(b) Use your answer in part (a) to solve the Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{in } \mathbb{O} \\ u(0, y, z) = 0, u(x, 0, z) = 0, \quad u(x, y, 0) = h(x, y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } x > 0, y > 0, z > 0. \end{cases}$$

19. Consider the four-dimensional laplacian $\Delta u = u_{xx} + u_{yy} + u_{zz} + u_{ww}$. Show that its fundamental solution is r^{-2}, where $r^2 = x^2 + y^2 + z^2 + w^2$.
20. Use Exercise 19 to find the Green's function for the half-hyperspace $\{(x, y, z, w)\colon w > 0\}$.
21. The *Neumann function* $N(x, y)$ for a domain D is defined exactly like the Green's function in Section 7.3 except that (ii) is replaced by the Neumann boundary condition

(ii)* $$\frac{\partial N}{\partial n} = c \quad \text{for } x \in \text{bdy } D.$$

for a suitable constant c.

(a) Show that $c = 1/A$, where A is the area of bdy D. ($c = 0$ if $A = \infty$)

(b) State and prove the analog of Theorem 7.3.1, expressing the solution of the Neumann problem in terms of the Neumann function.

22. Solve the Neumann problem in the half-plane: $\Delta u = 0$ in $\{y > 0\}$, $\partial u/\partial y = h(x)$ on $\{y = 0\}$ with $u(x, y)$ bounded at infinity. (*Hint:* Consider the problem satisfied by $v = \partial u/\partial y$.)
23. Solve the Neumann problem in the quarter-plane $\{x > 0, y > 0\}$.
24. Solve the Neumann problem in the half-space $\{z > 0\}$.
25. Let the nonconstant function $u(\mathbf{x})$ satisfy the inequality $\Delta u \geq 0$ in a domain D in three dimensions. Prove that it cannot assume its maximum inside D. This is the maximum principle for *subharmonic functions*. (*Hint:* Let $f = \Delta u$, and let h denote u restricted to the boundary bdy D. Let $B \subset D$ be any ball and let $\mathbf{x}_0$ be its center. Use (11) and (16) together with (7.3.7) in the ball B. Show that $u(\mathbf{x}_0)$ is at most the average of h on bdy B. Continue the proof as in Section 6.3.)