Theorem 0.1. *Given $\kappa > 2^{\aleph_0}$, it is consistent with ZFC that there is a Lindelöf first countable T_1 space of cardinality κ.*

Before we begin our construction, let us fix some sets. Fixed an infinite cardinal κ, let $\langle A_\alpha : \alpha < \kappa\rangle$ be such that

- each $A_\alpha = \{a_n^\alpha : n \in \omega\} \subset \kappa$;
- $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$;
- $\kappa = \bigcup_{\alpha<\kappa} A_\alpha$.

The sequence $\langle B_n : n \in \omega\rangle$ given by the next lemma will be used in our construction:

Lemma 0.2. *There is a sequence $\langle B_n : n \in \omega\rangle$ of infinite subsets of ω such that:*

1. *$\langle B_n : n \in \omega\rangle$ is almost disjoint;*
2. *For every $s \in 2^{<\omega}$ and every $n \in \omega$, there is an $m > n$ such that*

$$\chi_{B_m} \in [s] \wedge \forall i \leq n\ (B_m \cap B_i \cap (\omega \setminus |s|)) = \emptyset.$$

Note that, in particular, $\{\chi_{B_n} : n \in \omega\}$ is dense in the usual topology of 2^ω.

Proof. Let $\langle s_i : i < \omega\rangle$ be an enumeration of $2^{<\omega}$ such that each term is listed infinitely many times. Suppose that $\langle B_j : j < i\rangle$ is almost disjoint and $\omega \setminus \bigcup_{j<i} B_j$ is infinite. Let B_i be an infinite set such that

- $\chi_{B_i} \in [s_i]$;
- $B_i \cap B_j \cap (\omega \setminus |s_i|) = \emptyset$ for every $j < i$;
- $\omega \setminus \bigcup_{j \leq i} B_j$ is infinite.

Since every s_i is repeated infitely many times, the sequence is as desired. $\square$

Before the definition of the forcing, a last piece of notation: for each $\alpha \in \kappa$ and $n \in \omega$, define $A_{\alpha,n} = \{a_j^\alpha : j \in B_n\}$. Therefore, $A_\alpha = \bigcup_{n\in\omega} A_{\alpha,n}$ and, given $m, n \in \omega$, $A_{\alpha,n} \cap A_{\alpha,m}$ is finite if $m \neq n$.

Each condition of our forcing $\mathbb{P}$ will be of the form $p = \langle I, A, F, \mathcal{U}, \ell, r\rangle$ (as usual, when needed, we will add a p as an upper index, e.g., I^p) such that:

- $I \in [\kappa]^\omega$;
- $A = \bigcup_{\alpha \in I} A_\alpha$;
- $F = \{x_{\alpha,n} \in 2^A : \alpha \in I \wedge n \in \omega\}$ satisfying the following properties:
  - $x_{\alpha,n} \restriction A_\alpha = \chi_{A_{\alpha,n}}$ for every $\alpha \in I$ and $n \in \omega$;
  - $x_{\alpha,m} \restriction (A \setminus A_\alpha) = x_{\alpha,n} \restriction (A \setminus A_\alpha)$ for every $\alpha \in I$ and $m, n \in \omega$;
  - if $\alpha, \beta \in I$ are distinct, then for every $m, n \in \omega$, $x^{-1}_{\alpha,m}(1) \cap A_{\beta,n}$ is finite;
  - We are considering on F the topology generated by sets of the form $\{U_\alpha : \alpha \in A\}$, where $U_\alpha = \{x \in F : x(\xi) = 1\}$. Note that this topology is properly contained in the usual topology of 2^I. For $s \in [I]^{<\omega}$, define $U_s = \bigcap_{i \in s} U_i$. Note that the collection of all U_s is a base for the fixed topology.
- $\mathcal{U}$ is a countable family of countable open coverings of F made by basic open sets of the topology defined above;
- Given $S \subset A$ and $\xi \in A$, we say ξ requires S if, given $x \in F$, $x \restriction S \equiv 1 \Rightarrow x(\xi) = 1$. In this case, we will use the notation $S \vDash \xi$ (in other words, $S \vDash \xi$ means that $F \cap U_S \subset U_\xi$).
- Given $\alpha \in I$ and $n \in \omega$, let

  $$r(\alpha, n) = \{j \in B_n : \exists \beta \in I \setminus \{\alpha\} \ \exists m \in \omega \ (x^{-1}_{\alpha,n}(1) \cap A_{\beta,m}) \vDash a^\alpha_j\}.$$

  We impose that $r(\alpha, n)$ is finite;
- Given $x_{\alpha,n} \in F$ and $\xi = a^\beta_m \in A$ such that $x_{\alpha,n}(\xi) = 1$ and $\beta \neq \alpha$, there must be an $\ell(\alpha, n, \xi) \in \omega$ such that $S^{\alpha,n}_{\ell(\alpha,n,\xi)} \vDash \xi$, where $S^{\alpha,n}_k = \{a^\alpha_j : j \in B_n \wedge j \leq k\}$.

Given $p, q \in \mathbb{P}$, we define $q \leq p$ if, $I^q \supset I^p$, $\mathcal{U}^q \supset \mathcal{U}^p$, $\ell^q \supset \ell^p$, $r^q \supset r^p$ and $x^q_{\alpha,n} \restriction A^p = x^p_{\alpha,n}$ for every $x^q_{\alpha,n} \in F^q$ such that $\alpha \in A^p$.

Suppose that V is a model for GCH. Let G be a $\mathbb{P}$-generic over V. In $V[G]$, let $A = \bigcup_{p \in G} A^p$ (by density, we will have that $A = \kappa$ - see Proposition 0.9). For each $\alpha \in A$ and $n \in \omega$, let $x_{\alpha,n} = \bigcup_{p \in G} x^p_{\alpha,n}$ (if $\alpha \notin A^p$, $x^p_{\alpha,n} = \emptyset$). Note that if $\langle \alpha, n \rangle \neq \langle \beta, m \rangle$, then $x_{\alpha,n} \neq x_{\beta,m}$. This together with the density argument above imply that $X = \kappa$ in $V[G]$.

Our main theorem will follow from this:

Theorem 0.3. *In $V[G]$, X is a Lindelöf, first countable, T_1 space of cardinality κ.*

We will prove the previous theorem by a sequence of propositions. The first one follows directly from the definition of $\mathbb{P}$:

Proposition 0.4. *$\mathbb{P}$ is countably closed.*

Proposition 0.5. *X is T_1 and first countable.*

Proof. Note that the base made by sets of the form U_S generates a T_1 topology. Therefore, we only have to check the first countability. Fix $x_{\alpha,n} \in X$. We will show that $\{U_S : S \in [A_\alpha]^{<\aleph_0}\}$ contains a local base for $x_{\alpha,n}$. Let $H \subset \kappa$ be a finite set such that $x_{\alpha,n} \in U_H$. Let $\{\xi_1, ..., \xi_k\} = H \setminus A_\alpha$. By the defnition of the forcing, for each i, there is an $\ell(\alpha, n, \xi_i)$ such that $S^{\alpha,n}_{\ell(\alpha,n,\xi_i)} \vDash \xi_i$ - recall that this means that $U_{S^{\alpha,n}_{\ell(\alpha,n,\xi_i)}} \subset U_{\xi_i}$. Since each $S^{\alpha,n}_{\ell(\alpha,n,\xi_i)} \subset A_{\alpha,n}$ is finite,

$$S = (H \cap A_\alpha) \cup \bigcup_{i=1}^{k} S^{\alpha,n}_{\ell(\alpha,n,\xi_i)}$$

is finite and $x_{\alpha,n} \in U_S \subset U_H$ as required. □

Proposition 0.6. *X is Lindelöf.*

Proof. Let $p \in \mathbb{P}$ and $\dot{\mathcal{C}}$ be a name such that $p \Vdash \dot{\mathcal{C}}$ is a covering for $\dot{X}$. *** □

Lemma 0.7. *For every condition $p \in \mathbb{P}$ such that $\beta \in I^p$, for any $n' \in \omega$, $B_{n'} \cap \bigcup_{n\in\omega} r(\beta, n) = r(\beta, n')$.*

Proof. Since $r(\beta, n') \subset B_{n'}$ one of the inclusions is clear. For the other one, let $j \in B_{n'} \cap r(\beta, n)$ for some n. Let $\gamma \in I^p \setminus \{\beta\}$ and $m \in \omega$ be witnesses that $j \in r(\beta, n)$. Thus, by definition, $x^{-1}_{\beta,n}(1) \cap A_{\gamma,m} \vDash a^\beta_j$. Since $x^{-1}_{\beta,n'}(1) \cap A_{\gamma,m} = x^{-1}_{\beta,n}(1) \cap A_{\gamma,m}$ and since $j \in B_{n'}$, $j \in r(\beta, n')$ as desired. □

The following lemma will help us finding compatible conditions:

Lemma 0.8. *Let $p = \langle I, A, F, \mathcal{U}, \ell, r\rangle \in \mathbb{P}$. There is an $\tilde{x} \in 2^A$ such that:*

1. *$\tilde{x} \in \bigcup \mathcal{C}$ for every $\mathcal{C} \in \mathcal{U}$;*
2. *for all $m \in \omega$ and all $\alpha \in I$, $\tilde{x}^{-1}(1) \cap A_{\alpha,m}$ is finite;*

3. *suppose that for some $\eta \neq \gamma \in I$ and some $i, k, \ell \in \omega$, $S_\ell^{\eta,k} \vDash a_i^\gamma$. If $\tilde{x} \restriction S_\ell^{\eta,k} \equiv 1$, then $\tilde{x}(a_i^\gamma) = 1$.*

Proof. Fix $\beta \in I$. Define $\tilde{x} \restriction (A \setminus A_\beta) = x_{\beta,0} \restriction (A \setminus A_\beta)$.

So it only remains to define $\tilde{x} \restriction A_\beta$. By assumption,

$$r(\beta, k) = \{j \in B_k : \exists \gamma \in I^p \setminus \{\beta\} \ \exists m \in \omega \ (x_{\beta,k}^{-1}(1) \cap A_{\gamma,m}) \vDash a_j^\beta\}$$

is finite for every $k \in \omega$. Let $\mathcal{U}^p = \{\mathcal{C}_k : k \in \omega\}$. Let S_0 be such that $U_{S_0} \in \mathcal{C}_0$ and $x_{\beta,0} \in U_{s_0}$. By the definition of $x_{\beta,0}$, $S_0 \cap A_\beta \subset A_{\beta,0}$. Let $t_0 = \{j \in B_0 : a_j^\beta \in S_0\}$ and $t_0' = t_0 \cup r(\beta, 0)$. Note that $t_0' \subset B_0$ is finite. Let $n_0 = 0$.

By Lemma 0.2, there is an $n_1 > n_0$ such that $B_{n_1} \cap (\max t_0' + 1) = t_0'$ and $B_{n_1} \cap B_{n_0} \cap (\omega \setminus (\max t_0' + 1)) = \emptyset$.

Let S_1 be such that $U_{S_1} \in \mathcal{C}_1$ and $x_{\beta,n_1} \in U_{S_1}$. Then, by the definition of x_{β,n_1}, $S_1 \cap A_\beta \subset A_{\beta,n_1}$. Let $t_1 = \{j \in B_{n_1} : a_j^\beta \in S_1\}$. Let $t_1' = t_1 \cup \bigcup_{k \leq n_1} r(\beta, k)$. Note that t_1' is finite and $t_1' \cap B_0 \subset t_0' \cap B_0$ - to see this, note that $t_1 \cap B_0 \subset t_0'$ by the definition of t_1 and $B_0 \cap \bigcup_{k \leq n_1} r(\beta, k) \subset r(\beta, 0) \subset t_0'$ by the previous lemma.

In general, suppose that for some $k \geq 1$, we have sequences $\langle n_v : v \leq k \rangle$ and $\langle S_v : v \leq k \rangle$ such that $n_0 < n_1 < \cdots < n_k$ and, for each $v \leq k$:

- $U_{S_v} \in \mathcal{C}_v$, $x_{\beta,n_v} \in U_{S_v}$;
- $t_v = \{j \in B_{n_v} : a_j^\beta \in S_v \cap A_\beta\}$, $t_v' = t_v \cup \bigcup_{y \leq n_v} r(\beta, y)$.

Also, suppose that, for each $v \leq k$,

$$t_v' \cap B_y \subset t_\mu' \cap B_y \tag{1}$$

for all $\mu < v$ and $y \leq n_\mu$.

By Lemma 0.2, there is an $n_{k+1} > n_k$ such that $B_{n_{k+1}} \cap (\max t_k' + 1) = t_k'$ and $B_{n_{k+1}} \cap (\omega \setminus (\max t_k' + 1) \cap B_y = \emptyset$ for every $y \leq n_k$.

Let S_{k+1} be such that $U_{S_{k+1}} \in \mathcal{C}_{k+1}$ and $x_{\beta,n_{k+1}} \in U_{S_{k+1}}$. Let $t_{k+1} = \{j \in B_{n_{k+1}} : a_j^\beta \in S_{k+1}\}$. Let $t_{k+1}' = t_{k+1} \cup \bigcup_{y \leq n_{k+1}} r(\beta, y)$. We only need to check (1) for $v = k + 1$, which means

$$\left(t_{k+1} \cup \bigcup_{z \leq n_{k+1}} r(\beta, z) \right) \cap B_y \subset t_\mu' \cap B_y$$

for each $\mu \leq k$ and $y \leq n_\mu$. By the previous lemma, note that $B_y \cap \bigcup_{z \leq n_{k+1}} r(\beta, z) \subset r(\beta, y)$. Since $r(\beta, y) \subset t'_\mu$, it only remains to check that $t_{k+1} \cap B_y \subset t'_\mu \cap B_y$. For this, it is enough to prove

$$B_{n_{k+1}} \cap B_y \subset t'_\mu \cap B_y.$$

Note that $B_{n_{k+1}} \cap B_y \subset B_{n_{k+1}} \cap (\max t'_k + 1) \cap B_y = t'_k \cap B_y$. We have two cases. If $k = \mu$, we are done. Otherwise, if $\mu < k$, then, by the induction hypothesis, $t'_k \cap B_y \subset t'_\mu \cap B_y$ and we are done as well.

Let $T = \bigcup_{k \in \omega} t'_k$ and $T_\beta = \{a^\beta_j : j \in T\}$. Finally, define $\tilde{x} \restriction A_\beta = \chi_{T_\beta}$. Let us now check that $\tilde{x}$ satisfies all requirements.

1. $\tilde{x} \in \bigcup \mathcal{C}$ for every $\mathcal{C} \in \mathcal{U}$. Indeed, let $k \in \omega$ such that $\mathcal{C} = \mathcal{C}_k$. By construction, $U_{S_k} \in \mathcal{C}_k$. So it is enough to show that $\tilde{x} \in U_{S_k}$. Since $\tilde{x} \restriction (A \setminus A_\beta) = x_{\beta, n_k} \restriction (A \setminus A_\beta)$ and $x_{\beta, n_k} \in U_{S_k}$, $\tilde{x} \restriction (S_k \setminus A_\beta) \equiv 1$. On the other hand, $S_k \cap A_\beta = \{a^\beta_j : j \in t_k\}$. Thus $S_k \cap A_\beta \subset T_\beta$ which means that $\tilde{x} \restriction (S_k \cap A_\beta) \equiv 1$ as well.

2. let $m \in \omega$ and $\alpha \in I^p$. We need to show that $\tilde{x}^{-1}(1) \cap A_{\alpha,m}$ is finite. If $\alpha = \beta$, this is the same as showing that $T \cap B_m$ is finite. Let $k \geq m$. Then, for every $k' > k$, $t'_{k'} \cap B_m \subset t'_k \cap B_m$, which is finite.

   Now, if $\alpha \neq \beta$, then note that $\tilde{x} \restriction A_{\alpha,m} = x_{\beta,0} \restriction A_{\alpha,m}$. So, by hypothesis, $\tilde{x}^{-1}(1) \cap A_{\alpha,m}$ is finite.

3. suppose that for some $\eta \neq \gamma \in I$ and some $i, k, \ell \in \omega$, $S^{\eta,k}_\ell \vDash a^\gamma_i$. If $\tilde{x} \restriction S^{\eta,k}_\ell \equiv 1$, we need to prove that $\tilde{x}(a^\gamma_i) = 1$. First, suppose that $\gamma = \beta$. Let j be such that $i \in B_j$. Note that $S^{\eta,k}_\ell \subset A_{\eta,k}$. Also, $\tilde{x} \restriction A_\eta = x_{\beta,j} \restriction A_\eta$. Therefore $x_{\beta,j} \restriction S^{\eta,k}_\ell \equiv 1$ and then $x_{\beta,j}(a^\beta_i) = 1$. Therefore $i \in r(\beta, j) \subset t'_j$. Thus $\tilde{x}_\beta(a^\beta_i) = 1$ as required.

   If $\beta \neq \gamma$ and $\beta \neq \eta$, then it is enough to note that $\tilde{x} \restriction (A \setminus A_\beta) = x_{\beta,0} \restriction (A \setminus A_\beta)$. Finally, suppose that $\eta = \beta$. Since $x_{\beta,k} \restriction S^{\beta,k}_\ell \equiv 1$, $x_{\beta,k}(a^\gamma_i) = 1$. Therefore, $\tilde{x}(a^\gamma_i) = 1$, since $\tilde{x} \restriction A_\gamma = x_{\beta,k} \restriction A_\gamma$.

$\square$

Proposition 0.9. *For every $\alpha \in \kappa$, the set $\{p \in \mathbb{P} : \alpha \in I^p\}$ is dense.*

Proof. Let $p \in \mathbb{P}$ and suppose that $\alpha \notin I^p$. Define $I^q = I^p \cup \{\alpha\}$, $\mathcal{U}^q = \mathcal{U}^p$. We need to define the elements of F^q. For each $x^p_{\beta,n} \in F^p$, we extend it to

A_α doing $x^q_{\beta,n} \restriction A_\alpha \equiv 0$. Define $x_{\alpha,n} \restriction A_\alpha = \chi_{A_{\alpha,n}}$ and $x_{\alpha,n} \restriction I^p = \tilde{x}_\beta$, where $\tilde{x}_\beta$ is given by the previous lemma applied to a fixed $x_{\beta,0} \in F^p$.

Now, let us take care of the ℓ's. Since $x_{\gamma,k} \restriction A_\alpha \equiv 0$ for every $\gamma \neq \alpha$,

$$x_{\gamma,k} \restriction S^{\alpha,n}_{\ell(\alpha,n,\xi)} \equiv 1 \Rightarrow x_{\gamma,k}(\xi) = 1$$

is satisfied trivially, by noting that $S^{\alpha,n}_{\ell(\alpha,n,\xi)} \subset A_\alpha$. In particular, $\ell(\alpha, n, \xi) = 0$ works for every $\xi = a^\gamma_j$ with $\gamma \neq \alpha$. On the other hand, let $\gamma \in I^p$ and let $\xi = a^\eta_j$ with $\eta \neq \gamma$. By the third condition of the previous lemma, the same $\ell(\gamma, n, \xi)$ that worked in p works in q. Finally, let $\gamma \in I^p$ and let $\xi = a^\alpha_j$. Note that, since $x_{\gamma,n}(\xi) = 0$, there is need to verify $\ell(\gamma, n, \xi)$.

Finally, let us check the condition about r's. Fixed $\gamma \in I^p$ and $n \in \omega$, since $x_{\gamma,n} \restriction A_\alpha \equiv 0$, the same $r(\gamma, n)$ that worked in p works in q. By the same reason, all elements of p cannot require anything about a^α_j. Therefore, $r(\alpha, n)$ is finite since $\langle B_n : n \in \omega \rangle$ is almost disjoint.

$\square$