## EXERCISES

1. Verify the linearity and nonlinearity of the eight examples of PDEs given in the text, by checking whether or not equations (3) are valid.
2. Which of the following operators are linear?
   (a) $\mathscr{L}u = u_x + xu_y$
   (b) $\mathscr{L}u = u_x + uu_y$
   (c) $\mathscr{L}u = u_x + u_y^2$
   (d) $\mathscr{L}u = u_x + u_y + 1$
   (e) $\mathscr{L}u = \sqrt{1+x^2}\,(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$
3. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.
   (a) $u_t - u_{xx} + 1 = 0$
   (b) $u_t - u_{xx} + xu = 0$
   (c) $u_t - u_{xxt} + uu_x = 0$
   (d) $u_{tt} - u_{xx} + x^2 = 0$
   (e) $iu_t - u_{xx} + u/x = 0$
   (f) $u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$
   (g) $u_x + e^y u_y = 0$
   (h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$
4. Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same $g$ is a solution of the homogeneous equation $\mathscr{L}u = 0$.
5. Which of the following collections of 3-vectors $[a, b, c]$ are vector spaces? Provide reasons.
   (a) The vectors with $b = 0$.
   (b) The vectors with $b = 1$.
   (c) The vectors with $ab = 0$.
   (d) All the linear combinations of the two vectors [1, 1, 0] and [2, 0, 1].
   (e) All the vectors such that $c - a = 2b$.
6. Are the three vectors [1, 2, 3], [−2, 0, 1], and [1, 10, 17] linearly dependent or independent? Do they span all vectors or not?
7. Are the functions $1 + x$, $1 - x$, and $1 + x + x^2$ linearly dependent or independent? Why?
8. Find a vector that, together with the vectors [1, 1, 1] and [1, 2, 1], forms a basis of $\mathbb{R}^3$.
9. Show that the functions $(c_1 + c_2 \sin^2 x + c_3 \cos^2 x)$ form a vector space. Find a basis of it. What is its dimension?
10. Show that the solutions of the differential equation $u''' - 3u'' + 4u = 0$ form a vector space. Find a basis of it.
11. Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions $f$ and $g$ of one variable.

12. Verify by direct substitution that

$$u_n(x, y) = \sin nx \sinh ny$$

is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

## 1.2 FIRST-ORDER LINEAR EQUATIONS

We begin our study of PDEs by solving some simple ones. The solution is quite geometric in spirit.

The simplest possible PDE is $\partial u/\partial x = 0$ [where $u = u(x, y)$]. Its general solution is $u = f(y)$, where $f$ is any function of *one* variable. For instance, $u = y^2 - y$ and $u = e^y \cos y$ are two solutions. Because the solutions don't depend on $x$, they are constant on the lines $y =$ constant in the $xy$ plane.

### THE CONSTANT COEFFICIENT EQUATION

Let us solve

$$\boxed{au_x + bu_y = 0,} \tag{1}$$

where $a$ and $b$ are constants not both zero.

**Geometric Method** The quantity $au_x + bu_y$ is the directional derivative of $u$ in the direction of the vector $\mathbf{V} = (a, b) = a\mathbf{i} + b\mathbf{j}$. It must always be zero. This means that $u(x, y)$ must be constant in the direction of $\mathbf{V}$. The vector $(b, -a)$ is orthogonal to $\mathbf{V}$. The lines parallel to $\mathbf{V}$ (see Figure 1) have the equations $bx - ay =$ constant. (They are called the *characteristic lines*.) The solution is constant on each such line. Therefore, $u(x, y)$ depends on $bx - ay$ only. Thus the solution is

$$\boxed{u(x, y) = f(bx - ay),} \tag{2}$$

where $f$ is any function of one variable. Let's explain this conclusion more explicitly. On the line $bx - ay = c$, the solution $u$ has a constant value. Call

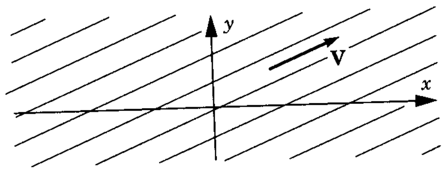


Figure 1

uniqueness). It is also unstable. To illustrate the instability further, consider a nonsingular matrix $A$ with one very small eigenvalue. The solution is unique but if $b$ is slightly perturbed, then the error will be greatly magnified in the solution $u$. Such a matrix, in the context of scientific computation, is called ill-conditioned. The ill-conditioning comes from the instability of the matrix equation with a singular matrix.

As a fourth example, consider Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $D = \{-\infty < x < \infty, 0 < y < \infty\}$. It is *not* a well-posed problem to specify both $u$ and $u_y$ on the boundary of $D$, for the following reason. It has the solutions

$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny. \tag{2}$$

Notice that they have boundary data $u_n(x, 0) = 0$ and $\partial u_n/\partial y(x, 0) = e^{-\sqrt{n}} \sin nx$, which tends to zero as $n \to \infty$. But for $y \neq 0$ the solutions $u_n(x, y)$ do not tend to zero as $n \to \infty$. Thus the stability condition (iii) is violated.

## EXERCISES

1. Consider the problem

$$\frac{d^2u}{dx^2} + u = 0$$
$$u(0) = 0 \quad \text{and} \quad u(L) = 0,$$

consisting of an ODE and a pair of boundary conditions. Clearly, the function $u(x) \equiv 0$ is a solution. Is this solution *unique, or not*? Does the answer depend on $L$?

2. Consider the problem

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \tfrac{1}{2}[u'(l) + u(l)],$$

with $f(x)$ a given function.
   (a) Is the solution *unique*? Explain.
   (b) Does a solution necessarily *exist*, or is there a condition that $f(x)$ must satisfy for existence? Explain.

3. Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

4. Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D.$$

(a) What can we surely add to any solution to get another solution? So we don't have uniqueness.
(b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = 0$$

is a necessary condition for the Neumann problem to have a solution.
(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

5. Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition $u(x, 0) = \phi(x)$.
(a) For $\phi(x) \equiv x$, show that no solution exists.
(b) For $\phi(x) \equiv 1$, show that there are many solutions.

6. Solve the equation $u_x + 2xy^2u_y = 0$.

## 1.6 TYPES OF SECOND-ORDER EQUATIONS

In this section we show how the Laplace, wave, and diffusion equations are in some sense typical among all second-order PDEs. However, these three equations are quite different from each other. It is natural that the Laplace equation $u_{xx} + u_{yy} = 0$ and the wave equation $u_{xx} - u_{yy} = 0$ should have very different properties. After all, the *algebraic* equation $x^2 + y^2 = 1$ represents a circle, whereas the equation $x^2 - y^2 = 1$ represents a hyperbola. The parabola is somehow in between.

In general, let's consider the PDE

$$\boxed{a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.} \tag{1}$$

This is a linear equation of order two in two variables with six real constant coefficients. (The factor 2 is introduced for convenience.)

**Theorem 1.** By a linear transformation of the independent variables, the equation can be reduced to one of three forms, as follows.

(i) *Elliptic case:* If $a_{12}^2 < a_{11}a_{22}$, it is reducible to

$$u_{xx} + u_{yy} + \cdots = 0$$

(where $\cdots$ denotes terms of order 1 or 0).

(ii) *Hyperbolic case:* If $a_{12}^2 > a_{11}a_{22}$, it is reducible to

$$u_{xx} - u_{yy} + \cdots = 0.$$