

turns out that if you solve the wave equation in N dimensions, signals propagate sharply (i.e., Huygens's principle is valid) only for dimensions $N = 3, 5, 7, \dots$. Thus three is the "best of all possible" dimensions, the smallest dimension in which signals propagate sharply!

In fact, the method of spherical means can be generalized to any odd dimension ≥ 5 . For each odd dimension $n = 2m + 1$ we can "descend" to the even dimension $2m$ below it to get a formula that shows that Huygens's principle is false in $2m$ dimensions [CH].

EXERCISES

1. Prove that $\Delta(\bar{u}) = \overline{(\Delta u)}$ for any function; that is, the laplacian of the average is the average of the laplacian. (*Hint:* Write Δu in spherical coordinates and show that the angular terms have zero average on spheres centered at the origin.)
2. Verify that (3) is correct in the case of the example $u(x, y, z, t) \equiv t$.
3. Solve the wave equation in three dimensions with the initial data $\phi \equiv 0$, $\psi(x, y, z) = y$, by use of (3).
4. Solve the wave equation in three dimensions with the initial data $\phi \equiv 0$, $\psi(x, y, z) = x^2 + y^2 + z^2$. (*Hint:* Use (5).)
5. Where does a three-dimensional wave have to vanish if its initial data ϕ and ψ vanish outside a sphere?
6. (a) Let S be the sphere of center \mathbf{x} and radius R . What is the surface area of $S \cap \{|\mathbf{x}| < \rho\}$, the portion of S that lies within the sphere of center $\mathbf{0}$ and radius ρ ?
- (b) Solve the wave equation in three dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) \equiv 0$, $\psi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, and $\psi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, where A is a constant. Sketch the regions in space-time that illustrate your answer. (This is like the hammer blow of Section 2.1.)
- (c) Sketch the graph of the solution (u versus $|\mathbf{x}|$) for $t = \frac{1}{2}, 1$, and 2 , assuming that $\rho = c = A = 1$. (This is a "movie" of the solution.)
- (d) Sketch the graph of u versus t for $|\mathbf{x}| = \frac{1}{2}$ and 2 , assuming that $\rho = c = A = 1$. (This is what a stationary observer sees.)
- (e) Let $|\mathbf{x}_0| < \rho$. Ride the wave along a light ray emanating from $(\mathbf{x}_0, 0)$. That is, look at $u(\mathbf{x}_0 + t\mathbf{v}, t)$ where $|\mathbf{v}| = c$. Prove that

$$t \cdot u(\mathbf{x}_0 + t\mathbf{v}, t) \text{ converges as } t \rightarrow \infty.$$

(*Hint:* (a) Divide into cases depending on whether one sphere contains the other or not. Use the law of cosines. (b) Use Kirchhoff's formula.)

7. (a) Solve the wave equation in three dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, and $\psi|\mathbf{x}| \equiv 0$, where A is a constant. (This is somewhat like the plucked string.) (*Hint:* Differentiate the solution in Exercise 6(b).)

- (b) Sketch the regions in space-time that illustrate your answer. Where does the solution have jump discontinuities?
- (c) Let $|\mathbf{x}_0| < \rho$. Ride the wave along a light ray emanating from $(\mathbf{x}_0, 0)$. That is, look at $u(\mathbf{x}_0 + t\mathbf{v}, t)$ where $|\mathbf{v}| = c$. Prove that

$$t \cdot u(\mathbf{x}_0 + t\mathbf{v}, t) \text{ converges as } t \rightarrow \infty.$$

8. Carry out the derivation of the second term in (3).
9. (a) For any solution of the three-dimensional wave equation with initial data vanishing outside some sphere, show that $u(x, y, z, t) = 0$ for fixed (x, y, z) and large enough t .
- (b) Prove that $u(x, y, z, t) = O(t^{-1})$ uniformly as $t \rightarrow \infty$; that is, prove that $t \cdot u(x, y, z, t)$ is a bounded function of x, y, z , and t . (*Hint*: Use Kirchhoff's formula.)
10. Derive the mean value property of harmonic functions $u(x, y, z)$ by the following method. A harmonic function is a wave that happens not to depend on time, so that its mean value $\bar{u}(r, t) = \bar{u}(r)$ satisfies (5). Deduce that $\bar{u}(r) = u(\mathbf{0})$.
11. Find all the spherical solutions of the three-dimensional wave equation; that is, find the solutions that depend only on r and t . (*Hint*: See (5).)
12. Solve the three-dimensional wave equation in $\{r \neq 0, t > 0\}$ with zero initial conditions and with the limiting condition

$$\lim_{r \rightarrow 0} 4\pi r^2 u_r(r, t) = g(t).$$

Assume that $g(0) = g'(0) = g''(0) = 0$.

13. Solve the wave equation in the half-space $\{(x, y, z, t): z > 0\}$ with the Neumann condition $\partial u / \partial z = 0$ on $z = 0$, and with initial data $\phi(x, y, z) \equiv 0$ and general $\psi(x, y, z)$. (*Hint*: See (3) and use the method of reflection.)
14. Why doesn't the method of spherical means work for two-dimensional waves?
15. Obtain the general solution formula (19) in two dimensions from the special case (18).
16. (a) Solve the wave equation in two dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) \equiv 0$, $\psi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, and $\psi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, where A is a constant. Do not carry out the integral.
- (b) Under the same conditions find a simple formula for the solution $u(\mathbf{0}, t)$ at the origin by carrying out the integral.
17. Use the result of Exercise 16 to compute the limit of $t \cdot u(\mathbf{0}, t)$ as $t \rightarrow \infty$.
18. For any solution of the two-dimensional wave equation with initial data vanishing outside some circle, prove that $u(x, y, t) = O(t^{-1})$ for fixed (x, y) as $t \rightarrow \infty$; that is, $t \cdot u(x, y, t)$ is a bounded function of t for fixed x and y . Note the contrast to three dimensions. (*Hint*: Use formula (19).)