Characterization of Strict Positive Definiteness on products of complex spheres

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Abstract

In this paper we consider Positive Definite functions on products $\Omega_{2q} \times \Omega_{2p}$ of complex spheres, and we obtain a condition, in terms of the coefficients in their disc polynomial expansions, which is necessary and sufficient for the function to be Strictly Positive Definite. The result includes also the more delicate cases in which p and/or q can be 1 or ∞.

The condition we obtain states that a suitable set in $\mathbb{Z}^2$, containing the indexes of the strictly positive coefficients in the expansion, must intersect every product of arithmetic progressions.

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1 Introduction

The main purpose of this paper is to obtain a characterization of Strictly Positive Definite functions on products of complex spheres, in terms of the coefficients in their disc polynomial expansions: these results are contained in the Theorems 1.1, 1.2 and 1.3.

Positive Definiteness and Strict Positive Definiteness are important in many applications, for example, Strict Positive Definiteness is required in certain interpolation problems in order to guarantee the unicity of their solution. From a theoretical point of view, the problem of characterizing both Positive Definiteness and Strict Positive Definiteness has been considered in many recent papers, in different contexts. More details on the applications and the literature related to this problem will be given in Section 1.2.

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Let Ω be a nonempty set. A kernel $K : \Omega \times \Omega \to \mathbb{C}$ is called *Positive Definite* (PD in the following) on Ω when

$$\sum_{\mu,\nu=1}^{L} c_\mu \overline{c_\nu} K(x_\mu, x_\nu) \geq 0, \tag{1.1}$$

for any $L \geq 1$, $c = (c_1, \dots, c_L) \in \mathbb{C}^L$ and any subset $X := \{x_1, \dots, x_L\}$ of distinct points in Ω. Moreover, K is *Strictly Positive Definite* (SPD in the following) when it is Positive Definite and the inequality above is strict for $c \neq 0$.

If S^q is the q-dimensional unit sphere in the Euclidean space $\mathbb{R}^{q+1}$, we say that a continuous function $f : [-1, 1] \to \mathbb{R}$ is PD (resp. SPD) on S^q, when the associated kernel $K(v, v') := f(v \cdot_{\mathbb{R}} v')$ is PD (resp. SPD) on S^q (here " $\cdot_{\mathbb{R}}$ " is the usual inner product in $\mathbb{R}^{q+1}$). In [36] it was proved that a continuous function f is PD on S^q, $q \geq 1$, if, and only if, it admits an expansion in the form

$$\begin{gathered} f(t) = \sum_{m \in \mathbb{Z}_+} a_m P_m^{(q-1)/2}(t), \quad t \in [-1, 1], \\ \text{where } \sum a_m P_m^{(q-1)/2}(1) < \infty \text{ and } a_m \geq 0 \text{ for all } m \in \mathbb{Z}_+. \end{gathered} \tag{1.2}$$

In (1.2), $P_m^{(q-1)/2}$ are the Gegenbauer polynomials of degree m associated to $(q-1)/2$ (see [37, page 80]) and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. In [8] it was proved that the function f in (1.2) is also SPD on S^q, $q \geq 2$ if, and only if, the set $\{m \in \mathbb{Z}_+ : a_m > 0\}$ contains an infinite number of odd and of even numbers. This condition is equivalent to asking that

$$\{m \in \mathbb{Z}_+ : a_m > 0\} \cap (2\mathbb{N} + x) \neq \emptyset \qquad \text{for every } x \in \mathbb{N}. \tag{1.3}$$

The complex case is defined in a similar way: if Ω_{2q} is the unit sphere in $\mathbb{C}^q$, $q \geq 2$, and $\mathbb{D}$ is the closed unit disc in $\mathbb{C}$, then a continuous function $f : \mathbb{D} \to \mathbb{C}$ is said to be PD (resp. SPD) on Ω_{2q} if the associated kernel $K(z, z') := f(z \cdot z')$ is PD (resp. SPD) on Ω_{2q}, where " $\cdot$ " is the usual inner product in $\mathbb{C}^q$. As proved in [31], a continuous function $f : \mathbb{D} \to \mathbb{C}$ is PD on Ω_{2q}, $q \geq 2$ if, and only if, it has the representation in series of the form

$$\begin{gathered} f(\xi) = \sum_{m,n \in \mathbb{Z}_+} a_{m,n} R_{m,n}^{q-2}(\xi), \quad \xi \in \mathbb{D}, \\ \text{where } \sum a_{m,n} < \infty \text{ and } a_{m,n} \geq 0 \text{ for all } m, n \in \mathbb{Z}_+. \end{gathered} \tag{1.4}$$

The functions $R_{m,n}^{q-2}$ in (1.4) are the *disc polynomials*, or *generalized Zernike polynomials* (see Equation (2.1)). The condition for f to be SPD was obtained in [17, 30]: f as in (1.4) is SPD on Ω_{2q} if, and only if, the set $\{m - n \in \mathbb{Z} : a_{m,n} > 0\}$ intersects every full arithmetic progression in $\mathbb{Z}$, that is,

$$\{m - n \in \mathbb{Z} : a_{m,n} > 0\} \cap (N\mathbb{Z} + x) \neq \emptyset \qquad \text{for every } N, x \in \mathbb{N}. \tag{1.5}$$

The characterization of SPD functions on the spheres S^1, Ω_2, S^∞ and Ω_∞ were also considered in [29, 28, 17], obtaining similar results.

Products of real spheres were considered in [18, 15, 19, 20]: a continuous PD function on $S^q \times S^p$, $q,p \geq 1$, associated to the kernel $K((u,v),(u',v')) := f(u \cdot_{\mathbb{R}} u', v \cdot_{\mathbb{R}} v')$, can be written as

$$\begin{gathered} f(t,s) = \sum_{m,k\in\mathbb{Z}_+} a_{m,k} P_m^{(q-1)/2}(t) P_k^{(p-1)/2}(s), \quad t,s \in [-1,1], \\ \text{where } \sum a_{m,k} P_m^{(q-1)/2}(1) P_k^{(p-1)/2}(1) < \infty \text{ and } a_{m,k} \geq 0 \text{ for all } m,k \in \mathbb{Z}_+. \end{gathered} \tag{1.6}$$

For $q,p \geq 2$, it is also SPD on $S^q \times S^p$ if, and only if, the following condition, obtained in [15], holds true: in each intersection of the set $\{(m,k) \in \mathbb{Z}_+^2 : a_{m,k} > 0\}$ with the four sets $(2\mathbb{Z}_+ + x) \times (2\mathbb{Z}_+ + y)$, $x,y \in \{0,1\}$, there exists a sequence (m_i,k_i) such that $m_i, k_i \to \infty$. In fact, this condition is equivalent to the following one:

$$\{(m,k) \in \mathbb{Z}_+^2 : a_{m,k} > 0\} \cap (2\mathbb{N} + x) \times (2\mathbb{N} + y) \neq \emptyset \qquad \text{for every } x,y \in \mathbb{N}. \tag{1.7}$$

Again, when considering S^1 in the place of S^q and/or S^p, similar (but not analogous) results are obtained: see [19, 20].

1.1 Main results

The purpose of this paper is to consider the same kind of problem described above for the case of the products $\Omega_{2q} \times \Omega_{2p}$ of two complex spheres.

The characterization of Positive Definiteness in this setting was obtained in [4, Theorem 7.1] for $q,p \in \mathbb{N}$, $q,p \geq 2$: it was proved that a continuous function $f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$, associated to the kernel $K((z,w),(z',w')) := f(z \cdot z', w \cdot w')$, is PD on $\Omega_{2q} \times \Omega_{2p}$ if, and only if, it admits an expansion in the form

$$\begin{gathered} f(\xi,\eta) = \sum_{m,n,k,l\in\mathbb{Z}_+} a_{m,n,k,l} R_{m,n}^{q-2}(\xi) R_{k,l}^{p-2}(\eta), \quad (\xi,\eta) \in \mathbb{D} \times \mathbb{D}, \\ \text{where } \sum a_{m,n,k,l} < \infty \text{ and } a_{m,n,k,l} \geq 0 \text{ for all } m,n,k,l \in \mathbb{Z}_+. \end{gathered} \tag{1.8}$$

If p and/or q can take the values 1 or ∞, a characterization of Positive Definiteness is also known (see in Section 2.2), except for the case $p = q = \infty$, which we address in Theorem 4.1. In fact, if we define $R_{m,n}^{\infty}(\xi) := \xi^m \overline{\xi}^n$, $\xi \in \mathbb{D}$, then the characterization (1.8) holds for $q,p \in \mathbb{N} \cup \{\infty\}$, $q,p \geq 2$.

Our main results are contained in the following theorems, where we characterize SPD functions on the product of two complex spheres $\Omega_{2q} \times \Omega_{2p}$, $q,p \in \mathbb{N} \cup \{\infty\}$, in terms of the coefficients in their expansions.

Theorem 1.1. *Let $q,p \in \mathbb{N} \cup \{\infty\}$, $q,p \geq 2$. A continuous function $f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$, which is PD on $\Omega_{2q} \times \Omega_{2p}$, is also SPD on $\Omega_{2q} \times \Omega_{2p}$ if, and only if, considering its expansion as in (1.8), the set*

$$J' := \left\{(m-n, k-l) \in \mathbb{Z}^2 : \ a_{m,n,k,l} > 0\right\}$$

intersects every product of full arithmetic progressions in $\mathbb{Z}$, that is,

$$J' \cap (N\mathbb{Z} + x) \times (M\mathbb{Z} + y) \neq \emptyset \qquad \textit{for every } N,M,x,y \in \mathbb{N}. \tag{1.9}$$

It is worth noting the similarities between the characterizations of SPD in the various cases described here. They can always be reduced to a condition on the intersection between a set constructed with the indexes of the strictly positive coefficients in the expansion of the function, and certain arithmetic progressions or products of them: compare the conditions (1.3-1.5-1.7-1.9).

When p and/or q can take the value 1, we obtain the following characterizations.

Theorem 1.2. *Let $2 \le p \in \mathbb{N} \cup \{\infty\}$. A continuous function $f : \partial\mathbb{D} \times \mathbb{D} \to \mathbb{C}$, which is PD on $\Omega_2 \times \Omega_{2p}$, is also SPD on $\Omega_2 \times \Omega_{2p}$ if, and only if, considering its expansion as*

$$\begin{gathered} f(\xi,\eta) = \sum_{m\in\mathbb{Z},\, k,l\in\mathbb{Z}_+} a_{m,k,l}\xi^m R^{p-2}_{k,l}(\eta), \quad (\xi,\eta) \in \partial\mathbb{D}\times\mathbb{D}, \\ \text{where } \sum a_{m,k,l} < \infty \text{ and } a_{m,k,l} \ge 0 \text{ for all } m \in \mathbb{Z},\ k,l \in \mathbb{Z}_+, \end{gathered} \tag{1.10}$$

the set

$$\{(m, k-l) \in \mathbb{Z}^2 :\ a_{m,k,l} > 0\}$$

intersects every product of full arithmetic progressions in $\mathbb{Z}$.

Theorem 1.3. *A continuous function $f : \partial\mathbb{D} \times \partial\mathbb{D} \to \mathbb{C}$, which is PD on $\Omega_2 \times \Omega_2$, is also SPD on $\Omega_2 \times \Omega_2$ if, and only if, considering its expansion as*

$$\begin{gathered} f(\xi,\eta) = \sum_{m,k\in\mathbb{Z}} a_{m,k}\xi^m \eta^k, \quad (\xi,\eta) \in \partial\mathbb{D}\times\partial\mathbb{D}, \\ \text{where } \sum a_{m,k} < \infty \text{ and } a_{m,k} \ge 0 \text{ for all } m,k \in \mathbb{Z}, \end{gathered} \tag{1.11}$$

the set

$$\{(m, k) \in \mathbb{Z}^2 :\ a_{m,k} > 0\}$$

intersects every product of full arithmetic progressions in $\mathbb{Z}$.

We observe that the Theorems 1.2 and 1.3 will follow immediately from the same proof as Theorem 1.1, after rewriting the expansions (1.10) and (1.11) in order to be formally identical to (1.8) (see Lemma 2.2). This is a remarkable fact considering that, in the real case, when the product involves the sphere S^1 (see [20, 19]) one had to use quite different arguments with respect to the higher dimensional case in [15]. We remark however that Theorem 1.3 is not new, as it is a particular case of the main result in [14].

As a consequence of Theorem 1.2 we will also obtain the following result for the product of S^1 with a complex sphere, where the definition of PD and SPD functions is as always given by associating the function with the kernel $\widehat{K}((u,w),(u',w')) := \widehat{f}(u \cdot_{\mathbb{R}} u', w \cdot w')$.

Theorem 1.4. *Let $2 \le p \in \mathbb{N} \cup \{\infty\}$. A continuous function $\widehat{f} : [-1,1] \times \mathbb{D} \to \mathbb{C}$, which is PD on $S^1 \times \Omega_{2p}$, is also SPD on $S^1 \times \Omega_{2p}$ if, and only if, considering its expansion as*

$$\begin{gathered} \widehat{f}(\cos(\phi),\eta) = \sum_{m,k,l\in\mathbb{Z}_+} \widehat{a}_{m,k,l}\cos(m\phi) R^{p-2}_{k,l}(\eta), \quad (\phi,\eta) \in [0,\pi]\times\mathbb{D}, \\ \text{where } \sum \widehat{a}_{m,k,l} < \infty \text{ and } \widehat{a}_{m,k,l} \ge 0 \text{ for all } m,k,l \in \mathbb{Z}_+, \end{gathered} \tag{1.12}$$

the set

$$\{(m, k-l) \in \mathbb{Z}^2 : \widehat{a}_{|m|,k,l} > 0\}$$

intersects every product of full arithmetic progressions in $\mathbb{Z}$.

This result represents a partial answer to an open problem stated in [16], where products of real spheres with a general set X were considered and a characterization of Strict Positive Definiteness was obtained only for the spheres S^d with $d \geq 2$: Theorem 1.4 provides a characterization for the case $d = 1$ and $X = \Omega_{2p}$.

By the same argument it is also possible to deduce, from Theorem 1.3, the characterization of SPD functions on $S^1 \times S^1$ proved in [20], namely, that a continuous function $f : [-1,1] \times [-1,1] \to \mathbb{C}$ which is PD on $S^1 \times S^1$ is also SPD on $S^1 \times S^1$ if, and only if, considering its expansion as in (1.6), the set $\{(m,k) \in \mathbb{Z}^2 : a_{|m|,|k|} > 0\}$ intersects every product of full arithmetic progressions in $\mathbb{Z}$.

This paper is organized in the following way. In Section 1.2 we discuss some further literature related to our problem. In Section 2, we set our notation and discuss some known results that will be used later. Theorem 1.1 is proved in Section 3. In Section 4 we state and prove the mentioned characterization of PD functions on $\Omega_\infty \times \Omega_\infty$. Finally, Section 5 is devoted to showing how one can deduce Theorem 1.4 from Theorem 1.2.

1.2 Literature

Since the first results on Positive Definite functions on real spheres, obtained by Schoenberg in his seminal paper ([36]), such functions were found to be relevant and have been studied in several distinct areas. In fact, they are both used by researchers directly interested in applied sciences, such as geostatistics, numerical analysis, approximation theory (cf. [9, 10, 12, 34]), and by theoretical researchers aiming at further generalizations that, along with their theoretical importance, could become useful in other practical problems.

One important motivation for characterizing Strictly Positive Definite functions comes from certain interpolation problems, where the interpolating function is generated by a Positive Definite kernel. Actually, the unicity of the solution of the interpolation problem is guaranteed only if the generating kernel is also Strictly Positive Definite (cf. [26, 40]): consider, for instance, the interpolation function

$$F(x) = \sum_{j=1}^{L} c_j K(x, x_j), \quad x \in \Omega,$$

where $X = \{x_1, \dots, x_L\} \subset \Omega$ is given and K is a known Strictly Positive Definite kernel in Ω; then the matrix of the system obtained from the interpolation conditions $F(x_i) = \lambda_i$, $i = 1, \dots, L$, is the matrix $[K(x_i, x_j)]$, whose determinant is positive, thus giving a unique solution for the system. In particular, the case where Ω is a real sphere is very important in applications where one needs to assure unicity for interpolation problems with data given on the Earth's surface (which can be identified with the real sphere S^2). Also, the case where Ω is the product of a sphere with some other set turns out to be of particular interest for its application to geostatistical problems in space and time, whose natural domain is $S^2 \times \mathbb{R}$ (see [34] and references therein). Immediate applications in the case of complex spheres are less obvious: we refer to [27], where parametric families of Positive Definite functions on complex spheres are provided. It is also worth noting that the Zernike polynomials are used in applications such as optics and optical engineering (cf. [35, 38] and references therein).

Motivated by these and other applications, several papers appeared dealing with the theoretical problem of characterizing Positive Definiteness and Strict Positive Definiteness: along with those already mentioned in the introduction, we cite [32], where a characterization of real-valued multivariate Positive Definite functions on S^q is obtained, and [41, 21, 7], where matrix-valued Positive Definite functions are investigated.

In [6], the characterization in [36] is extended to the case of Positive Definite functions on the cartesian product of S^q times a locally compact group G, which includes the mentioned case $S^q \times \mathbb{R}$ and also generalizes the result obtained in [18] about Positive Definite functions on products of real spheres. Also, the Positive Definite functions on Gelfand pairs and on products of them were characterized in [4], while those on the product of a locally compact group with Ω_∞ in [5]. In [4] it was observed that the characterizations in (1.2) and (1.4) can be viewed as special cases of the Bochner-Godement Theorem for Gelfand pairs, see [13].

Concerning the characterization of Strictly Positive Definite functions, we cite also the cases of compact two-point homogeneous spaces and products of them ([1, 2]) and the case of a torus ([14]).

2 Notation and known results

We first give a brief introduction on the disc polynomials that appear in the Equations (1.4) and (1.8): for a real number $\alpha > -1$, the function $R^\alpha_{m,n}$, defined in the disc $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$, is called *disc polynomial* (or *generalized Zernike polynomial*) of degree m in ξ and n in $\overline{\xi}$ associated to α, and can be written as (see [24])

$$R^\alpha_{m,n}(\xi) = r^{|m-n|}\, e^{i(m-n)\phi}\, P^{(\alpha,\,|m-n|)}_{\min\{m,n\}}(2r^2 - 1), \quad \xi = re^{i\phi} \in \mathbb{D},\ m, n \in \mathbb{Z}_+, \tag{2.1}$$

where $P^{(\alpha,\beta)}_k$ is the usual Jacobi polynomial of degree k with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on $[-1,1]$, $\alpha, \beta > -1$, normalized by $P^{(\alpha,\beta)}_k(1) = 1$ (see [37, page 58]).

For future use we also define

$$\begin{aligned} R^\infty_{m,n}(\xi) &:= \xi^m \overline{\xi}^n, && \xi \in \mathbb{D}\,, \\ R^{-1}_{m,n}(\xi) &:= \xi^m \overline{\xi}^n, && \xi \in \partial\mathbb{D}\,. \end{aligned} \tag{2.2}$$

Observe that $R^\infty_{m,n}$ is actually the limit of $R^\alpha_{m,n}$ as $\alpha \to \infty$ (see [39, (2.12)]), while the expression for $R^{-1}_{m,n}$ is identical to the restriction of $R^\alpha_{m,n}$, $\alpha > -1$, to $\partial\mathbb{D}$.

It is well known (see [24, 23]) that the disc polynomials, as well as those defined in (2.2), satisfy, for $q \in \mathbb{N} \cup \{\infty\}$, $\xi \in \mathbb{D}$, and $m, n \in \mathbb{Z}_+$,

$$R_{m,n}^{q-2}(1) = 1, \quad |R_{m,n}^{q-2}(\xi)| \leq 1, \tag{2.3}$$

$$R_{m,n}^{q-2}(e^{i\phi}\xi) = e^{i(m-n)\phi} R_{m,n}^{q-2}(\xi), \qquad \phi \in \mathbb{R}, \tag{2.4}$$

$$R_{m,n}^{q-2}\left(\overline{\xi}\right) = \overline{R_{m,n}^{q-2}(\xi)}. \tag{2.5}$$

Observe that, by (2.3), the series in (1.4) and (1.8) converge uniformly in their domain. Moreover, the characterization in (1.8) implies that the functions $(\xi, \eta) \mapsto R_{m,n}^{q-2}(\xi) R_{k,l}^{p-2}(\eta)$ are PD on $\Omega_{2q} \times \Omega_{2p}$ for all $m, n, k, l \in \mathbb{Z}_+$ (and, by (1.4), the functions $\xi \mapsto R_{m,n}^{q-2}(\xi)$ are PD on Ω_{2q}).

Another important property, proved in [17], is contained in the following lemma.

Lemma 2.1. *Let $2 \leq q \in \mathbb{N} \cup \{\infty\}$ and*

$$\begin{cases} \xi \in \mathbb{D}' & if \;\; q > 2, \\ \xi \in \mathbb{D}' \setminus \{0\} & if \;\; q = 2, \end{cases}$$

where $\mathbb{D}' = \{\xi \in \mathbb{C} : |\xi| < 1\}$; then

$$\lim_{m+n \to \infty} R_{m,n}^{q-2}(\xi) = 0\,. \tag{2.6}$$

In the special case $q = 2,\ \xi = 0,$ one has

$$\lim_{\substack{m+n \to \infty \\ m \neq n}} R_{m,n}^{0}(0) = 0\,. \tag{2.7}$$

2.1 Positive Definiteness on complex spheres

As we anticipated in the introduction, it is known by [31] that a continuous function $f : \mathbb{D} \to \mathbb{C}$ is PD on Ω_{2q}, $2 \leq q \in \mathbb{N}$, if, and only if, the coefficients $a_{m,n}$ in the series representation (1.4) satisfy $\sum a_{m,n} < \infty$ and $a_{m,n} \geq 0$ for all $m, n \in \mathbb{Z}_+$.

In the case of the complex sphere Ω_2, when associating a continuous function f to a kernel via the formula $K(z, z') := f(z \cdot z')$, one has that $z \cdot z' \in \partial\mathbb{D}$ for every $z, z' \in \Omega_2$, then it becomes natural to consider functions f defined in $\partial\mathbb{D}$. The PD functions on Ω_2 were also characterized in [31], namely, $f : \partial\mathbb{D} \to \mathbb{C}$ is PD on Ω_2 if, and only if,

$$\begin{gathered} f(\xi) = \sum_{m \in \mathbb{Z}} a_m \xi^m, \quad \xi \in \partial\mathbb{D}, \\ \text{where } \sum a_m < \infty \text{ and } a_m \geq 0 \text{ for all } m \in \mathbb{Z}. \end{gathered} \tag{2.8}$$

In order to write this formula as (1.4), and then to be able to use the same expansion for all $q \in \mathbb{N}$, we use the polynomials $R_{m,n}^{-1}$ defined in (2.2) and we rearrange the coefficients in (2.8) so that

$$f(\xi) = \sum_{m,n \in \mathbb{Z}_+} a_{m,n} R_{m,n}^{-1}(\xi), \quad \xi \in \partial\mathbb{D}, \tag{2.9}$$

with the additional requirement that $a_{m,n} = 0$ if $mn > 0$, implying that

$$\begin{cases} a_{m,0} := a_m, & m \geq 0, \\ a_{0,m} := a_{-m}, & m \geq 0. \end{cases}$$

In this way, f is PD on Ω_2 if, and only if, it satisfies the characterization (1.4) with $a_{m,n} = 0$ for $mn > 0$ and $\partial\mathbb{D}$ in the place of $\mathbb{D}$.

The complex sphere Ω_∞ is defined as the sphere of the sequences in the complex Hilbert space $\ell^2(\mathbb{C})$ with unit norm. In [11], it was proved that a continuous function $f : \mathbb{D} \to \mathbb{C}$ is PD on Ω_∞ if, and only if, it admits the series representation

$$\begin{gathered} f(\xi) = \sum_{m,n \in \mathbb{Z}_+} a_{m,n} \xi^m \overline{\xi}^n, \quad \xi \in \mathbb{D}, \\ \text{where } \sum a_{m,n} < \infty \text{ and } a_{m,n} \geq 0 \text{ for all } m, n \in \mathbb{Z}_+, \end{gathered} \tag{2.10}$$

which becomes analogous to the characterization (1.4) if we use the definition of $R^\infty_{m,n}$ in (2.2). It is also worth noting that f is PD on Ω_∞ if, and only if, f is PD on Ω_{2q} for every $q \geq 2$, as it is shown in [6] for the real case.

2.2 Positive Definiteness on products of spheres

From now on, in order to simplify the exposition, we will use the symbol $\varXi$ to designate either $\partial\mathbb{D}$ or $\mathbb{D}$, depending if we are considering, respectively, the sphere Ω_2 or a higher dimensional sphere.

When considering products of spheres $\Omega_{2q} \times \Omega_{2p}$, $q, p \in \mathbb{N} \cup \{\infty\}$, a continuous function $f : \varXi \times \varXi \to \mathbb{C}$ is said to be PD (resp. SPD) on $\Omega_{2q} \times \Omega_{2p}$, if the associated kernel

$$K : [\Omega_{2q} \times \Omega_{2p}] \times [\Omega_{2q} \times \Omega_{2p}] \ni (\,(z, w), (z', w')\,) \mapsto f(z \cdot z', w \cdot w') \tag{2.11}$$

is PD (resp. SPD) on $\Omega_{2q} \times \Omega_{2p}$.

In this section we will justify the following claim:

Lemma 2.2. *A continuous function $f : \varXi \times \varXi \to \mathbb{C}$ is PD on $\Omega_{2q} \times \Omega_{2p}$, $q, p \in \mathbb{N} \cup \{\infty\}$, if and only if, it admits an expansion in the form*

$$\begin{gathered} f(\xi, \eta) = \sum_{m,n,k,l \in \mathbb{Z}_+} a_{m,n,k,l} R^{q-2}_{m,n}(\xi) R^{p-2}_{k,l}(\eta), \quad (\xi, \eta) \in \varXi \times \varXi, \\ \textit{where } \sum a_{m,n,k,l} < \infty \textit{ and } a_{m,n,k,l} \geq 0 \textit{ for all } m, n, k, l \in \mathbb{Z}_+, \end{gathered} \tag{2.12}$$

adding the requirement that $a_{m,n,k,l} = 0$ if $q = 1$ and $mn > 0$ (resp. $p = 1$ and $kl > 0$).

Lemma 2.2 is a generalization of the characterization (1.8) to include the cases when q, p can take the values 1 or ∞, replacing $\mathbb{D}$ with $\varXi$ and redefining the coefficients in the series, where p or q is 1, as we did in Equation (2.9).

In order to justify the claim, we will use results from [4] and [5], which are stated in a more general setting. Let $U(p)$ be the locally compact group of the unitary $p \times p$ complex matrices. A continuous function $\widetilde{\Phi} : U(p) \to \mathbb{C}$ is called Positive Definite on $U(p)$ if the kernel $(A, B) \mapsto \widetilde{\Phi}(B^{-1}A)$ is Positive Definite on $U(p)$ (see [3, page 87]).

The following remark will be useful to translate from this setting to the case of complex spheres in which we are interested (see also [4, Section 6]).

Remark 2.3. Let $\Phi : \Xi \to \mathbb{C}$ and $\widetilde{\Phi} : U(p) \to \mathbb{C}$ be related by $\widetilde{\Phi}(A) = \Phi(Ae_p \cdot e_p)$, where $e_p = (1, 0, \ldots, 0) \in \Omega_{2p}$.

Then $\widetilde{\Phi}(A)$ depends only on the upper-left element $[A]_{1,1}$ and it can be seen by the definition of Positive Definiteness that $\widetilde{\Phi}$ is PD on $U(p)$ if, and only if, Φ is PD on Ω_{2p}.

Moreover, $\widetilde{\Phi}$ is continuous if, and only if, Φ is, since $M : U(p) \to \Xi : A \mapsto [A]_{1,1}$ is continuous and admits a continuous right inverse

$$M^{-} : \Xi \to U(p) : \xi \mapsto M^{-}(\xi) \ \text{ such that } \ [M^{-}(\xi)]_{1,1} = \xi \,.$$

Now Lemma 2.2 is obtained as follows:

1. When $q, p \in \mathbb{N}$, $q, p \geq 2$, the lemma is exactly the characterization (1.8).
2. When $q = 1$ and $p \in \mathbb{N}$ (or vice-versa) we can use Corollary 3.5 in [4], observing that we can identify functions on Ω_2 with periodic functions on $\mathbb{R}$, and we can take the locally compact group $L = U(p)$, obtaining a characterization for PD functions on $\Omega_2 \times U(p)$. Then we can translate the characterization from $U(p)$ to Ω_{2p}, using Remark 2.3.
3. When $q = \infty$ and $p \in \mathbb{N}$ (or vice-versa) we can use Theorem 1.3 in [5], taking the locally compact group $L = U(p)$ and proceeding as above.
4. When $q = p = \infty$ the claim is a consequence of Theorem 4.1 in Section 4.

3 Proof of the main results

In the following we will need to consider matrices whose elements are described by many indexes: for this we will write

$$[b_{i,j,k,l,\ldots}]_{i=1,..,I,\ j=i,..,J,\ \ldots}^{k=1,..,K,\ l=1,..,L,\ \ldots} \ ,$$

where the indexes in the lower line are intended to be row indexes and those in the above line are column indexes. Also, we will specify the indexes alone when their ranges are clear.

Let $q, p \in \mathbb{N} \cup \{\infty\}$. From (1.1) and (2.11), the definition of Positive Definiteness on $\Omega_{2q} \times \Omega_{2p}$, for a continuous function $f : \Xi \times \Xi \to \mathbb{C}$, takes the form

$$\sum_{\mu,\nu=1}^{L} c_\mu \overline{c_\nu} f(z_\mu \cdot z_\nu, w_\mu \cdot w_\nu) \geq 0 \tag{3.1}$$

for all $L \geq 1$, $(c_1, c_2, \ldots, c_L) \in \mathbb{C}^L$ and $X = \{(z_1, w_1), (z_2, w_2), \ldots, (z_L, w_L)\} \subset \Omega_{2q} \times \Omega_{2p}$. As a consequence, if we define the matrix A_X associated to the function f and to the set X by

$$A_X := [f(z_\mu \cdot z_\nu, w_\mu \cdot w_\nu)]_{\nu=1,\ldots,L}^{\mu=1,\ldots,L}, \tag{3.2}$$

then:

- f is PD if, and only if, for every choice of L, X, and $c^t = (c_1, c_2, \ldots, c_L)$,

$$\overline{c}^t A_X c \geq 0,$$

  that is, A_X is a Hermitian and positive semidefinite matrix (see [22, page 430]);

- f is also SPD if, and only if, for every choice of L and X,

$$\overline{c}^t A_X c = 0 \Longleftrightarrow c = 0,$$

  that is, A_X is a positive definite matrix.

Let now f be a continuous function, PD on $\Omega_{2q} \times \Omega_{2p}$, which we can write uniquely as in Lemma 2.2. If we define the set

$$J = \left\{(m, n, k, l) \in \mathbb{Z}_+^4 : \ a_{m,n,k,l} > 0\right\}, \tag{3.3}$$

then, for a finite set $X = \{(z_1, w_1), (z_2, w_2), \ldots, (z_L, w_L)\} \subseteq \Omega_{2q} \times \Omega_{2p}$, we can write

$$A_X = \sum_{(m,n,k,l)\in J} a_{m,n,k,l} B_X^{m,n,k,l} \tag{3.4}$$

where

$$B_X^{m,n,k,l} := [R_{m,n}^{q-2}(z_\mu \cdot z_\nu)\, R_{k,l}^{p-2}(w_\mu \cdot w_\nu)]_{\nu=1,\ldots,L}^{\mu=1,\ldots,L} \tag{3.5}$$

is the positive semidefinite matrix associated to X and to the function $R_{m,n}^{q-2}(\xi) R_{k,l}^{p-2}(\eta)$.

With these definitions, the following lemma holds.

Lemma 3.1. *The matrix A_X is a positive definite matrix if, and only if, the equivalence*

$$\overline{c}^t B_X^{m,n,k,l} c = 0 \ \ \forall\ (m, n, k, l) \in J \quad \Longleftrightarrow \quad c = 0 \tag{3.6}$$

holds true.

Lemma 3.1 is a consequence of the following one.

Lemma 3.2. *Let $A = \sum_j A_j$, where A_j are positive semidefinite matrices. Then A is positive semidefinite and the condition that A is positive definite is equivalent to*

$$\overline{c}^t A_j c = 0 \ \ \forall j \quad \Longleftrightarrow \quad c = 0\,.$$

Proof. First, $\overline{c}^t Ac = \sum_j \overline{c}^t A_j c \geq 0$, then one has that A is positive semidefinite too. If A is positive definite and $\overline{c}^t A_j c = 0$ for every j, then of course $\overline{c}^t Ac = 0$ and so $c = 0$. Finally, if $\overline{c}^t Ac = 0$ then (sum of nonnegative terms) $\overline{c}^t A_j c = 0$ $\forall j$; if we assume that this system implies $c = 0$ then A is positive definite. $\square$

In the following proposition we prove one of the two implications of Theorem 1.1.

Proposition 3.3. *Let $q, p \in \mathbb{N} \cup \{\infty\}$, f be a continuous function which is PD on $\Omega_{2q} \times \Omega_{2p}$ and consider*

$$J' = \left\{(m-n, k-l) \in \mathbb{Z}^2 : \ (m,n,k,l) \in J\right\} . \tag{3.7}$$

If f is SPD on $\Omega_{2q} \times \Omega_{2p}$ then

$$J' \cap (N\mathbb{Z} + x) \times (M\mathbb{Z} + y) \neq \emptyset \ \text{ for every } N, M, x, y \in \mathbb{N} . \tag{3.8}$$

Proof. Assume $J' \cap (N\mathbb{Z} + x) \times (M\mathbb{Z} + y) = \emptyset$ for some $N, M, x, y \in \mathbb{N}$. Without loss of generality we may assume $M, N \geq 2$.
Fix a point $(z, w) \in \Omega_{2q} \times \Omega_{2p}$ and take the set of points

$$X = \left\{(e^{i2\pi\tau/N} z, e^{i2\pi\sigma/M} w) \in \Omega_{2q} \times \Omega_{2p} : \ \tau = 1, .., N, \ \sigma = 1, .., M\right\} ;$$

then, using the Equations (2.3-2.4), the matrix in (3.5) reads as

$$B_X^{m,n,k,l} = \left[e^{i2\pi(m-n)(\tau-\lambda)/N} e^{i2\pi(k-l)(\sigma-\zeta)/M}\right]_{\lambda=1,..,N, \ \zeta=1,..,M}^{\tau=1,..,N, \ \sigma=1,..,M} .$$

Observe that this matrix factors as the product $B_X^{m,n,k,l} = \overline{b}^t b$ where b is the row vector

$$b = \left[e^{i2\pi(m-n)\tau/N} e^{i2\pi(k-l)\sigma/M}\right]^{\tau,\sigma}$$

(we omit the dependence on X and (m, n, k, l) in the notation for b). Then each equation of the system in (3.6) reads as $\overline{c}^t B_X^{m,n,k,l} c = \overline{c}^t \overline{b}^t bc = 0$ and is equivalent to $bc = 0$.

At this point we take $c = \left[e^{-i2\pi\tau x/N} e^{-i2\pi\sigma y/M}\right]_{\tau,\sigma}$, so that

$$bc = \sum_{\tau,\sigma} e^{i2\pi(m-n-x)\tau/N} e^{i2\pi(k-l-y)\sigma/M} = \sum_{\tau} e^{i2\pi(m-n-x)\tau/N} \sum_{\sigma} e^{i2\pi(k-l-y)\sigma/M} . \tag{3.9}$$

By our assumption, for every $(m, n, k, l) \in J$, either $m - n - x$ is not a multiple of N or $k - l - y$ is not a multiple of M. This implies that one of the two sums in (3.9) is zero and then $bc = 0$.
Then c is a nontrivial solution of the system in (3.6). We have thus proved that $J' \cap (N\mathbb{Z} + x) \times (M\mathbb{Z} + y) = \emptyset$ implies that f is not SPD. $\square$

The rest of this section is dedicated to proving the following proposition, which contains the remaining implication of Theorem 1.1.

Proposition 3.4. *Let q, p, f and J' be as in Proposition 3.3. If condition (3.8) holds true, then f is SPD on $\Omega_{2q} \times \Omega_{2p}$.*

First of all, we prove the following consequence of condition (3.8).

Lemma 3.5. *If $A \subset \mathbb{Z}^2$ satisfies*

$$I_{M,N,x,y} := A \cap (N\mathbb{Z} + x) \times (M\mathbb{Z} + y) \neq \emptyset \quad \text{for every } N, M, x, y \in \mathbb{N}\,, \tag{3.10}$$

then, for every $N, M, x, y \in \mathbb{N}$, the set

$$\{\min\{|\alpha|, |\beta|\} : (\alpha, \beta) \in I_{M,N,x,y}\}$$

is unbounded and $I_{M,N,x,y}$ is infinite.

Proof. Suppose $\{\min\{|\alpha|, |\beta|\} : (\alpha, \beta) \in I_{M,N,x,y}\} \subseteq [0, C]$.
Let $(\widehat{x}, \widehat{y}) \in (N\mathbb{Z} + x) \times (M\mathbb{Z} + y)$ with $\widehat{x}, \widehat{y} > C$ and D be a multiple of M and of N such that $\widehat{x} - D, \widehat{y} - D < -C$. Then $(D\mathbb{Z} + \widehat{x}) \times (D\mathbb{Z} + \widehat{y}) \cap I_{M,N,x,y} = \emptyset$ and

$$(D\mathbb{Z} + \widehat{x}) \times (D\mathbb{Z} + \widehat{y}) \subseteq (N\mathbb{Z} + x) \times (M\mathbb{Z} + y).$$

As a consequence $(D\mathbb{Z} + \widehat{x}) \times (D\mathbb{Z} + \widehat{y}) \cap A = \emptyset$, which contradicts (3.10). □

The next step will be to prove that we need to verify Strict Positive Definiteness only on certain special sets $X \subseteq \Omega_{2q} \times \Omega_{2p}$ (see Lemma 3.9).

In view of Lemma 2.1 and Equation (2.4), when calculating $R^{q-2}_{m,n}(z_\mu \cdot z_\nu)$ and considering the limit for $m + n \to \infty$, the obtained behavior is quite different depending on whether $|z_\mu \cdot z_\nu| < 1$ or $|z_\mu \cdot z_\nu| = 1$. In particular, we will have to treat carefully the cases when $|z_\mu \cdot z_\nu| = 1$. This happens either if $z_\mu = z_\nu$ (observe that the points in the set X must be distinct but they can have one of the two components in common), or if $z_\mu = e^{i\theta} z_\nu$ with $\theta \in (0, 2\pi)$. In this last case we say that the two points $z_\mu, z_\nu \in \Omega_{2q}$ are *antipodal*. Our strategy to deal with antipodal points is inspired by [17]. We will say that a set of (distinct) points $Y = \{(z_\mu, w_\mu) : \mu = 1, \ldots, L\}$ in $\Omega_{2q} \times \Omega_{2p}$ is *antipodal-free* if the following property holds:

(AF) if $\mu \neq \nu$ then $|z_\mu \cdot z_\nu| < 1$ unless $z_\mu = z_\nu$ and $|w_\mu \cdot w_\nu| < 1$ unless $w_\mu = w_\nu$.

Of course, since the points in Y are distinct, if $z_\mu = z_\nu$ then $|w_\mu \cdot w_\nu| < 1$ (resp. if $w_\mu = w_\nu$ then $|z_\mu \cdot z_\nu| < 1$).

Remark 3.6. Note that two distinct points in Ω_2 are always antipodal. So if, for instance, $q = 1$, then, in an antipodal-free set Y in $\Omega_2 \times \Omega_{2p}$, all the z_μ are the same and then $|w_\mu \cdot w_\nu| < 1$ for $\mu \neq \nu$. When $p = q = 1$ then an antipodal-free set Y in $\Omega_2 \times \Omega_2$ contains a unique point (z, w).

Consider now an antipodal-free set $Y \subseteq \Omega_{2q} \times \Omega_{2p}$ and two sets of angles $\Theta = \{\theta_\tau : \tau = 1, \ldots, t\}$ and $\Delta = \{\delta_\sigma : \sigma = 1, \ldots, s\}$ in $[0, 2\pi)$. We define the *enhanced set associated to* Y, Θ and Δ as the set

$$X = \left\{(e^{i\theta_\tau} z_\mu, e^{i\delta_\sigma} w_\mu) : \mu = 1, \ldots, L, \ \tau = 1, \ldots, t, \ \sigma = 1, \ldots, s\right\}. \tag{3.11}$$

Observe that, by construction, the points that appear in X are all distinct (but now there exist many antipodal points among them).

The following lemma provides a sort of inverse construction.

Lemma 3.7. *Given a finite set $S \subseteq \Omega_{2q} \times \Omega_{2p}$ one can always obtain an antipodal-free set $Y \subseteq \Omega_{2q} \times \Omega_{2p}$ and two sets Θ and Δ of angles in $[0, 2\pi)$, such that S is contained in the enhanced set X associated to Y, Θ and Δ.*

Proof. For a finite set $X_1 \subseteq \Omega_{2q}$ one can select a maximal subset Y_1 not containing antipodal points and then define the set Θ containing 0 and all the distinct $\theta \in (0, 2\pi)$ that are needed to produce the remaining points as $e^{i\theta} z_\mu$ with $z_\mu \in Y_1$.

For the set $S \subseteq \Omega_{2q} \times \Omega_{2p}$ one produces with this algorithm a maximal subset Y_1 not containing antipodal points along with a corresponding set of angles Θ from all the first coordinates z in S, then a maximal subset Y_2 not containing antipodal points along with a corresponding set of angles Δ from all the second coordinates w in S.

Then $Y := Y_1 \times Y_2$ will be such that S is contained in the enhanced set associated to Y, Θ and Δ. □

The following two lemmas will make clear why it is useful to consider antipodal-free sets.

Lemma 3.8. *Let $Y = \{(z_\mu, w_\mu) : \mu = 1, \dots, L\}$ in $\Omega_{2q} \times \Omega_{2p}$ be antipodal-free. Then the matrix*

$$[R^{q-2}_{m,n}(z_\mu \cdot z_\nu) R^{p-2}_{k,l}(w_\mu \cdot w_\nu)]^\mu_\nu$$

is positive definite provided $n \neq m$, $k \neq l$ and $m + n, k + l$ are large enough.

Proof. Actually, the diagonal elements of the matrix are all equal to $R^{q-2}_{m,n}(1) R^{p-2}_{k,l}(1) = 1$, moreover, condition (AF) implies that if $z_\mu \cdot z_\nu = 1$ then $|w_\mu \cdot w_\nu| < 1$ and if $w_\mu \cdot w_\nu = 1$ then $|z_\mu \cdot z_\nu| < 1$. As a consequence, the non-diagonal elements converge to zero by Lemma 2.1, when $n \neq m$, $k \neq l$ and $min\,\{m + n, k + l\} \to \infty$. Then the matrix, which is Hermitian and with real positive diagonal, becomes strictly diagonally dominant, thus positive definite ([22, Theorem 6.1.10]). □

Lemma 3.9. *Let $q, p \in \mathbb{N} \cup \{\infty\}$ and f be a continuous function which is PD on $\Omega_{2q} \times \Omega_{2p}$. Then the following assertions are equivalent:*

(i) f is SPD on $\Omega_{2q} \times \Omega_{2p}$;

(ii) the matrix A_X defined in (3.2) is positive definite for every finite set X being the enhanced set associated to some antipodal-free set $Y \subseteq \Omega_{2q} \times \Omega_{2p}$ and two sets Θ and Δ of angles in $[0, 2\pi)$.

Proof. First observe that (i) is equivalent to:

(iii) A_S is a positive definite matrix for every finite set $S \subseteq \Omega_{2q} \times \Omega_{2p}$.

The implication $(iii) \implies (ii)$ is trivial. In order to prove that $(ii) \implies (iii)$ observe that, given S, one can obtain X as described in Lemma 3.7: since $S \subseteq X$, then A_S is a principal submatrix of the positive definite matrix A_X and then it is a positive definite matrix itself. □

At this point we can prove Proposition 3.4.

Proof of Proposition 3.4. Let X (finite) be the enhanced set associated to an antipodal-free set $Y \subseteq \Omega_{2q} \times \Omega_{2p}$ and two sets Θ and Δ of angles in $[0, 2\pi)$ and consider the system

$$\overline{c}^t B_X^{m,n,k,l} c = 0 \text{ for every } (m,n,k,l) \in J. \tag{3.12}$$

In view of the Lemmas 3.1 and 3.9, all we have to do is to prove that this system implies $c = 0$.

Using the property in (2.4), with the notation introduced in (3.11) for the elements of X, we have

$$B_X^{m,n,k,l} = \left[e^{i(m-n)(\theta_\tau - \theta_\lambda)} e^{i(k-l)(\delta_\sigma - \delta_\zeta)} R_{m,n}^{q-2}(z_\mu \cdot z_\nu) R_{k,l}^{p-2}(w_\mu \cdot w_\nu)\right]_{\lambda,\zeta,\nu}^{\tau,\sigma,\mu}.$$

It is convenient to write this matrix as a block matrix as follows:

$$B_X^{m,n,k,l} = [R_{m,n}^{q-2}(z_\mu \cdot z_\nu) R_{k,l}^{p-2}(w_\mu \cdot w_\nu) A^{m,n,k,l}]_\nu^\mu$$

where

$$A^{m,n,k,l} = [e^{i(m-n)(\theta_\tau - \theta_\lambda)} e^{i(k-l)(\delta_\sigma - \delta_\zeta)}]_{\lambda,\zeta}^{\tau,\sigma}.$$

The vector c will be correspondingly split as

$$c = [c_\mu]_\mu \quad \text{ where } \quad c_\mu = [c_\mu^{\tau\sigma}]_{\tau,\sigma}.$$

We have then

$$\overline{c}^t B_X^{m,n,k,l} c = \sum_{\mu,\nu} R_{m,n}^{q-2}(z_\mu \cdot z_\nu) R_{k,l}^{p-2}(w_\mu \cdot w_\nu) \overline{c_\nu}^t A^{m,n,k,l} c_\mu.$$

As in the proof of Proposition 3.3, the matrix $A^{m,n,k,l}$ factors as $A^{m,n,k,l} = \overline{b}^t b$ where

$$b = [e^{i(m-n)\theta_\tau} e^{i(k-l)\delta_\sigma}]^{\tau\sigma},$$

then we may write

$$\overline{c}^t B_X^{m,n,k,l} c = \sum_{\mu,\nu} \overline{bc_\nu}^t bc_\mu R_{m,n}^{q-2}(z_\mu \cdot z_\nu) R_{k,l}^{p-2}(w_\mu \cdot w_\nu). \tag{3.13}$$

Observe that since Y is antipodal-free we will be able to use Lemma 3.8 in order to discuss this quadratic form.

We suppose now for the sake of contradiction that $c \neq 0$. Without loss of generality we assume that $c_1^{1,1} \neq 0$ and we first aim to prove that

$$bc_1 = \sum_{\tau,\sigma} e^{i(m-n)\theta_\tau} e^{i(k-l)\delta_\sigma} c_1^{\tau,\sigma} \neq 0 \tag{3.14}$$

for certain $(m,n,k,l) \in J$.

Actually, by the Theorem 2.4 and the Lemmas 2.5 and 2.6 in [20], which use the theory of linear recurrence sequences, and in particular a generalization of the Skolem-Mahler-Lech Theorem due to Laurent [25, Theorem 1] (see also [33]), we know that given the angles $\theta_\tau, \delta_\sigma$ and the vector c_1, with $c_1^{1,1} \neq 0$, there exist $N, M, x, y \in \mathbb{N}$ such that the function defined in $\mathbb{Z}^2$

$$L(\alpha, \beta) := \sum_{\tau,\sigma} e^{i\,\alpha\,\theta_\tau} e^{i\,\beta\,\delta_\sigma} c_1^{\tau,\sigma}$$

is not zero for all (α, β) in the set $P := (N\mathbb{Z} + x) \times (M\mathbb{Z} + y)$.

By Lemma 3.5 applied to J', there exists a sequence $S := \{(\alpha_i, \beta_i)\} \subseteq P \cap J'$ such that $|\alpha_i|, |\beta_i| \to \infty$. As a consequence, (3.14) holds true for every $(m, n, k, l) \in J$ such that $(m - n, k - l) \in S$.

Now we can select $(m - n, k - l) \in S$ with $|m - n|, |k - l|$ as large as we want (which implies that $m \neq n$, $k \neq l$ and that $m + n$ and $k + l$ are also large). For the corresponding $(m, n, k, l) \in J$, the equation in (3.12) cannot be zero in view of Equation (3.14) and Lemma 3.8.

We have then proved that a nontrivial solution of system (3.12) cannot exist. □

Remark 3.10. Observe that in the case $p = q = 1$, in view of Remark 3.6, the sum in Equation (3.13) has only one term which is $|bc_1|^2$. Then the contradiction follows readily after proving (3.14).

At this point, Theorem 1.1 is a consequence of the Propositions 3.3 and 3.4. The Theorems 1.2 and 1.3 follow from the same two propositions after translating back from the expansion in Lemma 2.2 to the usual ones in the Equations (1.10) and (1.11) (see in the Sections 2.1 and 2.2).

4 Characterization of Positive Definiteness on $\Omega_\infty \times \Omega_\infty$

In this section we aim to prove the following:

Theorem 4.1. *Let $f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be a continuous function. Then f is PD on $\Omega_\infty \times \Omega_\infty$ if, and only if,*

$$\begin{aligned} f(\xi, \eta) &= \sum_{m,n,k,l \in \mathbb{Z}_+} a_{m,n,k,l} R^{\infty}_{m,n}(\xi) R^{\infty}_{k,l}(\eta) \\ &= \sum_{m,n,k,l \in \mathbb{Z}_+} a_{m,n,k,l} \xi^m \overline{\xi}^n \eta^k \overline{\eta}^l, \qquad (\xi, \eta) \in \mathbb{D} \times \mathbb{D}, \end{aligned} \tag{4.1}$$

where $\sum a_{m,n,k,l} < \infty$ and $a_{m,n,k,l} \geq 0$ for all $m, n, k, l \in \mathbb{Z}_+$.

Moreover, the series in Equation (4.1) is uniformly convergent on $\mathbb{D} \times \mathbb{D}$.

In the proof we will use ideas from [5] and we will need the following lemma, whose proof is analogous to that of Lemma 4.1 in [5] and will be omitted.

Lemma 4.2. *Let $q, p \in \mathbb{N} \cup \{\infty\}$, $q, p \geq 2$ and $f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be a continuous and PD function on $\Omega_{2q} \times \Omega_{2p}$. Given points $w_1, \dots, w_L \in \Omega_{2p}$ and numbers $c_1, \dots, c_L \in \mathbb{C}$, the function $F : \mathbb{D} \to \mathbb{C}$ defined by*

$$F(\xi) = \sum_{j,k=1}^{L} f(\xi, w_j \cdot w_k) c_j \overline{c_k} \tag{4.2}$$

is continuous and PD on Ω_{2q}.

Proof of Theorem 4.1. First observe that, as for a single sphere, f is PD on $\Omega_\infty \times \Omega_\infty$ if, and only if, f is PD on $\Omega_{2q} \times \Omega_{2p}$ for every $q, p \geq 2$.

It is also easy to see that the function $g(\xi) = \xi$, $\xi \in \mathbb{D}$, is PD on Ω_{2q} for every $q \geq 2$, as well as its conjugate. By the Schur Product Theorem for Positive Definite kernels, cf. [3, Theorem 3.1.12], one obtains that also $h(\xi) = \xi^m \overline{\xi}^n$ is PD on Ω_{2q} for $q \geq 2$ and $m, n \in \mathbb{Z}_+$, and that $\xi^m \overline{\xi}^n \eta^k \overline{\eta}^l$ is PD on $\Omega_{2q} \times \Omega_{2p}$ for $q, p \geq 2$ and $m, n, k, l \in \mathbb{Z}_+$. As a consequence, any function of the form (4.1) is continuous and PD on $\Omega_{2q} \times \Omega_{2p}$ for every $q, p \geq 2$, and then on $\Omega_\infty \times \Omega_\infty$ too.

Now let the continuous function $f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be PD on $\Omega_\infty \times \Omega_\infty$. For $\eta \in \mathbb{D}$, $c \in \mathbb{C}$, consider the special case of (4.2) with $L = 2, q = \infty, p = 2$, $w_1 = (\eta, w), w_2 = (1, 0) \in \Omega_4$, $c_1 = 1, c_2 = c$, that is,

$$F_{\eta,c}(\xi) = f(\xi, 1)(1 + |c|^2) + f(\xi, \eta)\overline{c} + f(\xi, \overline{\eta})c. \tag{4.3}$$

By Lemma 4.2, $F_{\eta,c}$ is a continuous PD function on Ω_∞. Then, using a theorem due to Christensen and Ressel, see [11], it can be written as

$$F_{\eta,c}(\xi) = \sum_{m,n \in \mathbb{Z}_+} a_{m,n}(\eta, c)\, \xi^m \overline{\xi}^n,$$

where $a_{m,n}(\eta, c) \geq 0$ are uniquely determined and satisfy $\sum_{m,n \in \mathbb{Z}_+} a_{m,n}(\eta, c) < \infty$.

By using $c = 1, -1, i$ and proceeding as in the end of the proof of [5, Theorem 1.2], one obtains that

$$f(\xi, \eta) = \frac{1-i}{4} F_{\eta,1}(\xi) - \frac{1+i}{4} F_{\eta,-1}(\xi) + \frac{i}{2} F_{\eta,i}(\xi) = \sum_{m,n \in \mathbb{Z}_+} \varphi_{m,n}(\eta) \xi^m \overline{\xi}^n, \tag{4.4}$$

where

$$\varphi_{m,n}(\eta) := \frac{1-i}{4} a_{m,n}(\eta, 1) - \frac{1+i}{4} a_{m,n}(\eta, -1) + \frac{i}{2} a_{m,n}(\eta, i), \quad \eta \in \mathbb{D},$$

and then

$$\left| \sum_{m,n \in \mathbb{Z}_+} \varphi_{m,n}(\eta) \right| < \infty, \qquad \eta \in \mathbb{D}. \tag{4.5}$$

Consider now $p \geq 2$ and the function $\widetilde{f}_p : \mathbb{D} \times U(p) : (\xi, A) \mapsto f(\xi, Ae_p \cdot e_p)$, where $e_p = (1, 0, \ldots, 0) \in \Omega_{2p}$. By construction, $\widetilde{f}_p$ is continuous and PD on $\Omega_\infty \times U(p)$. By Theorem 1.3 in [5], we can expand $\widetilde{f}_p$ as

$$\widetilde{f}_p(\xi, A) = \sum_{m,n \in \mathbb{Z}_+} \widetilde{\varphi}^{(p)}_{m,n}(A) R^\infty_{m,n}(\xi) = \sum_{m,n \in \mathbb{Z}_+} \widetilde{\varphi}^{(p)}_{m,n}(A) \xi^m \overline{\xi}^n ,$$

where $\widetilde{\varphi}^{(p)}_{m,n}$ are continuous PD functions on $U(p)$.

By differentiation one has that

$$\widetilde{\varphi}^{(p)}_{m,n}(A) = \frac{1}{m!n!} \frac{\partial^{m+n} \widetilde{f}_p(0, A)}{\partial \xi^m \partial \overline{\xi}^n}$$

and

$$\varphi_{m,n}(\eta) = \frac{1}{m!n!} \frac{\partial^{m+n} f(0, \eta)}{\partial \xi^m \partial \overline{\xi}^n}, \tag{4.6}$$

but by construction

$$\widetilde{\varphi}^{(p)}_{m,n}(A) = \frac{1}{m!n!} \frac{\partial^{m+n} \widetilde{f}_p(0, A)}{\partial \xi^m \partial \overline{\xi}^n} = \frac{1}{m!n!} \frac{\partial^{m+n} f(0, Ae_p \cdot e_p)}{\partial \xi^m \partial \overline{\xi}^n} = \varphi_{m,n}(Ae_p \cdot e_p).$$

By Remark 2.3 we deduce that $\varphi_{m,n}$ is continuous and PD on Ω_{2p}, for every $p \geq 2$. As a consequence, $\varphi_{m,n}$ is PD on Ω_∞ and thus we can again use the theorem by Christensen and Ressel, in order to conclude that for every m, n,

$$\varphi_{m,n}(\eta) = \sum_{k,l \in \mathbb{Z}_+} a_{m,n,k,l}\, \eta^k \overline{\eta}^l, \quad \eta \in \mathbb{D} ,$$

where $a_{m,n,k,l} \geq 0$, for every $k, l \in \mathbb{Z}_+$, and $\sum_{k,l \in \mathbb{Z}_+} a_{m,n,k,l} < \infty$. Thus,

$$f(\xi, \eta) = \sum_{m,n \in \mathbb{Z}_+} \sum_{k,l \in \mathbb{Z}_+} a_{m,n,k,l}\, \xi^m \overline{\xi}^n \eta^k \overline{\eta}^l ,$$

and then $\displaystyle\sum_{m,n,k,l \in \mathbb{Z}_+} a_{m,n,k,l} < \infty.$ □

5 From Ω_2 to S^1

In this section we aim to show that one can deduce, from the results obtained on the complex sphere Ω_2, corresponding results for the real sphere S^1.

For instance, it is possible to relate the characterizations of SPD functions on Ω_2 from [29], with that of SPD functions on S^1 (see [29, 1]) and it is possible to obtain the characterization of Strict Positive Definiteness on $S^1 \times S^1$ proved in [20], as a consequence of Theorem 1.3.

Actually, the known conditions for Strict Positive Definiteness on S^1 exhibit more similarities with the conditions we obtain here in the Theorems 1.1, 1.2 and 1.3 for the complex spheres, where an intersection with every product of full arithmetic progressions in $\mathbb{Z}$ is required, rather than with the known conditions for real spheres in higher dimensions, where only progressions of step 2 are involved (see Equations (1.3) and (1.7)).

We will show here how to deduce Theorem 1.4 from Theorem 1.2. The two characterizations mentioned above can be obtained in the same way.

First, we will show how one can establish a correspondence between PD (and between SPD) functions on S^1 and a subset of those on Ω_2.

Lemma 5.1. *There exists a bijection between PD (resp. SPD) functions on S^1 and PD (resp. SPD) functions on Ω_2 which are invariant under conjugation, that is, $f(e^{i\phi}) = f(e^{-i\phi})$, $\phi \in [0, 2\pi)$.*

Proof. Let $f : \partial\mathbb{D} \to \mathbb{C}$ be a PD function on Ω_2 satisfying $f(e^{i\phi}) = f(e^{-i\phi})$, then it is real valued and it only depends on the real part.
Consider the bijection

$$A : \Omega_2 \to S^1 : e^{i\phi} \mapsto (\cos(\phi), \sin(\phi))$$

and the surjective map

$$C : \partial\mathbb{D} \to [-1, 1] : e^{i\phi} \mapsto \cos(\phi)\,,$$

which admits a right inverse $C^- : x \mapsto e^{i\arccos(x)}$. Then $C \circ C^- = id_{[-1,1]}$ and since f only depends on the real part,

$$f(C^- \circ C(e^{i\phi})) = f(e^{i\phi}), \quad e^{i\phi} \in \partial\mathbb{D}. \tag{5.1}$$

Also observe that

$$C(w \cdot w') = Aw \cdot_{\mathbb{R}} Aw', \quad w, w' \in \Omega_2. \tag{5.2}$$

Therefore, the bijection in the claim is the following:

$$B : f \mapsto \widehat{f} := f \circ C^-\,,$$

whose inverse is given by

$$B^{-1} : \widehat{f} \mapsto f := \widehat{f} \circ C\,.$$

Actually, for kernels K and $\widehat{K}$ associated, respectively, to f and $\widehat{f}$, it holds, by (5.1-5.2),

$$\widehat{K}(Aw, Aw') = \widehat{f}(Aw \cdot_{\mathbb{R}} Aw') = f(C^- \circ C(w \cdot w')) = f(w \cdot w') = K(w, w'),$$

then the definition of PD (resp. SPD) in (1.1) becomes equivalent for the two kernels. $\square$

Using the argument in Lemma 5.1 one can obtain

Lemma 5.2. *There exists a bijection between PD (resp. SPD) functions on $S^1 \times \Omega_{2p}$ and PD (resp. SPD) functions on $\Omega_2 \times \Omega_{2p}$ that are invariant under conjugation in the first variable, that is, $f(e^{i\phi}, \eta) = f(e^{-i\phi}, \eta)$ $\phi \in [0, 2\pi)$, $\eta \in \mathbb{D}$.*

The bijection in Lemma 5.2 is given by

$$B : f \mapsto \widehat{f}(\triangle, \eta) := f(C^-(\triangle), \eta)\,,$$

where C^- is defined in the proof of Lemma 5.1.

Proof of Theorem 1.4. Let f be a function as in Theorem 1.2 which is also invariant under conjugation in the first variable. This implies that $a_{m,k,l} = a_{-m,k,l}$, $m \in \mathbb{Z}$, $k, l \in \mathbb{Z}_+$ in the expansion (1.10), then

$$f(e^{i\phi}, \eta) = \sum_{m\in\mathbb{Z},\, k,l\in\mathbb{Z}_+} a_{m,k,l} e^{im\phi} R_{k,l}^{p-2}(\eta) = \sum_{k,l\in\mathbb{Z}_+} R_{k,l}^{p-2}(\eta) \left(a_{0,k,l} + \sum_{m\in\mathbb{N}} 2a_{m,k,l} \cos(m\phi) \right)$$

and the function $\widehat{f}$ corresponding to f in the bijection from Lemma 5.2 can be written as

$$\widehat{f}(\cos\phi, \eta) = f(C^-(\cos\phi), \eta) = f(e^{i\phi}, \eta) = \sum_{k,l\in\mathbb{Z}_+} R_{k,l}^{p-2}(\eta) \left(a_{0,k,l} + \sum_{m\in\mathbb{N}} 2a_{m,k,l} \cos(m\phi) \right),$$

which can be rewritten as in (1.12), where the coefficients $\widehat{a}_{m,k,l}$ are such that

$$a_{m,k,l} > 0 \text{ (Resp. } \geq 0) \iff \widehat{a}_{|m|,k,l} > 0 \text{ (Resp. } \geq 0), \quad m \in \mathbb{Z},\ k, l \in \mathbb{Z}_+ \tag{5.3}$$

and

$$\sum a_{m,k,l} < \infty \iff \sum \widehat{a}_{|m|,k,l} < \infty\,. \tag{5.4}$$

As a consequence one obtains, from Lemma 5.2, both the characterization (1.12) for Positive Definiteness on $S^1 \times \Omega_{2p}$ (which can also be obtained from the results in [16]) and the characterization of Strict Positive Definiteness in Theorem 1.4. □

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