one of the eigenvalues is zero and all the others have the same sign, the PDE is called *parabolic*.

Ultrahyperbolic equations occur quite rarely in physics and mathematics, so we shall not discuss them further. Just as each of the three conic sections has quite distinct properties (boundedness, shape, asymptotes), so do each of the three main types of PDEs. □

More generally, if the coefficients are variable, that is, the a_{ij} are functions of **x**, the equation may be elliptic in one region and hyperbolic in another.

Example 2.

Find the regions in the *xy* plane where the equation

$$yu_{xx} - 2u_{xy} + xu_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Indeed, $\mathscr{D} = (-1)^2 - (y)(x) = 1 - xy$. So the equation is parabolic on the hyperbola ($xy = 1$), elliptic in the two convex regions ($xy > 1$), and hyperbolic in the connected region ($xy < 1$). □

If the equation is nonlinear, the regions of ellipticity (and so on) may depend on which solution we are considering. Sometimes nonlinear transformations, instead of linear transformations such as B above, are important. But this is a complicated subject that is poorly understood.

EXERCISES

1. What is the type of each of the following equations?
   (a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$.
   (b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.
2. Find the regions in the *xy* plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

   is elliptic, hyperbolic, or parabolic. Sketch them.
3. Among all the equations of the form (1), show that the only ones that are unchanged under all rotations (*rotationally invariant*) have the form $a(u_{xx} + u_{yy}) + bu = 0$.
4. What is the *type* of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

   Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.
5. Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$ by a change of dependent variable $u = ve^{\alpha x + \beta y}$ and then a change of scale $y' = \gamma y$.

6. Consider the equation $3u_y + u_{xy} = 0$.
   (a) What is its type?
   (b) Find the general solution. (*Hint:* Substitute $v = u_y$.)
   (c) With the auxiliary conditions $u(x, 0) = e^{-3x}$ and $u_y(x, 0) = 0$, does a solution exist? Is it unique?