degeneracy* (PLD) the set of point $(x, y) \in U$ where $(F^2 - EG)(x, y) = 0$, and by the *Locus of Degeneracy* (LD) its image by ϕ. The LD is the locus of points on M where the induced metric is degenerate.

We decompose $U = U_1 \cup U_2 \cup PLD$, where $\phi(U_1)$ is the Riemannian part of M and $\phi(U_2)$ is its Lorentzian part. One can define the de Sitter Gauss map $U_1 \to S^2_1(1)$ on the Riemannian part of the surface and the hyperbolic Gauss map $U_2 \to H^2(-1)$ on its Lorentzian part. Both maps are given by $\boldsymbol{N} = \phi_x \times \phi_y / ||\phi_x \times \phi_y||$. The map $A_p(\boldsymbol{u}) = -d\boldsymbol{N}_p(\boldsymbol{u})$ is a self-adjoint operator on $M \setminus LD$. We denote by

$$\begin{array}{rcccl} l &=& -\langle \boldsymbol{N}_x, \phi_x \rangle &=& \langle \boldsymbol{N}, \phi_{xx} \rangle, \\ m &=& -\langle \boldsymbol{N}_x, \phi_y \rangle &=& \langle \boldsymbol{N}, \phi_{xy} \rangle, \\ n &=& -\langle \boldsymbol{N}_y, \phi_y \rangle &=& \langle \boldsymbol{N}, \phi_{yy} \rangle \end{array}$$

the coefficients of the second fundamental form on $M \setminus LD$. At points on the LD, we multiply the above coefficients by $||\phi_x \times \phi_y||$ and set

$$\bar{l} = \langle \phi_x \times \phi_y, \phi_{xx} \rangle, \quad \bar{m} = \langle \phi_x \times \phi_y, \phi_{xy} \rangle, \quad \bar{n} = \langle \phi_x \times \phi_y, \phi_{yy} \rangle.$$

The Gaussian curvature K of M at $p=\phi(q)\in M\setminus LD$ is given by

$$K(q)=\det(A_p)=\frac{ln-m^2}{EG-F^2}(q).$$

The (closure of) the pre-parabolic set is defined as the set of points in U where $(\bar{l}\bar{n}-\bar{m}^2)(q)=0$. Its image under ϕ is defined as the parabolic set on M (this is the closure of the set of points where the Gaussian curvature vanishes).

We are interested in the $\mathcal{R}$-singularities of germs of functions $f:\mathbb{R}^2,0\to\mathbb{R},0$, where $\mathcal{R}$ denotes the group of germs of diffeomorphisms $h:\mathbb{R}^2,0\to\mathbb{R}^2,0$ which acts on the set of germs of such functions by $f\circ h^{-1}$. We shall use representatives (up to a sign) of the following orbits of this action (see [1]):

$$\begin{array}{ll} A_k & x^2\pm y^{k+1}, k\geq 0\\ D_k & x^2y\pm y^{k-1}, k\geq 4\\ X_{1,0} & \left\{\begin{array}{l} x^4+ax^2y^2+y^4,\ a^2-4\neq 0\\ xy(x^2+bxy+y^2),\ b^2-4<0\end{array}\right.\end{array}$$

In the complex case, the singularity $X_{1,0}$ has one normal form given by $x^4+ax^2y^2+y^4$, $a^2-4\neq 0$. However, this form does not include the case when the quartic has two real roots. This case is represented by the normal form $xy(x^2+bxy+y^2)$, $b^2-4<0$.

3 Special parametrisations of the cross-cap

It is shown in [23] that a parametrisation of a cross-cap in the Euclidean 3-space can be taken, by a suitable choice of coordinates in the source and isometries in the target, in the form

$$\phi(x,y)=(x,xy+p(y),y^2+ax^2+q(x,y)),\tag{1}$$

where $p\in\mathcal{M}^3(y)$ and $q\in\mathcal{M}^3(x,y)$ ($\mathcal{M}(u)$ denotes the maximal ideal in the ring of germs of functions in the variables u).

We consider now a cross-cap in the Minkowski 3-space and seek parametrisations in a simplified form, allowing any smooth changes of coordinates in the source and changes of coordinates given by Lorentzian isometries in $\mathbb{R}^3_1$. At the cross-cap point the tangent plane to the surface degenerates to a line. We label it the *tangent line* to the cross-cap. This line can be spacelike, timelike or lightlike.

Definition 3.1 *A cross-cap in the Minkowski 3-space is called spacelike, timelike or lightlike if its tangent line is, respectively, spacelike, timelike or lightlike.*

Theorem 3.2 *A parametrisation of the cross-cap can be taken, by suitable choice of coordinates in the sources and Lorentzian isometries in the target, in one of the following forms.*

(a) *Timelike cross-cap:*

$$(x, y^2 + p_{20}x^2 + p(x), q_{20}x^2 + q_{21}xy + q_{22}y^2 + q(x, y)) \tag{2}$$

(b) *Spacelike cross-cap:*

$$(y^2 + p_{20}x^2 + p(x), x, q_{20}x^2 + q_{21}xy + q_{22}y^2 + q(x, y)) \tag{3}$$

(c) *Lightlike cross-cap:*

$$(y^2 + x + p_{20}x^2 + p(x), x, q_{20}x^2 + q_{21}xy + q_{22}y^2 + q(x, y)) \tag{4}$$

where $p \in \mathcal{M}^3(x)$, $q \in \mathcal{M}^3(x, y)$ and $q_{21} \neq 0$ (as the singularity is a cross-cap).

Proof (a) When the tangent line is timelike, we can make a Lorentzian motion in the target and take it to be along $(1, 0, 0)$. We can then write $\phi(x, y) = (x, f(x, y), g(x, y))$, where f and g are germs of functions with zero 1-jets. As the singularity is a cross-cap, we have $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$ or $\frac{\partial^2 g}{\partial y^2}(0, 0) \neq 0$. We can suppose that $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$ (if it vanishes, we make the isometric change of coordinates $(u, v, w) \to (u, w, v)$ in the target to get back to the case where it does not vanish).

We consider $f(x, y)$ as a 1-parameter unfolding of the function $f(0, y)$. Since $\frac{\partial^2 f}{\partial y^2}(0, 0) \neq 0$, $f(x, y)$ is $\mathcal{R}^+$-equivalent to the germ y^2, that is, there exist a germ of a diffeomorphism $H : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ of the form $H(x, y) = (h(x), k(x, y))$ and a germ of a function c such that

$$y^2 = f(h(x), k(x, y)) + c(x)$$

(see [1]). We have $\phi \circ H(x, y) = (h(x), y^2 - c(x), g(k(x), h(x, y)))$.

Let K be a change of coordinate in the source with $K(u, v) = (x, y) = (h^{-1}(u), v)$. Then

$$\phi \circ H \circ K(u, v) = (u, v^2 - c(k^{-1}(u)), g(u, h(k^{-1}(u), v)).$$

We revert back to the original notation and write x for u and y for v, so that the cross-cap is parametrised in the form

$$(x, y^2 + p_{20}x^2 + p(x), q_{20}x^2 + q_{21}xy + q_{22}y^2 + q(x, y)),$$

where p and q are germs of functions with zero 2-jets.

The case (b) follows in a similar way. For the case (c), we make a Lorentzian motion in the target to set the lightlike tangent line along the direction $(1, 1, 0)$ and take the parametrisation of the surface in the form $\phi(x, y) = (x + f(x, y), x, g(x, y))$, where f and g have zero 1-jets. We then proceed as in case (a). $\square$

Table 1: The $\mathcal{G}$-classes of pairs of quadratic forms.

$\mathcal{G}$-class	Name
(y^2+x^2, xy)	hyperbolic
(y^2-x^2, xy)	elliptic
(x^2, xy)	parabolic
$(x^2 \pm y^2, 0)$	inflection
$(x^2, 0)$	degenerate inflection
$(0, 0)$	degenerate inflection

Corollary 3.3 *With a parametrisation of the cross-cap as in* Theorem 3.2(b), (c) *the limiting tangent direction to the double point curve is lightlike on a spacelike or lightlike cross-cap if and only if* $1 - q_{22}^2 = 0$.

The limiting tangent direction to the double point curve is always spacelike on a timelike cross-cap.

Proof A point is on the double point curve if there exists two distinct points (x_1, y_1) and (x_2, y_2) in the source such that $\phi(x_1, y_1) = \phi(x_2, y_2)$. Taking a parametrisation of the cross-cap as in Theorem 3.2, one can show that $x_1 = x_2 = 0$ and $y_2 = -y_1$, so the double point curve in the source is parametrised by $(0, y)$ and the double point (on the cross-cap) is given by $\phi(0, y)$. For a timelike cross-cap, the limiting tangent direction to the double point curve is along $(0, 1, q_{22})$ so is spacelike. In the case of spacelike and lightlike cross-caps, the limiting tangent direction is along $(1, 0, q_{22})$ and is lightlike if and only if $1 - q_{22}^2 = 0$. $\square$

In the rest of the paper, we shall always take the parametrisations of the cross-cap as in Theorem 3.2 and write the homogeneous part of degree n in p and q, respectively, in the form $p_{n0}x^n$ and $q_{n0}x^n + q_{n1}x^{n-1}y + \ldots + q_{nn}y^n$.

4 Affine properties of the cross-cap

We associate to the parametrisations in Theorem 3.2 the pair of quadratic forms

$$(Q_1(x, y), Q_2(x, y)) = (y^2 + p_{20}x^2, q_{20}x^2 + q_{21}xy + q_{22}y^2)$$

given by the 2-jet of the parametrisation with its linear part removed.

We consider the action of the group $\mathcal{G} = GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$ on the pairs of binary forms (Q_1, Q_2), where $GL(2, \mathbb{R})$ denotes the general linear group (see for example [10]). If $H = (h, k) \in \mathcal{G}$, then $H.(Q_1, Q_2) = k.(Q_1 \circ h^{-1}, Q_2 \circ h^{-1})$. The $\mathcal{G}$-orbits are listed in Table 1.

Lemma 4.1 *The pair of quadratic forms* (Q_1, Q_2) *is*

$$\begin{array}{ccc} \textit{hyperbolic} & \Leftrightarrow & q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 > 0, \\ \textit{elliptic} & \Leftrightarrow & q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 < 0, \\ \textit{parabolic} & \Leftrightarrow & q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 = 0. \end{array}$$

Proof The action in the target by $(u, v) \mapsto (u, v - q_{22}u)$ gives

$$(Q_1, Q_2) \sim_{\mathcal{G}} (y^2 + p_{20}x^2, (q_{20} - p_{20}q_{22})x^2 + q_{21}xy).$$

The action by $(x, y) \mapsto (x, y - \frac{q_{20}-p_{20}q_{22}}{q_{21}}x)$ in the source followed by an action in the target of the form $(u, v) \mapsto (u - \alpha v, \frac{1}{q_{21}}v)$ gives

$$(Q_1, Q_2) \sim_{\mathcal{G}} (y^2 + \frac{q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2}{q_{21}^2}x^2, xy)$$

and the result follows. □

A cross-cap is hyperbolic/elliptic/parabolic if the singularity of its pre-parabolic set is $A_1^+/A_1^-/A_{\geq 2}$ ([5, 23]). The singularity of the pre-parabolic set depends on the contact of the surface with planes, so is affine invariant (in particular, they do not depend on the metric in the ambient space). We have the following for the cross-cap in $\mathbb{R}^3_1$; see [18] for an analogous result for a cross-cap in the Euclidean 3-space.

Proposition 4.2 *The cross-cap is hyperbolic/elliptic/parabolic if and only if its associated pair of quadratic forms* (Q_1, Q_2) *is elliptic/hyperbolic/parabolic.*

Proof The 2-jet of $\bar{m}^2 - \bar{n}\bar{l}$ for the three cross-caps in Theorem 3.2 is given by

$$-4q_{21}(p_{20}q_{21}x^2 + 2(q_{22}p_{20} - q_{20})xy - q_{21}y^2).$$

We have $q_{21} \neq 0$, so the discriminant of the above quadratic form is given, up to a non-zero factor, by $q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2$. The result follows by Lemma 4.1. (When $q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 = 0$, the parabolic set has an A_k-singularity, with $k \geq 2$.) □

In view of Proposition 4.2, we label the property hyperbolic/elliptic/parabolic of a cross-cap as its affine property. This property can also be detected by considering the following curves in the source. We consider the intersection of the cross-cap parametrised by $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ with the planes $f(x, y, z) = ax + by + cz - d = 0$, $d \neq 0$, parallel to a plane containing the unique tangent direction to the cross-cap. We analyse the limit of the curves $f \circ \phi(x, y) = 0$ as d tends to zero and call them the *Dupin indicatrices in the source* associated to the tangent plane $ax + by + cz = 0$. These are approximated by the zero set of the 2-jet of $f \circ \phi$.

Consider the case of the timelike cross-cap with ϕ as in Theorem 3.2(a) (the other two cases follow similarly). Then $a = 0$ and the Dupin indicatrices in the source are given by

$$bQ_1(x, y) + cQ_2(x, y) - d = 0.$$

We identify a quadratic form $Q = Ax^2 + Bxy + Cy^2$ by its coefficients $(A : B : C)$ in the projective plane $\mathbb{R}P^2$. We denote by Γ the conic $\{Q : B^2 - 4AC = 0\}$ of degenerate quadratic forms. Then the $\mathcal{G}$-orbit of a pair of quadratic forms (Q_1, Q_2) is completely determined by the pencil $bQ_1(x, y) + cQ_2(x, y)$ in $\mathbb{R}P^2$. The pair (Q_1, Q_2) is hyperbolic (resp. elliptic) if and only if its associated pencil intersects the conic Γ in 2 (resp. 0) points. It is parabolic if the pencil is tangent to Γ.

Proposition 4.3 *At a hyperbolic cross-cap, the Dupin indicatrices in the source associated to any tangent plane are hyperbolae.*

At an elliptic cross-cap, there are two tangent planes whose associated Dupin indicatrices is a pair of parallel lines. The remaining Dupin indicatrices are either hyperbolae or ellipses.

At an parabolic cross-cap, there is a unique tangent plane whose associated Dupin indicatrices is a pair of parallel lines. The remaining Dupin indicatrices are all hyperbolae.

Remark 4.4 The height function on the cross-cap along a normal direction (a, b, c) is given by $\langle \phi(x, y), (a, b, c) \rangle$. On a hyperbolic cross-cap the singularities of the height function in any normal direction is A_1^-. On an elliptic cross-cap, there are two normal directions along which the singularities of the height function is $A_k, k \geq 2$, and for the remaining directions it is A_1^+ or A_1^-. On a parabolic cross-cap, there is a unique normal direction along which the singularity of the height function is $A_k, k \geq 2$, and for the remaining directions it is A_1^- ([5, 23]). As the 2-jet of $\langle \phi(x, y), (a, b, c) \rangle$ is the pencil associated to the pair of quadratic forms (Q_1, Q_2), the statement of Proposition 4.3 is a reformulation of the result on the singularities height functions in terms of the Dupin indicatrices in the source.

We turn now to an aspect of the contact of a cross-cap parametrised by $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ with pseudo-spheres in $\mathbb{R}^3_1$. This contact is measured by the singularities of the distance squared functions $d : (\mathbb{R}^2, 0) \times \mathbb{R}^3_1 \to \mathbb{R}$ with

$$d(x, y, u) = \langle \phi(x, y) - u, \phi(x, y) - u \rangle.$$

We set $d_u(x, y) = d(x, y, u)$ and consider its $\mathcal{R}$-singularities at the cross-cap point, that is at the origin in the xy-plane.

The plane orthogonal to the tangent line at the cross-cap is labelled the *normal plane to the cross-cap*. It is not difficult to show that the function d_u is singular at the origin if and only if u is on the normal plane to the cross-cap.

Theorem 4.5 (1) *For a spacelike or timelike cross-cap, the singularities of d_u are always of type A_k, $k \geq 1$. There is a conic in the normal plane of the cross-cap, labelled the focal conic, where the singularities of d_u are of type A_k, $k \geq 2$. The focal conic contains the cross-cap point and is as follows:*

an ellipse	$\Leftrightarrow$	$q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 < 0$	$\Leftrightarrow$	(Q_1, Q_2) *is elliptic,*
a hyperbola	$\Leftrightarrow$	$q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 > 0$	$\Leftrightarrow$	(Q_1, Q_2) *is hyperbolic,*
a parabola	$\Leftrightarrow$	$q_{21}^2 p_{20} + (q_{22}p_{20} - q_{20})^2 = 0$	$\Leftrightarrow$	(Q_1, Q_2) *is parabolic.*

(2) *For a lightlike cross-cap, the singularity of d_0 at the cross-cap point is of type D_4 if and only if $p_{20} \neq 0$. Those of d_u, $u \neq 0$, are of type A_k, $k \geq 1$. The focal conic is a pair of transverse lines intersecting at the cross-cap point if and only if the cross-cap is not parabolic (equivalently, if and only if (Q_1, Q_2) is hyperbolic or elliptic).*

Proof (1) We consider only the timelike cross-cap and take ϕ as in Theorem 3.2(a). The case of the spacelike cross-cap follows similarly and we get the same conditions which identify the focal conic. We write $u = (u_0, u_1, u_2)$, so

$$d_u(x, y) = -(u_0 - x)^2 + (u_1 - y^2 - p_{20}x^2 - p(x))^2 + \left(u_2 - q_{20}x^2 - q_{21}xy - q_{22}y^2 - q(x, y)\right)^2.$$

We have $j^1 d_u(x, y) = -u_0^2 + u_1^2 + u_2^2 + 2u_0 x$, so d_u is singular at the origin (i.e., at the cross-cap point) if and only if $u_0 = 0$, that is, if and only if u is on the normal plane of the cross-cap.

We take now $u_0 = 0$. Then the 2-jet of d_u, without the constant terms, is given by

$$-(1 + 2u_1 p_{20} + 2u_2 q_{20})x^2 - 2u_2 q_{21} xy - 2(u_1 - u_2 q_{22})y^2. \tag{5}$$

The quadratic form (5) can never vanish identically as $q_{21} \neq 0$, so the singularities of d_u are always of type A_k, $k \geq 1$.

The singularity of d_u is of type A_k, $k \geq 2$, if and only of the quadratic form (5) is degenerate, that is, if and only if

$$2p_{20}u_1^2 + 2(q_{20} + p_{20}q_{22})u_1 u_2 + (2q_{20}q_{22} - \frac{1}{2}q_{21}^2)u_2^2 + u_1 + u_2 q_{22} = 0. \tag{6}$$

The above equation is that of a non-degenerate conic (the focal conic), in the normal plane $(0, u_1, u_2)$. The discriminant of its quadratic part is

$$\delta = p_{20}q_{21}^2 + (q_{20} - p_{20}q_{22})^2.$$

The focal conic is a parabola if and only if $\delta = 0$, equivalently, if and only if (Q_1, Q_2) is parabolic (Lemma 4.1).

When $\delta \neq 0$, the linear terms in (6) can be removed by a translation to obtain a new equation in the form

$$2p_{20}U_1^2 + 2(q_{20} + p_{20}q_{22})U_1 U_2 + (2q_{20}q_{22} - \frac{1}{2}q_{21}^2)U_2^2 = \frac{1}{8}\frac{q_{21}^2}{p_{20}q_{21}^2 + (q_{20} - p_{20}q_{22})^2}.$$

This is a hyperbola if and only if $p_{20}q_{21}^2+(q_{20}-p_{20}q_{22})^2>0$ and an ellipse if and only if $p_{20}q_{21}^2+(q_{20}-p_{20}q_{22})^2<0$. The interpretation of these inequalities in terms of the pair of quadratic forms (Q_1,Q_2) is given in Lemma 4.1. (It is worth observing that for a spacelike cross-cap, the focal conic is tangent to a lightlike line if and only if the limiting tangent direction to the double point curve is lightlike.)

(2). When the cross-cap is lightlike, we take ϕ as in Theorem 3.2(c), so that

$$d_u=-(u_0-y^2-x-p_{20}x^2-p(x))^2+(u_1-x)^2+(u_2-q_{20}x^2-q_{21}xy-q_{22}y^2-q(x,y))^2.$$

We have $j^1d_u(x,y)=-u_0^2+u_1^2+u_2^2+2(u_0-u_1)x$, so d_u is singular at the origin if and only if $u_0=u_1$, that is, if and only if u is on the lightlike normal plane of the cross-cap.

We suppose now that $u_0=u_1$. Then the 2-jet of d_u, without the constant term, is given by

$$(u_0p_{20}-u_2q_{20})x^2-u_2q_{21}xy+(u_0-u_2q_{22})y^2. \tag{7}$$

The quadratic form (7) vanishes identically if $u_0=u_2=0$, that is, if $u=0$. Then the 3-jet of d_0, without the constant terms, is given by $-2x(y^2+p_{20}x^2)$. Thus, d_0 has a D_4-singularity if and only if $p_{20}\neq 0$.

Suppose that $u\neq 0$. Then the singularity of d_u is of type A_k, $k\geq 2$, if and only if the quadratic form (7) is degenerate, that is, if and only if

$$-4p_{20}u_0^2+4(q_{20}+p_{20}q_{22})u_0u_2+(q_{21}^2-4q_{20}q_{22})u_2^2=0. \tag{8}$$

This is a pair of transverse lines if and only if $q_{21}^2p_{20}+(q_{22}p_{20}-q_{20})^2\neq 0$, that is, if and only if (Q_1,Q_2) is not parabolic. $\square$

We have the following results about more degenerate singularities of the distance squared functions (these are not affine invariant).

Proposition 4.6 *There are generically* $1,3$ *or* 5 *points on the focal conic of a timelike or a spacelike cross-cap where* d_u *has an* A_3*-singularity at the cross-cap point, and one of these is always at the cross-cap point.*

For a lightlike cross-cap, the cross-cap point is a D_4*-singularity of* d_0 *and there is one point on each line of the focal conic where* d_u *has an* A_3*-singularity at the cross-cap point.*

The A_2 *and* A_3*-singularities of* d_u*,* $u\neq 0$*, at the cross-cap point are versally unfolded by the family* d*. The singularity of* d_0 *at the cross-cap point is not versally unfolded by the family* d*.*

Proof We take u on the focal conic. Then the 2-jet of d_u is a perfect square L^2. The singularity of d_u is of type A_k, $k\geq 3$ if and only if L divides C, where C is the homogeneous cubic part of d_u. For the timelike cross-cap, we take $L=u_2q_{21}x+2(u_1-$

$u_2q_{22})y$ (up to a constant factor), see the proof of Theorem 4.5. The cubic C divides L if and only if $C(2(u_1-u_2q_{22}),-u_2q_{21})=0$, that is, if and only if

$$\begin{array}{rcl} 8p_{30}u_1^4-8(3p_{30}q_{22}+q_{30})u_1^3u_2+4(q_{21}q_{31}-6p_{30}q_{22}^2-6q_{30}q_{22})u_1^2u^2 & & \\ +2(4q_{31}q_{21}q_{22}-q_{21}^2q_{32}-12q_{30}q_{22}^2-4p_{30}q_{22}^3)u_1u_2^3 & & \\ +(q_{21}^3q_{33}+4q_{31}q_{21}q_{22}^2-2q_{32}q_{21}^2q_{22}-8q_{30}q_{22}^3)u_2^4 & = & 0. \end{array}$$

This is a homogeneous quartic in u_1,u_2, so is generically a union of 0,2,4 real lines meeting at the origin. Thus, the singularity of d_u is of type A_k, $k\geq 3$ if and only if $u=(0,u_1,u_2)$ is a point of intersection of these lines with the conic (6), so we get generically $1,3$ or 5 such singularities, and they are generically of type A_3. We proceed similarly for the spacelike cross-cap.

For the lightlike cross-cap, proceeding as above, we show that d_u has an A_k, $k\geq 3$-singularity if and only if u is a point of intersection of the conic (8) with the following non-homogeneous quartic

$$\begin{array}{rl} Q(u_0,u_2)= & (2q_{21}^2q_{32}q_{22}-4q_{31}q_{21}q_{22}^2-q_{21}^3q_{33}+8q_{30}q_{22}^3)u_2^4 \\ & +2(4q_{31}q_{21}q_{22}-q_{32}q_{21}^2-4p_{30}q_{22}^3-12q_{30}q_{22}^2)u_0u_2^3 \\ & +4(6q_{30}q_{22}+6p_{30}q_{22}^2-q_{21}q_{31})u_2^2u_0^2-8(q_{30}+3p_{30}q_{22})u_2u_0^3+8p_{30}u_0^4 \\ & +2(q_{21}^2q_{22}+4p_{20}q_{22}^3)u_2^3-2(12p_{20}q_{22}^2+q_{21}^2)u_2^2u_0+24p_{20}q_{22}u_0^2u_2-8p_{20}u_0^3. \end{array}$$

Denote by $u_0=\lambda_iu_2$, $i=1,2$, the lines of the conic (8). Then $Q(\lambda_iu_2,u_2)=u_2^3(A_iu_2+B_i)$, $i=1,2$, where A_i and B_i depend on λ_i as well as p_{j0} and q_{jk}, $j,k=2,3$. Generically, $A_i\neq 0$ and $B_i\neq 0$, so we have a single point $u\neq 0$ on each line of the focal conic where the singularity of d_u at the origin is generically of type A_3. (When $u=0$, the singularity of of type D_4.)

The statement about the versality of the family d follows by standard calculations and are omitted (see for example [7] for detailed calculations for the cross-cap in the Euclidean 3-space). □

5 The induced metric on the cross-cap

Let E,F,G denote, as in §2, the coefficients of the first fundamental form of a cross-cap parametrised by ϕ. The induced metric on the cross-cap is given by $ds^2=Edx^2+2Fdxdy+Gdy^2$. This metric is Riemannian if $F^2-EG<0$, Lorentzian if $F^2-EG>0$ and degenerate if $F^2-EG=0$. The Pre-Locus of Degeneracy (PLD) is defined as the set of points (x,y) in the source where $(F^2-EG)(x,y)=0$. Its image by ϕ is labelled the *Locus of Degeneracy* (LD).

Proposition 5.1 *The PLD of a timelike cross-cap has an A_1^+-singularity and that of a spacelike cross-cap has A_1^--singularity. The PLD of a lightlike cross-cap has generically (when $p_{20}\neq 0$) an A_2-singularity.*

Proof We compute the coefficients of the first fundamental form. Suppose that the unique tangent direction is timelike and take a parametrisation of the surface as in Theorem 3.2(a). Then,

$$\begin{array}{rcl} j^2E & = & -1 + 4(p_{20}^2 + q_{20}^2)x^2 + 4q_{21}q_{20}xy + q_{21}^2y^2, \\ j^2F & = & 2q_{20}x^2q_{21} + (4p_{20} + q_{21}^2 + 4q_{20}q_{22})xy + 2q_{21}q_{22}y^2, \\ j^2G & = & q_{21}^2x^2 + 4q_{22}q_{21}xy + 4(1 + q_{22}^2)y^2, \end{array} \tag{9}$$

so that

$$j^2(F^2 - EG) = q_{21}^2x^2 + 4q_{21}q_{22}xy + 4(1 + q_{22}^2)y^2.$$

The discriminant of the above quadratic form is $-16q_{21}^2$ and is strictly negative, so the PLD has a Morse singularity of type A_1^+, i.e., it is an isolated point.

We take a parametrisation of a spacelike cross-cap as in Theorem 3.2(b). Then,

$$\begin{array}{rcl} j^2E & = & 1 - 4(p_{20}^2 - q_{20}^2)x^2 + 4q_{21}q_{20}xy + q_{21}^2y^2, \\ j^2F & = & 2q_{20}q_{21}x^2 + (4q_{20}q_{22} - 4p_{20} + q_{21}^2)xy + 2q_{21}q_{22}y^2, \\ j^2G & = & q_{21}^2x^2 + 4q_{22}q_{21}xy - 4(1 - q_{22}^2)y^2, \end{array} \tag{10}$$

and

$$j^2(F^2 - EG) = -q_{21}^2x^2 - 4q_{21}q_{22}xy + 4(1 - q_{22}^2)y^2.$$

The discriminant of the above quadratic form is $16q_{21}^2$ so the PLD has a Morse singularity of type A_1^-, i.e., it is a pair of transverse crossing curves.

For a lightlike cross-cap parametrised as in Theorem 3.2(c),

$$\begin{array}{rcl} j^2E & = & -4p_{20}x - 2(2p_{20}^2 + 3p_{30} - 2q_{20}^2)x^2 + 4q_{21}q_{20}xy + q_{21}^2y^2, \\ j^2F & = & -2y + 2q_{20}q_{21}x^2 + (4q_{20}q_{22} - 4p_{20} + q_{21}^2)xy + 2q_{21}q_{22}y^2, \\ j^2G & = & q_{21}^2x^2 + 4q_{22}q_{21}xy - 4(1 - q_{22}^2)y^2. \end{array} \tag{11}$$

Then the 3-jet of $F^2 - EG$ is, up to a scalar multiple, given by

$$y^2 + q_{21}^2p_{20}x^3 + 2q_{21}(2q_{22}p_{20} - q_{20})x^2y + (4p_{20}q_{22}^2 - 4q_{20}q_{22} - q_{21}^2)xy^2 - 2q_{21}q_{22}y^3. \tag{12}$$

Therefore, the PLD has an A_2-singularity if $p_{20} \neq 0$. Observe that the condition $p_{20} \neq 0$ is the same as that for the distance squared function d_0 to have a D_4-singularity (Theorem 4.5). □

Remark 5.2 The singularities of the PLD can be explained geometrically as follows. There is a pencil of planes containing the tangent line of the cross-cap which are tangent to the cross-cap. When the tangent direction is timelike all the planes in the pencil are timelike so all nearby tangent planes to the surface are timelike, i.e., the PLD must be an isolated point. For the spacelike (resp. lightlike) cross-cap, there are two (resp. one) tangent planes in the pencil which are lightlike and this indicates that there are two (resp. one) branches of the PLD.

We consider the integral curves of the lightlike directions on a cross-cap, which we label the lightlike curves. (These are the isotropic geodesics, i.e., those with identically zero length, [20].) The lightlike curves are the images by the parametrisation ϕ of the solution curves of the binary quadratic differential equation (BDE)

$$\omega : Edu^2 + 2Fdudv + Gdv^2 = 0. \tag{13}$$

We identify a BDE by its coefficients and write $\omega = (E, F, G)$. We call the solutions of (13) the pre-lightlike curves in the source. There are two pre-lightlike curves at each point in the region mapped by ϕ to the Lorentzian region of the cross-cap and none at points mapped to its Riemannian region. The PLD, which is the *discriminant curve* of the BDE (13) (the discriminant curve of a BDE is the set of points where the equation determines a unique solution direction) separates the two regions. We have the following result about the generic configurations of the pre-lightlike curves at the cross-cap point. (See [14] for the generic configurations of the lightlike curves on a smooth surface.)

Theorem 5.3 *The BDE* (13) *of the pre-lightlike curves of a cross-cap in the Minkowski 3-space is topologically equivalent to one of the following topological normal forms:*

(i) $(1, 0, -x^2 - y^2)$ *at a timelike cross-cap;*

(ii) $(1, 0, -x^2 + y^2)$ *at a spacelike cross-cap if the limiting tangent direction to the double point curve is spacelike* ($q_{22}^2 - 1 > 0$)*, and to* $(1, 0, x^2 - y^2)$ *if it is timelike* ($q_{22}^2 - 1 < 0$)*.*

(iii) (x, y, x^2) *at a lightlike cross-cap if* $p_{20} > 0$ *and to* $(x, -y, x^2)$ *if* $p_{20} < 0$*.*

See Table 2*, second column for figures.*

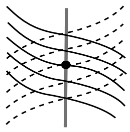
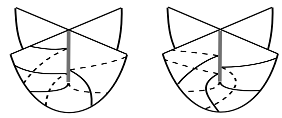
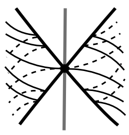
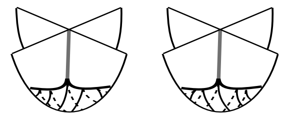
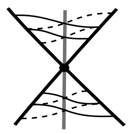
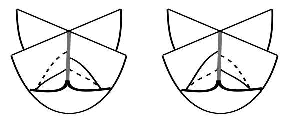
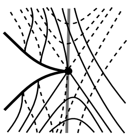
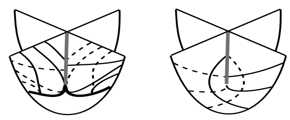
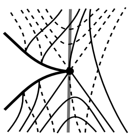
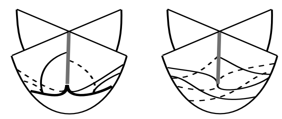
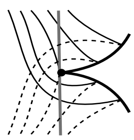
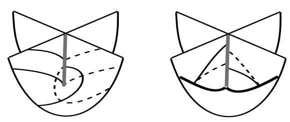
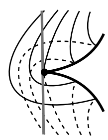
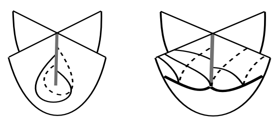
Proof (i) The 2-jets of the coefficients of the first fundamental form are given in (9). Dividing the coefficients of the BDE (13) by E and making changes of coordinates in the source, transforms the 2-jet of the BDE (13) to

$$dx^2 + (-4(1 + q_{22}^2)y^2 - \frac{q_{21}^2}{(1 + q_{22}^2)}x^2)dy^2.$$

Following [2] (see also [21]), the BDE (13) is topologically equivalent to $dx^2 + (-x^2-y^2)dy^2 = 0$, i.e., it has a Morse Type 1 A_1^+-singularity of type saddle (because its discriminant, which is the PLD, has an A_1^+-singularity and the coefficient $-4(1+q_{22}^2)$ of y^2 is negative). Thus, the configuration of the pre-lightlike curves are as in Table 2, left second column.

There is another Morse Type 1 A_1^+-singularity of type focus modeled by $dx^2 + (y^2 + x^2)dy^2 = 0$. Its integral curves consist of an isolated point. In our context, the timelike cross-cap is a Lorentzian surface, so it has two lightlike curves at each point

Table 2: Pre-lightlike lines and lightlike lines on cross-caps in $\mathbb{R}^3_1$ viewed from two opposite directions.

Cross-cap type	Pre-lightlike curves	Lightlike curves
Timelike		
Spacelike $q_{22}^2 - 1 > 0$		
Spacelike $q_{22}^2 - 1 < 0$		
Lightlike $p_{20} < 0$		
		
Lightlike $p_{20} > 0$		
		

away from the cross-cap point. Therefore, the Morse Type 1 A_1^+-singularity of type focus cannot occur in the BDE (13).

(ii) Proceeding as in (i) and using (9), we can transform the 2-jet of the BDE (13) to

$$dx^2 + (q_{21}^2 x^2 + 4q_{22}q_{21}xy - 4(1 - q_{22}^2)y^2)dy^2.$$

Following [2], the BDE (13) is topologically equivalent to $dx^2 \pm (x^2 - y^2)dy^2 = 0$ (i.e., it has a Morse Type 1 A_1^--singularity) if $q_{22}^2 - 1 \neq 0$, that is, if the limiting tangent direction to the double point curve is not lightlike (Corollary 3.3). When this is the case, we can reduce further the 2-jet of the BDE (13) to

$$dx^2 + (4(q_{22}^2 - 1)y^2 - \frac{q_{21}^2}{(q_{22}^2 - 1)}x^2)dy^2.$$

We have the saddle type Morse Type 1 A_1^- model $dx^2 + (-x^2 + y^2)dy^2 = 0$ (resp. the focus type model $dx^2 + (x^2 - y^2)dy^2 = 0$) if $q_{22}^2 - 1 > 0$ (resp. $q_{22}^2 - 1 < 0$), that is, if the limiting tangent direction to the double point curve is spacelike (resp. timelike). The configuration of the pre-lightlike curves are as in Table 2, second column.

(iii) Here the 2-jets of the coefficients of the BDE (13) are as in (11). We observe that all the coefficients of the BDE vanish at the origin and the 1-jet of the BDE is equivalent to $xdx^2 \pm ydxdy$ (assuming, generically, that $p_{20} \neq 0$). This case is not studied previously and we deal with it in details in section 7. Using Theorem 7.1 in section 7, we deduce that the BDE (13) is topologically equivalent to $xdx^2 + 2ydxdy + x^2dy^2 = 0$ if $p_{20} > 0$ and to $xdx^2 - 2ydxdy + x^2dy^2 = 0$ if $p_{20} < 0$; see Table 2, second column. (The sign of p_{20} determines the D_4 type of the singularity of the distance squared function d_0 at the origin, D_4^+ if $p_{20} > 0$ and D_4^- if $p_{20} < 0$; see the proof of Theorem 4.5.) □

Remark 5.4 The normal forms in Theorem 5.3 do not take into consideration the double point curve. When mapping the pre-lightlike curves to the cross-cap, the double point curve plays a key role. For the lightlike cross-cap, one can show that one of the separatrices of the pre-lightlike curves is tangent to the double point curve and is parametrised by $x = -\frac{1}{2}(-1 + q_{22}^2)y^2 + h.o.t.$ The relative position of this separatrice, the double point curve and of the PLD (12) depends only on the sign of $p_{20}(-1 + q_{22}^2)$. Thus, we have two cases for each case in Theorem 5.3(iii) when taking into consideration the relative position of the above three curves; see Table 2 second column.

One can make a cross-cap from a rectangular piece of paper as follows. Label one side of the paper A and the other B. Draw a line that divides the piece of paper into two equal rectangles. This line is the double point curve. Cut the piece of paper along half of the double point. Fold one free edge of the cut and seller tape it to the other fixed half of the double point curve on the side A of the paper. Take the remaining free edge and fold it along the fixed half of the curve double point curve on the side B of the paper.

When every pre-lightlike curve intersects the double point curve in at most one point, their images on the cross-cap do not self-intersect. Then one can draw the pre-lightlike curves on a piece of paper and determine by the above procedure the configurations of the lightlike curves on the cross-cap itself (Table 2 third column). This can be done for all the figures in Table 2 second column except for the last case. For this last case there are pre-lightlike curves which intersect the double point curve twice. One needs to show that these two points are not mapped to the same image on the cross-cap (see for example [22] for proofs for some pairs of foliations on a cross-cap). We conjecture that this is the case and that the configurations on the cross-cap are as in the last figure in Table 2 third column.

6 The lines of principal curvature

When the shape operator A_p has real eigenvalues at a point $p \in M \setminus LD$, we call them the *principal curvatures* and their associated eigenvectors the *principal directions* of M at p. (There are always two principal curvatures at each point on the Riemannian part of M but this is not always true on its Lorentzian part.) The lines of principal curvature, which are the integral curves of the principal directions, are the images by the parametrisation ϕ of the solutions of the BDE

$$(Gm - Fn)dy^2 + (Gl - En)dydx + (Fl - Em)dx^2 = 0. \tag{14}$$

One can extend the lines of principal curvature across the LD as follows ([14]). As equation (14) is homogeneous in l, m, n, we substitute these by $\bar{l}, \bar{m}, \bar{n}$. This substitution does not alter the pair of foliations on $M \setminus LD$. The new equation is defined on the LD and defines the same pair of foliations associated to the de Sitter (resp. hyperbolic) Gauss map on the Riemannian (resp. Lorentzian) part of M. The extended lines of principal curvature are the images by ϕ of the solution curves of the BDE

$$(G\bar{m} - F\bar{n})dy^2 + (G\bar{l} - E\bar{n})dydx + (F\bar{l} - E\bar{m})dx^2 = 0. \tag{15}$$

We call the solutions of the BDE (15) the *pre-lines of principal curvature* and label its discriminant the *Pre-Lightlike Principal Locus* ($PLPL$). The image of the $PLPL$ by ϕ is labelled the *Lightlike Principal Locus* (LPL) (see [13, 14] for smooth Lorentzian surface and smooth surface with varying signature metric).

The $PLPL$ is the zero set of the function

$$(G\bar{l} - E\bar{n})^2 - 4(G\bar{m} - F\bar{n})(F\bar{l} - E\bar{m}). \tag{16}$$

We have the following about the singularities of the $PLPL$.

Proposition 6.1 *The $PLPL$ of a timelike cross-cap has an A_3^--singularity.*

The $PLPL$ of a spacelike cross-cap has an A_3-singularity if the limiting tangent direction to the double point curve is not lightlike. The singularity is of type A_3^+ if the limiting tangent direction to the double point curve in the target is spacelike and A_3^- if it is timelike.

For both spacelike and timelike cross-caps, when the $PLPL$ has an A_3^--singularity, its two branches are tangent to the double point curve in the source.

The $PLPL$ of a lightlike cross-cap has generically and $X_{1,0}$-singularity with two or four real branches. The double point curve has generically an ordinary tangency with one of the branches of the $PLPL$.

Proof We compute the relevant jets of the coefficients $\bar{l}, \bar{m}, \bar{n}$ and find that the 3-jet of the $PLPL$ is given by

$$4q_{21}^2x^2 + 8q_{21}q_{31}x^3 - 24q_{21}q_{33}xy^2.$$

Therefore, its singularity is of type $A_{\geq 3}$. We eliminate the term xy^2 by a change of coordinates of the form $x \to x + ay^2$ and find that the 4-jet of the $PLPL$ is $\mathcal{R}$-equivalent to

$$4q_{21}^2(x^2 - 16(1 + q_{22}^2)y^4),$$

which is an A_3^--singularity.

Similar calculation to the timelike case shows that the 4-jet of the $PLPL$ is $\mathcal{R}$-equivalent to

$$4q_{21}^2(x^2 - 16(1 - q_{22}^2)y^4).$$

This is an A_3^- if $1 - q_{22}^2 > 0$ and an A_3^+-singularity if $1 - q_{22}^2 < 0$ (see the proof of Corollary 3.3 for a geometric interpretation of the sign of $1 - q_{22}^2$).

For a lightlike cross-cap, the 4-jet of the $PLPL$ is given by

$$64q_{21}x(q_{21}p_{20}^2x^3 + 3q_{21}p_{20}xy^2 + 2(q_{22}p_{20} - q_{20})y^3).$$

This is an $X_{1,0}$-singularity if $p_{20}(q_{21}^2p_{20} + (q_{22}p_{20} - q_{20})^2) \neq 0$ (see Lemma 4.1 and Theorem 4.5 for a geometric interpretation of this condition). The above quartic has always two or four real roots. One of the roots has tangent direction $x = 0$ so is tangent to the double point curve. The tangency is ordinary if and only if the coefficient of y^5 in the Taylor expansion of the $PLPL$ is not zero, that is, if and only if

$$\Lambda = (2q_{21}(q_{22}^2 - 1) + 3q_{33})(q_{22}p_{20} - q_{20}) \neq 0.$$

$\square$

We seek to determine the generic topological configurations of the pre-lines of principal curvature and their images on the cross-cap. We start with the timelike and spacelike cross-cap. Then the $PLPL$, the discriminant of the BDE (15), has an A_3^+-singularity for the timelike case and an A_3^--singularity for the spacelike case. We take a parametrization of the surface as in Theorem 3.2. Then the 1-jet of the coefficients of the BDE (15) is $(0, b_0 x, y)$, with $b_0 = -1/2$.

Proposition 3.2 in [22] asserts that the 3-jet of the BDE (15) is equivalent to $(a_3y^3, b_0x + b_2y^2 + b_3y^3, y)$ and Theorem 3.3 in [22] states that if the discriminant has an $A_3^\pm$-singularity, then this BDE is topologically equivalent to

$$(\mp y^3, b_0x + b_2y^2, y),$$

with (b_0, b_2) a fixed value in an open region delimited by some exceptional curves in the b_0b_2-plane. The exceptional curves are the parabola $1 + b_0 - b_2^2 = 0$, and the lines $b_0 = -1, b_0 = 0, 2 + b_0 - 2b_2 = 0, 2 + b_0 + 2b_2 = 0$ (Figure 2). There are 4 generic topological models when the singularity is A_3^+ and 9 when it is A_3^-.

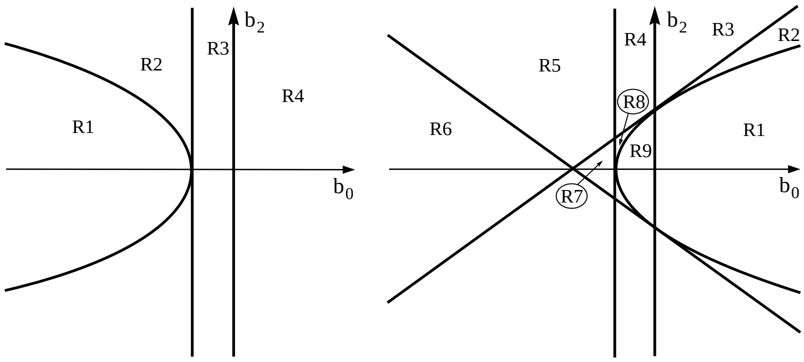


Figure 2: Partition of the (b_0, b_2)-plane, A_3^+ left and A_3^- right. The topological type for $(b_0, -b_2)$ is the same as that for (b_0, b_2).

Theorem 6.2 (1) *Suppose that the PLPL has an A_3^--singularity. Then the BDE* (15) *of the pre-lines of principal curvature of a timelike or spacelike cross-cap is topologically equivalent to*

$$(y^3, -\frac{1}{2}x + b_2y^2, y)$$

if $b_2 \neq 3/4, \pm\sqrt{2}/2$, where $b_2 = 3q_{33}/(4q_{21}\sqrt{1 + q_{22}^2})$ for a timelike cross-cap and $b_2 = 3q_{33}/(4q_{21}\sqrt{1 - q_{22}^2})$ for a spacelike cross-cap. The topological configuration of the pre-lines of principal curvature is as in

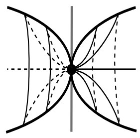
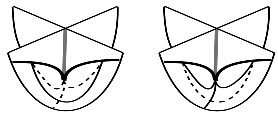
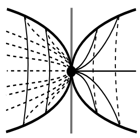
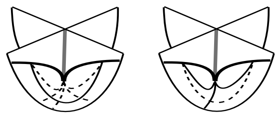
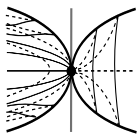
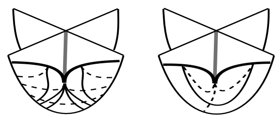
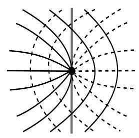
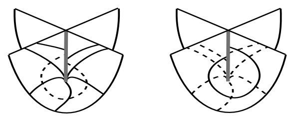
Table 3 second column first figure (region 9 in Figure 2 right) *if* $|b_2| < \frac{\sqrt{2}}{2}$,

Table 3 second column second figure (region 8 in Figure 2 right) *if* $-\frac{3}{4} < b_2 < -\frac{\sqrt{2}}{2}$ or $\frac{\sqrt{2}}{2} < b_2 < \frac{3}{4}$,

Table 3 second column third figure (region 4 in Figure 2 right) *if* $|b_2| > \frac{3}{4}$.

(1) *When the PLPL of a spacelike cross-cap has an* A_3^+*-singularity, the BDE* (15) *is topologically equivalent to* $(-y^3, -\frac{1}{2}x, y)$*;* Table 3 second column last figure (region 3 in Figure 2 left).

Table 3: Pre-lines and lines of principal curvature on a timelike and spacelike cross-cap in $\mathbb{R}^3_1$ viewed from two opposite directions.

Cross-cap type	Pre-lines of principal curvature	Lines of principal curvaure
A_3^--singularity of the $PLPL$		
		
		
A_3^+-singularity of the $PLPL$		

Proof We take a parametrization of the timelike cross-cap as in Theorem 3.2(a). Then we can reduce the 3-jet of the BDE (15) to

$$(4(1+q_{22}^2)y^3, -\frac{x}{2} + \frac{3q_{33}}{2q_{21}}y^2 + \beta y^3, y),$$

where β is a constant depending on the coefficients of the 4-jet of the parametrisation of the surface. We divide the BDE by $4(1+q_{22}^2)$ and make smooth changes of coordinates in the source of the form $x = 2\sqrt{1+q_{22}^2}X$, $y = Y$. The 3-jet of the new BDE is given by

$$(Y^3, b_0X + b_2Y^2 + \tilde{\beta}Y^3, Y), \tag{17}$$

where $b_0 = -1/2$, $b_2 = 3q_{33}/4q_{21}\sqrt{1+q_{22}^2}$ and $\tilde{\beta}$ is a new constant. The result follows by apply Theorem 3.3 in [22]. Similar calculations give the result for the spacelike case. □

The double point curve is tangent to the branches of the $PLPL$ when the latter has an A_3^- singularity (Proposition 6.1), so following the discussion at the end of §5, we can easily map the configurations in Table 3 second column to the cross-cap surface and these are as shown in Table 3 third column. The case when $PLPL$ has an A_3^+-singularity can be mapped to the surface using Theorem 2.7 in [22], where it is shown that, generically, the two points of intersection of a solution curve with the double point curve are not mapped to the same point on the cross-cap. The configurations of the lines of principal curvatures on the cross-cap are thus as in Table 3 third column, last figures.

We turn now to the lightlike cross-cap.

Theorem 6.3 *The BDE* (15) *of the pre-lines of principal curvature of a lightlike cross-cap parametrised as in* Theorem 3.2(c) *is topologically equivalent to one of the following normal forms:*

$(xy, -x^2, 3xy + 3y^2)$ *if* $p_{20} < 0$, Table 4 first column second row for $|c| > 2$
$(xy, -x^2, 3xy + y^2)$ *if* $p_{20} < 0$, Table 4 first column third row for $|c| < 2$
$(xy, x^2, -3xy + y^2)$ *if* $p_{20} > 0$, Table 4 first colum fourth row,

provided that $c = 2(q_{20} - p_{20}q_{22})/q_{21}\sqrt{|p_{20}|} \neq 0, \pm 2$ *and* $(q_{22}p_{20} - q_{20})(q_{21}(q_{22}^2 - 1) + 2q_{33}) \neq 0$.

Proof The 2-jet of the BDE (15) is given by

$$(4q_{21}xy, 4q_{21}p_{20}x^2, -12q_{21}p_{20}xy + 8(q_{20} - p_{20}q_{22})y^2).$$

When $p_{20} < 0$ we can change coordinates and multiply by a non zero function and reduce it to

$$(xy, -x^2, 3xy + cy^2),$$

with $c = 2(q_{20} - p_{20}q_{22})/q_{21}\sqrt{-p_{20}}$. Similarly, when $p_{20} > 0$ we can write the 2-jet in the form

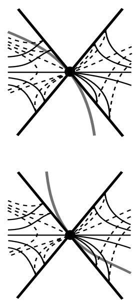
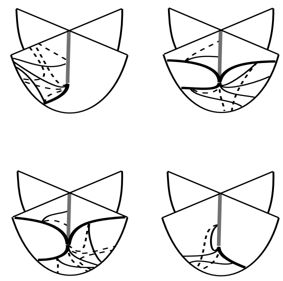
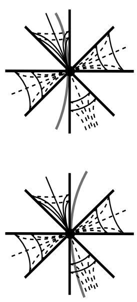
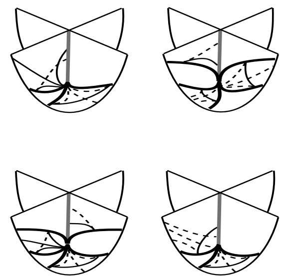
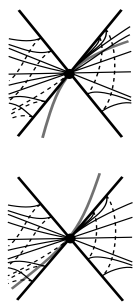
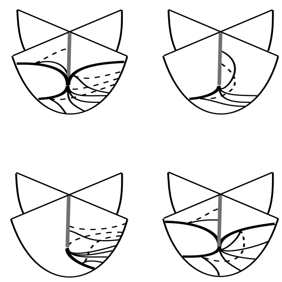
$$(xy, x^2, -3xy + cy^2),$$

with $c = 2(q_{20} - p_{20}q_{22})/q_{21}\sqrt{p_{20}}$.

The above cases are not studied previously. We deal with them in details in section 7. The result follows by applying Theorem 7.2 in section 7. □

In this case too it is not difficult to map the configurations in Table 4 first column to the cross-cap as all pre-lightlike curves intersect the double point curve in at most one point. As pointed out in Remark 5.4 the topological configuration in the source do not take into consideration the double point curve. This curve is drawn in grey in Table 4 first column and we have two possible configurations for its relative position with respect to the $PLPP$ for the cases in the second and fourth rows in Table 4 first column (the cases are determined by the sign of Λ in the proof of Proposition 6.1). The configurations of the lines of principal curvatures on the lightlike cross-cap are as in Table 4 second column.

Table 4: Pre-lines and lines of principal curvature on a lightlike cross-cap $\mathbb{R}^3_1$ viewed from two opposite directions.

Pre-lines of principal curvature	Lines of principal curvaure
	
	
	

7 Normal forms of certain BDEs

We obtain here topological normal forms of BDEs needed in the previous sections. A germ of a BDE is an equation in the form

$$\omega : \; a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0,$$

where a, b, c are germs of smooth functions (say, at the origin) $\mathbb{R}^2, 0 \to \mathbb{R}$. We denote a BDE by $\omega = (a, b, c)$. BDEs are extensively studied with many applications including control theory and differential geometry; see for example [6, 15] and [21] for a survey article. A BDE determines a pair of transverse foliations away from the discriminant curve which is the set of points where the function $\delta = b^2 - ac$ vanishes. The pair of foliations together with the discriminant curve are called the *configuration* of the solutions of the BDE.

Following the notation in [11], let $f_i(w)$, $i = 1, 2$, denote the foliation associated to ω which is tangent to the vector field

$$\xi_i(\omega) = a\frac{\partial}{\partial u} + (-b + (-1)^i\sqrt{b^2 - ac}\,)\frac{\partial}{\partial v}.$$

If ψ is a diffeomorphism and $\lambda(x, y)$ is a non-vanishing real valued function, then ([11]) for $k = 1, 2$,

1. $\psi(f_k(w)) = f_k(\psi^*(\omega))$ if ψ is orientation preserving;
2. $\psi(f_k(w)) = f_{3-k}(\psi^*(\omega))$ if ψ is orientation reserving;
3. $f_k(\lambda w) = f_k(\omega)$ if $\lambda(x, y)$ is positive;
4. $f_k(\lambda w) = f_{3-k}(\omega)$ if $\lambda(x, y)$ is negative.

7.1 BDEs with 1-jet $(0, \pm y, x)$

We consider BDEs ω with 1-jet equivalent to $(0, \pm y, x)$ and whose discriminants have an A_2-singularity (see section 5). We shall take $j^1\omega = (0, \pm y, x)$. Similar calculation to those carried out in [3, 4, 21] show that any k-jet, $k \geq 3$, of ω can be reduced by smooth changes of coordinates in $\mathbb{R}^2, 0$ and multiplication by a non-zero polynomial to one in the form

$$(M_1(x), \pm y, x + M_2(y)), \tag{18}$$

where $M_1(x) = a_2x^2 + a_3x^3 + \ldots + a_kx^k$ with $a_2 \neq 0$ and $M_2(y) = b_3y^3 + \ldots + b_ky^k$. When $a_2 \neq 0$, we can re-scale and set $a_2 = 1$.

Theorem 7.1 *Suppose that $j^1\omega = (0, \varepsilon y, x), \varepsilon = \pm 1$ and that discriminant of ω has an A_2-singularity. Then ω is topologically determined by the 2-jet of its coefficients and is topologically equivalent to one of the following normal forms*

(i)	(x^2, y, x)	*Figure 3, bottom left*
(ii)	$(x^2, -y, x)$	*Figure 3, bottom right.*

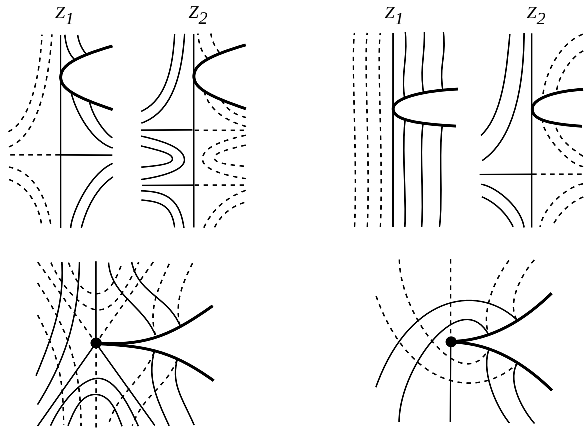


Figure 3: Configurations of the integral curves of a BDE ω with $j^1\omega = (0, \varepsilon y, x)$ and whose discriminant has an A_2-singularity, together with their associated blowing up models: $\varepsilon = -1$ left and $\varepsilon = 1$ right.

Proof We take ω as in (18) and consider the directional blowing-up $x = u, y = uv$ and $x = uv, y = v$.

The blowing-up $x = u, y = uv$.

The new BDE is given by $\omega_0 = (u,v)^*\omega = \bar{a}dv^2 + 2\bar{b}dudv + \bar{c}du^2$ with

$$\begin{array}{rcl} \bar{a} & = & u^2M_1(u), \\ \bar{b} & = & uv(\varepsilon v + M_1(u)), \\ \bar{c} & = & u + 2\varepsilon uv^2 + M_1(u)v^2 + M_2(uv). \end{array}$$

We can write $\omega_0 = u(u^2A_1, uB_1, C_1)$ with

$$\begin{array}{rcl} A_1 & = & uN_1(u), \\ B_1 & = & \varepsilon v + uvN_1(u), \\ C_1 & = & 1 + 2\varepsilon v^2 + u(N_1(u)v^2 + N_2(uv)) \end{array}$$

where $M_1(u) = u^2N_1(u)$ and $M_2(uv) = u^2N_2(uv)$.

The quadratic form $\omega_1 = (u^2A_1, uB_1, C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vectors fields

$$Z_i = \Big(-uB_1 + (-1)^i u\sqrt{B_1^2 - A_1C_1}\Big)\frac{\partial}{\partial u} + C_1\frac{\partial}{\partial v},\ i = 1, 2.$$

The blowing-up transformation is orientation preserving if u is positive and orientation reserving if u is negative. As we factored out u once, it follows that Z_1 is tangent to the foliation associated to $f_1(\omega)$ and Z_2 is tangent to the foliation associated to $f_2(\omega)$.

The fields $Z_i, i = 1,2$ are defined in the region where $B_1^2 - A_1C_1 > 0$. The set $B_1^2 - A_1C_1 = 0$ is a smooth curve tangent to the exceptional fibre at $u = 0$ and we have $(B_1^2 - A_1C_1)(0,v) = v^2$, so the whole exceptional fibre is an integral curve for both Z_1 and Z_2.

We study the vector fields Z_i in a neighbourhood of the exceptional fibre $u = 0$. The singularities of Z_i on $u = 0$ occur when $1 + 2\varepsilon v^2 = 0$. Thus, the vector fields Z_1 and Z_2 have singularities at $v = \pm\sqrt{2}/2$ when $\varepsilon = -1$ and have no singularities when $\varepsilon = 1$.

Consider $\varepsilon = -1$. At $v = \sqrt{2}/2$, we have $B_1(0, \sqrt{2}/2) = -\sqrt{2}/2$, so that

$$\begin{aligned} -uB_1 - u\sqrt{B_1^2 - A_1C_1} &= -uB_1 + uB_1\sqrt{1 - A_1C_1/B_1^2} \\ &= \frac{A_1C_1}{2B_1} + C_1^2 g(u,v) \end{aligned}$$

for some germ of a smooth function g with a zero 1-jet at the origin. Therefore Z_1 is singular along the curve $C_1(u,v) = 0$. We replace Z_1 with the vector field $\tilde{Z}_1 = Z_1/C_1$, which is regular along the exceptional fibre.

At $v = -\sqrt{2}/2$, the eigenvalues of the linear part of Z_1 at a singularity are $2\sqrt{2}$ and $-\sqrt{2}$, so Z_1 has a saddle singularity at this point.

Similar calculations to those for Z_1 show that Z_2 has a saddle singularity at $v = \sqrt{2}/2$ and its regular at the $v = -\sqrt{2}/2$.

The blowing-up $x = uv, y = v$

The new BDE is given by $\omega_0 = (u,v)^*\omega = \bar{a}dv^2 + 2\bar{b}dudv + \bar{c}du^2$ with

$$\begin{aligned} \bar{a} &= u^3v + 2\varepsilon uv + u^2M_2(v) + M_1(uv), \\ \bar{b} &= u^2v^2 + \varepsilon v^2 + uvM_2(v), \\ \bar{c} &= uv^3 + v^2M_2(v). \end{aligned}$$

We can write $\omega_0 = v(A_1, vB_1, v^2C_1)$ with

$$\begin{aligned} A_1 &= u^3 + 2\varepsilon u + v(u^2N_2(v) + N_1(uv)), \\ B_1 &= u^2 + \varepsilon + vuN_2(v), \\ C_1 &= u + vN_2(v) \end{aligned}$$

where $M_1(uv) = v^2N_1(uv)$ and $M_2(v) = v^2N_2(v)$.

The quadratic form $\omega_1 = (A_1, vB_1, v^2C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vectors fields

$$Z_i = \Big(- B_1 + (-1)^i\sqrt{B_1^2 - A_1C_1}\Big)\frac{\partial}{\partial u} + vC_1\frac{\partial}{\partial v},\ i = 1,2$$

with

$$\begin{array}{rcl} A_1 & = & u^3+2\epsilon u+v(N_1(uv)+u^2N_2(v)),\\ B_1 & = & u^2+\epsilon+v(uN_2(v)),\\ C_1 & = & u+vN_2(v). \end{array}$$

We only need to study the vector fields Z_i at origin. Similar calculations to the first blowing-up show that Z_1 has a saddle singularity (resp. has no singularity) and Z_2 has no singularity (resp. has a saddle singularity) when $\varepsilon=-1$ (resp. $\varepsilon=1$). Therefore, the integral curves of Z_1 and Z_2 are in as Figure 3, top left. Blowing down yields the configuration in Figure 3, bottom left □

7.2 BDEs with 2-jet $(xy, \varepsilon x^2, -3\varepsilon xy+cy^2)$

We take $j^2\omega=(xy,\varepsilon x^2,-3\varepsilon xy+cy^2)$, $\varepsilon=\pm1$. Then the 4-jet of the discriminant of ω is given by

$$x(x^3+3\varepsilon xy^2-cy^3).$$

The discriminant has an $X_{1,0}$-singularity if $c\neq 0,\pm2$ when $\varepsilon=-1$, and if $c\neq0$ when $\varepsilon=1$.

We write

$$\omega=(a,b,c)=(xy+M_1(x,y),\varepsilon x^2+M_2(x,y),-3\varepsilon xy+cy^2+M_3(x,y)), \qquad (19)$$

where $M_i(x,y)$, $i=1,2,3$, are germs of smooth functions with zero 2-jets at the origin. We set

$$\begin{array}{rcl} j^3M_1 & = & a_{30}x^3+a_{31}x^2y+a_{32}xy^2+a_{33}y^3,\\ j^3M_2 & = & b_{30}x^3+b_{31}x^2y+b_{32}xy^2+b_{33}y^3,\\ j^3M_3 & = & c_{30}x^3+c_{31}x^2y+c_{32}xy^2+c_{33}y^3. \end{array}$$

Theorem 7.2 *Suppose that $j^2\omega=(xy,\varepsilon x^2,-3\varepsilon xy+cy^2),\varepsilon=\pm1$, $c\neq0,\pm2$ if $\varepsilon=-1$, $c\neq0$ if $\varepsilon=1$. Suppose further that $(b_{33}-2ca_{33})\neq0$. Then ω is topologically determined by the 2-jet of its coefficients and is topologically equivalent to one of the following normal forms*

(i)	$(xy,-x^2,3xy+3y^2)$	*Table 4 first colum second row,*
(ii)	$(xy,-x^2,3xy+y^2)$	*Figure 4 first colum third row,*
(iii)	$(xy,x^2,-3xy+y^2)$	*Figure 4 first colum fourth row.*

Proof We start with the case the case $\varepsilon=-1$. We consider the directional blowing-up $x=uv$, $y=v$ and $x=u$, $y=uv$.

The blowing-up $x=uv$, $y=v$:

The new BDE is given by $\omega_0=(u,v)^*\omega=\bar a dv^2+2\bar b dudv+\bar c du^2$ with

$$\begin{array}{rcl} \bar a & = & (u^3+cu^2+u)v^2+M_3(uv,v)u^2+2M_2(uv,v)u+M_1(uv,v),\\ \bar b & = & (2u^2+cu)v^3+M_2(uv,v)v+M_3(uv,v)uv,\\ \bar c & = & (3u+c)v^4+M_3(uv,v)v^2. \end{array}$$

We can write $\omega_0 = v^2(A_1, vB_1, v^2C_1)$ with

$$\begin{array}{rcl} A_1 & = & u^3 + cu^2 + u + v(N_1(u,v) + 2N_2(u,v)u + N_3(u,v)u^2), \\ B_1 & = & u(2u+c) + v(N_2(u,v) + N_3(u,v)u), \\ C_1 & = & 3u + c + vN_3(u,v) \end{array}$$

where $M_i(uv, u) = v^2 N_i(u,v)$, $i = 1,2,3$. The quadratic form $\omega_1 = (A_1, vB_1, v^2C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vectors fields

$$Y_i = A_1 \frac{\partial}{\partial u} + \Big(- vB_1 + (-1)^i |v| \sqrt{(B_1^2 - A_1C_1)} \Big) \frac{\partial}{\partial v}, \quad i = 1,2. \tag{20}$$

Here, as we factored out v twice, it follows that Y_1 is tangent to the foliation associated to $f_1(\omega)$ if v is positive and to that associated to $f_2(\omega)$ if v is negative; while Y_2 is tangent to the foliation associated to $f_2(\omega)$ if v is positive and to the associate to $f_1(\omega)$ if v is negative.

We study the vector fields Y_i in a neighbourhood of the exceptional fibre $v = 0$. The fields Y_i are only defined in the regions where the discriminant $\delta = B_1^2 - A_1C_1 \geq 0$. On $v = 0$, this means that

$$u(u^3 - 3u - c) \geq 0.$$

The above segment of the exceptional fibre is an integral curve of both fields $Y_i, i = 1,2$. The discriminant δ has two roots if $|c| > 2$ and four roots if $|c| < 2$.

We start with the case $|c| > 2$ and take $c > 2$ (the case $c < -2$ is topologically equivalent to the case $c > 2$). The singularities of Y_1 on $v = 0$ occur when $A_1(u, 0) = 0$, that is, when

$$u(u^2 + cu + 1) = 0.$$

Thus, Y_1 has singularities at $u = 0$ and $u_\pm = (-c \pm \sqrt{c^2-4})/2$.

At $u_+ = (-c + \sqrt{c^2-4})/2$, we have $B_1(u_+, 0) = \sqrt{c^2-4}\,(-c + \sqrt{c^2-4})/2 < 0$, so that

$$\begin{array}{rcl} -vB_1 - |v|\sqrt{B_1^2 - A_1C_1} & = & -vB_1 - |vB_1|\sqrt{1 - A_1C_1/B_1^2} \\ & = & -vB_1 + |v|B_1 - \dfrac{A_1C_1|v|}{2B_1} + A_1^2 g(u,v) \end{array}$$

for some germ of a smooth function g with a zero 1-jet at the origin. When $v > 0$, Y_1 is singular along the curve $A_1(u,v) = 0$. We consider the vector field $\tilde{Y}_1 = Y_1/A_1$. Then $\tilde{Y}_1$ has no singularity. When $v < 0$, Y_1 has a saddle singularity at $(u_+, 0)$.

Similarly, Y_1 has a saddle singularity at $(u_-, 0)$ if $v > 0$ and no singularities if $v < 0$.

The singularity of Y_1 at $u = 0$ occur at the point of intersection of the exceptional fibre with the branches of the blown-up discriminant. We change variables and set

$t = v$, $s^2 = B_1^2 - A_1C_1$, with $s \geq 0$. The 2-jet of the vector field $(s,t)^*Y_1$ is equivalent to

$$(-s + \Lambda_1 t)\frac{\partial}{\partial s} + (2ts)\frac{\partial}{\partial t} \quad \text{if} \quad v > 0$$
$$(s + \Lambda_1 t)\frac{\partial}{\partial s} + (2ts)\frac{\partial}{\partial t} \quad \text{if} \quad v < 0, \tag{21}$$

where $\Lambda_1 = b_{33} - 2ca_{33}$. The singularity of $(s,t)^*Y_1$ is a saddle-node provided $\Lambda_1 \neq 0$, and its integral curves are as in Figure 4.

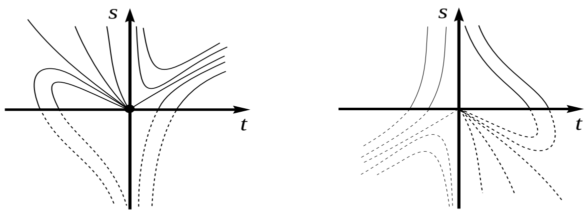


Figure 4: Integral curves of $(s,t)^*Y_1(s \geq 0)$, $\Lambda_1 > 0$ left, and $\Lambda_1 < 0$ right.

The singularities of Y_2 on $v = 0$ occur when $A_1(u,0) = 0$, that is, when

$$u(u^2 + cu + 1) = 0.$$

Therefore, Y_2 has singularities at $u = 0$ and $u_\pm = (-c \pm \sqrt{c^2-4})/2$.

At $u_+ = (-c + \sqrt{c^2-4})/2$, we have $B_1(u_+,0) = \sqrt{c^2-4}\,(-c+\sqrt{c^2-4})/2 < 0$. Following the same arguments for Y_1, we can write

$$-vB_1 - |v|\sqrt{B_1^2 - A_1C_1} = -vB_1 + |v|B_1 - \frac{A_1C_1|v|}{2B_1} + A_1^2 g(u,v)$$

for some germ of a smooth function g with a zero 1-jet at the origin. When $v < 0$, Y_2 is singular along the curve $A_1(u,v) = 0$. We consider the vector field $\tilde{Y}_2 = Y_2/A_1$. Then $\tilde{Y}_2$ has no singularities. When $v > 0$, Y_2 has a saddle singularity at $(u_+, 0)$.

Similarly, Y_2 has a saddle singularity at $(u_-, 0)$ if $v < 0$ and no singularities if $v > 0$.

At $u = 0$, we change variables and set $t = v$, $s^2 = B_1^2 - A_1C_1$, with $s \geq 0$. The 2-jet of the vector field $(s,t)^*Y_2$ is equivalent to

$$(s + \Lambda_1 t)\frac{\partial}{\partial s} + (2ts)\frac{\partial}{\partial t} \quad \text{if} \quad v > 0$$
$$(-s + \Lambda_1 t)\frac{\partial}{\partial s} + (2ts)\frac{\partial}{\partial t} \quad \text{if} \quad v < 0 \tag{22}$$

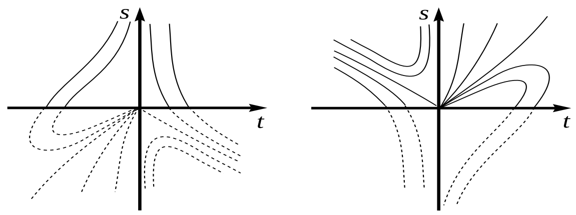


Figure 5: Integral curves of $(s,t)^*Y_2(s \geq 0)$, $\Lambda_1 > 0$ left, and $\Lambda_1 < 0$ right.

where $\Lambda_1 = b_{33} - 2ca_{33}$, as for Y_1. The singularity of $(s,t)^*Y_2$ is a saddle-node provided $\Lambda_1 \neq 0$, and its integral curves are as in Figure 5.

The blowing-up $x = u, y = uv$:

Consider the coefficients of the BDE as in (19). Then the new BDE is given by $\omega_0 = u^2(u^2A_1, uB_1, C_1)$ with

$$\begin{array}{rcl} A_1 & = & v + uN_1(u, uv), \\ B_1 & = & v^2 - 1 + u(N_1(u, uv)v + N_2(u, uv)), \\ C_1 & = & v^3 + cv^2 + v + u(N_1(u, uv)v^2 + 2N_2(u, uv)v + N_3(u, uv)) \end{array}$$

where $M_i(uv, u) = v^2 N_i(u, v)$, $i = 1, 2, 3$. The quadratic form $\omega_1 = (u^2A_1, uB_1, C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vectors fields

$$X_i = (u^2A_1)\frac{\partial}{\partial u} + \Big(- uB_1 + (-1)^i |u| \sqrt{(B_1^2 - A_1C_1)}\Big)\frac{\partial}{\partial v}, \quad i = 1, 2.$$

These vector fields are tangent to the foliations defined by ω_1. It is clear that we can factor out the term u in X_i, with an appropriate sign change when $u < 0$. The vector fields

$$Y_i = (uA_1)\frac{\partial}{\partial u} + \Big(- B_1 + (-1)^i \sqrt{(B_1^2 - A_1C_1)}\Big)\frac{\partial}{\partial v}, \quad i = 1, 2,$$

are then considered. It is easy to see that Y_1 (resp. Y_2) has a node singularity (resp. has no singularities) at the origin.

We can now draw the integral curves of the fields Y_1 and Y_2, as illustrated in Figure 6, top figures, and blow down to obtain the configurations of the integral curves of the associated BDE (Figure 6, bottom figures).

We consider now the case $|c| < 2$. The singularities of $Y_i, i = 1, 2$ on $v = 0$ occur only at $u = 0$. At $u = 0$, the vector fields Y_i have a saddle-node singularity as in (21) and (22). The configurations of the integral curves of Y_i are as in Figure 6 right, top figures. Blowing-down yields the configuration of the integral curves of the original BDE.

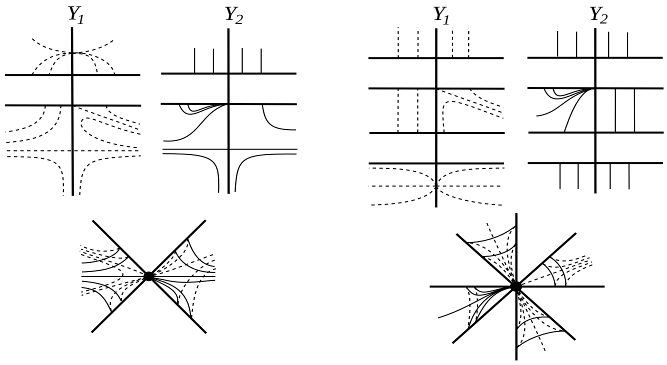


Figure 6: Configurations of the integral curves of the BDEs when $\varepsilon = -1$: $|c| > 2$ left, and $|c| < 2$ right and their associated blowing up.

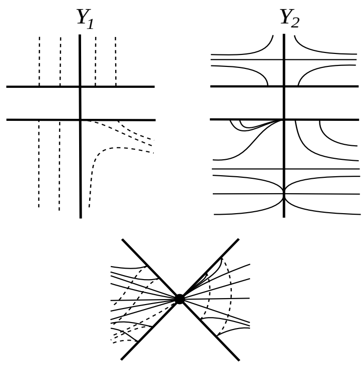


Figure 7: Configuration of the integral curves of the BDEs when $\varepsilon = 1$ and its associated blowing up.

The case $\varepsilon = 1$ follows similarly and the configuration of the integral curves is given in Figure 7.

□

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