EXERCISES

1. Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = e^x$, $u_t(x, 0) = \sin x$.
2. Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = \log(1 + x^2)$, $u_t(x, 0) = 4 + x$.
3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?
4. Justify the conclusion at the beginning of Section 2.1 that *every* solution of the wave equation has the form $f(x + ct) + g(x - ct)$.
5. (*The hammer blow*) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c$, a/c, $3a/2c$, $2a/c$, and $5a/c$. [*Hint:* Calculate

$$u(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds = \frac{1}{2c}\{\text{length of } (x - ct, x + ct) \cap (-a, a)\}.$$

Then $u(x, a/2c) = (1/2c)$ {length of $(x - a/2,\ x + a/2) \cap (-a,\ a)$}. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]
6. In Exercise 5, find the greatest displacement, $\max_x u(x, t)$, as a function of t.
7. If both ϕ and ψ are odd functions of x, show that the solution $u(x, t)$ of the wave equation is also odd in x for all t.
8. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \quad \text{(“spherical wave equation”)}.$$

   (a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.
   (b) Solve for v using (3) and thereby solve the spherical wave equation.
   (c) Use (8) to solve it with initial conditions $u(r, 0) = \phi(r)$, $u_t(r, 0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.
9. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x, 0) = x^2$, $u_t(x, 0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)
10. Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$.
11. Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$.

is the cornerstone of the theory of relativity. It means that a signal located at the position x_0 at the instant t_0 cannot move faster than the speed of light. The domain of influence of (x_0, t_0) consists of all the points that can be reached by a signal of speed c starting from the point x_0 at the time t_0. It turns out that the solutions of the *three*-dimensional wave equation always travel at speeds exactly equal to c and never slower. Therefore, the causality principle is sharper in three dimensions than in one. This sharp form is called *Huygens's principle* (see Chapter 9).

Flatland is an imaginary two-dimensional world. You can think of yourself as a waterbug confined to the surface of a pond. You wouldn't want to live there because Huygens's principle is not valid in two dimensions (see Section 9.2). Each sound you make would automatically mix with the "echoes" of your previous sounds. And each view would be mixed fuzzily with the previous views. Three is the best of all possible dimensions.

EXERCISES

1. Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$. (*Hint:* Use the first vanishing theorem in Section A.1.)
2. For a solution $u(x, t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.
   (a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.
   (b) Show that both $e(x, t)$ and $p(x, t)$ also satisfy the wave equation.
3. Show that the wave equation has the following invariance properties.
   (a) Any translate $u(x - y, t)$, where y is fixed, is also a solution.
   (b) Any derivative, say u_x, of a solution is also a solution.
   (c) The dilated function $u(ax, at)$ is also a solution, for any constant a.
4. If $u(x, t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

   $$u(x + h, t + k) + u(x - h, t - k) = u(x + k, t + h) + u(x - k, t - h)$$

   for all x, t, h, and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.
5. For the *damped* string, equation (1.3.3), show that the energy decreases.
6. Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

   $$u_{tt} = c^2 \left(u_{rr} + \frac{n-1}{r} u_r \right),$$

   where r is the spherical coordinate. Consider such a wave that has the special form $u(r, t) = \alpha(r) f(t - \beta(r))$, where $\alpha(r)$ is called the

This is one of the few fortunate examples that can be integrated. The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

Completing the square in the y variable, it is

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

We let $p = (y + 2kt - x)/\sqrt{4kt}$ so that $dp = dy/\sqrt{4kt}$. Then

$$u(x, t) = e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{\pi}} = e^{kt-x}.$$

By the maximum principle, a solution in a bounded interval cannot grow in time. However, this particular solution grows, rather than decays, in time. The reason is that the left side of the rod is initially very hot [$u(x, 0) \to +\infty$ as $x \to -\infty$] and the heat gradually diffuses throughout the rod. □

EXERCISES

1. Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

   Write your answer in terms of $\mathscr{E}\text{rf}(x)$.
2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.
3. Use (8) to solve the diffusion equation if $\phi(x) = e^{3x}$. (You may also use Exercises 6 and 7 below.)
4. Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.
5. Prove properties (a) to (e) of the diffusion equation (1).
6. Compute $\int_0^\infty e^{-x^2}\,dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)
7. Use Exercise 6 to show that $\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^{\infty} S(x, t)\,dx = 1.$$

8. Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \le |x| < \infty} S(x, t) \to 0 \qquad \text{as } t \to 0.$$

For the wave equation we have seen most of these properties already. That there is no maximum principle is easy to see. Generally speaking, the wave equation just moves information along the characteristic lines. In more than one dimension we'll see that it spreads information in expanding circles or spheres.

For the diffusion equation we discuss property (ii), that singularities are immediately lost, in Section 3.5. The solution is differentiable to all orders even if the initial data are not. Properties (iii), (v), and (vi) have been shown already. The fact that information is gradually lost [property (vii)] is clear from the graph of a typical solution, for instance, from $S(x, t)$.

As for property (i) for the diffusion equation, notice from formula (2.4.8) that the value of $u(x, t)$ depends on the values of the initial datum $\phi(y)$ for *all* y, where $-\infty < y < \infty$. Conversely, the value of ϕ at a point x_0 has an *immediate effect everywhere* (for $t > 0$), even though most of its effect is only for a short time near x_0. Therefore, the *speed of propagation is infinite.* Exercise 2(b) shows that solutions of the diffusion equation can travel at any speed. This is in stark contrast to the wave equation (and all hyperbolic equations).

As for (iv), there are several ways to see that *the diffusion equation is not well-posed for* $t < 0$ ("backward in time"). One way is the following. Let

$$u_n(x, t) = \frac{1}{n} \sin nx\, e^{-n^2 kt}. \tag{1}$$

You can check that this satisfies the diffusion equation for all x, t. Also, $u_n(x, 0) = n^{-1} \sin nx \to 0$ uniformly as $n \to \infty$. But consider any $t < 0$, say $t = -1$. Then $u_n(x, -1) = n^{-1} \sin nx\, e^{+kn^2} \to \pm\infty$ uniformly as $n \to \infty$ except for a few x. Thus u_n is close to the zero solution at time $t = 0$ but not at time $t = -1$. This violates the stability, in the uniform sense at least.

Another way is to let $u(x, t) = S(x, t + 1)$. This is a solution of the diffusion equation $u_t = ku_{xx}$ for $t > -1$, $-\infty < x < \infty$. But $u(0, t) \to \infty$ as $t \searrow -1$, as we saw above. So we cannot solve backwards in time with the perfectly nice-looking initial data $(4\pi k)^{-1} e^{-x^2/4}$.

Besides, any physicist knows that heat flow, brownian motion, and so on, are irreversible processes. Going backward leads to chaos.

EXERCISES

1. Show that there is no maximum principle for the wave equation.
2. Consider a traveling wave $u(x, t) = f(x - at)$ where f is a given function of one variable.
   (a) If it is a solution of the wave equation, show that the speed must be $a = \pm c$ (unless f is a linear function).
   (b) If it is a solution of the diffusion equation, find f and show that the speed a is arbitrary.