Schoenberg's theorem for real and complex Hilbert spheres revisited

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Abstract

Schoenberg's theorem for the complex Hilbert sphere proved by Christensen and Ressel in 1982 by Choquet theory is extended to the following result: Let L denote a locally compact group and let $\overline{\mathbb{D}}$ denote the closed unit disc in the complex plane. Continuous functions $f : \overline{\mathbb{D}} \times L \to \mathbb{C}$ such that $f(\xi \cdot \eta, u^{-1}v)$ is a positive definite kernel on the product of the unit sphere in $\ell_2(\mathbb{C})$ and L are characterized as the functions with a uniformly convergent expansion

$$f(z,u) = \sum_{m,n=0}^{\infty} \varphi_{m,n}(u) z^m \overline{z}^n,$$

where $\varphi_{m,n}$ is a double sequence of continuous positive definite functions on L such that $\sum \varphi_{m,n}(e_L) < \infty$ (e_L is the neutral element of L). It is shown how the coefficient functions $\varphi_{m,n}$ are obtained as limits from expansions for positive definite functions on finite dimensional complex spheres via a Rodrigues formula for disc polynomials.

Similar results are obtained for the real Hilbert sphere.

2010 MSC: 43A35,33C45,33C55

Keywords: Positive definite functions, spherical harmonics for real and complex spheres, Gegenbauer polynomials, disc polynomials.

1 Introduction and main results

Characterizations of positive definite functions on spheres or on products of spheres with locally compact groups can be found in the literature, see [4], [5], [6], [7], [10], [15], [19]. The spheres can be real or complex and of finite or countably infinite dimension. In this paper we obtain a characterization of positive definite

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functions on the product of the unit sphere in the complex Hilbert space $\ell_2(\mathbb{C})$ with a locally compact group via a power series expansion. This is the only missing case in the above picture. We show how the coefficients of this expansion are related with those of the expansions on the product of finite dimensional complex spheres with locally compact groups.

In the case of real spheres there has been several statistical applications of such results, see [1], [9], [17], but we do not know of statistical applications in the case of complex spheres. The interested reader is referred to [13], where parametric families of positive definite functions on complex spheres are provided.

In order to arrive quickly at the main results of the paper, we postpone precise definitions to Section 2.

Schoenberg's theorems in [19] for real spheres $\mathbb{S}^d, d = 1, 2, \ldots, \infty$, give uniformly convergent expansions

$$f(x) = \sum_{n=0}^{\infty} \varphi_{n,d} c_n(d, x), \quad x \in [-1, 1] \tag{1}$$

for certain classes $\mathcal{P}(\mathbb{S}^d)$ of continuous functions $f : [-1, 1] \to \mathbb{R}$. Here, $c_n(d, x)$ are normalized ultraspherical polynomials when $d \in \mathbb{N}$, see (13), while $c_n(\infty, x) = x^n$. Furthermore, $(\varphi_{n,d})_{n\geq 0}$ is a sequence of non-negative numbers satisfying $\sum_n \varphi_{n,d} < \infty$.

Schoenberg's theorems were extended in [5] to classes $\mathcal{P}(\mathbb{S}^d, L)$ of continuous functions $f : [-1, 1] \times L \to \mathbb{C}$, where L is an arbitrary locally compact group. In this case the uniformly convergent expansions are

$$f(x, u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d, x), \quad x \in [-1, 1], u \in L, \tag{2}$$

and the expansion coefficients $\varphi_{n,d}$ from (2) belong to the class $\mathcal{P}(L)$ of continuous positive definite functions on the group L. They are called the d-Schoenberg functions associated with f. The extension was motivated by problems in geostatistics in the particular case, where L is the additive group $\mathbb{R}$ of real numbers representing time.

In the special case where L is reduced to the neutral element e_L, the results of [5] yield Schoenberg's theorems.

The sets $\mathcal{P}(\mathbb{S}^d, L)$ are decreasing:

$$\mathcal{P}(\mathbb{S}^{d+1}, L) \subseteq \mathcal{P}(\mathbb{S}^d, L), \quad \mathcal{P}(\mathbb{S}^\infty, L) = \bigcap_{d=1}^{\infty} \mathcal{P}(\mathbb{S}^d, L),$$

so a function $f \in \mathcal{P}(\mathbb{S}^\infty, L)$ has expansion coefficient functions $\varphi_{n,d}$ for $d = 1, 2, \ldots, \infty$. In Schoenberg's paper [19] the scalar coefficients $\varphi_{n,\infty}$ have been obtained as accumulation points of $\varphi_{n,d}$ as $d \to \infty$, and this has been sharpened in [5] to pointwise limits

$$\lim_{d\to\infty} \varphi_{n,d}(u) = \varphi_{n,\infty}(u) \tag{3}$$

for each $u \in L, n \geq 0$.

In this paper we have a different approach which in (3) yields the local uniform convergence with respect to $u \in L$. It is based on a sharpening of the differentiability properties of $f \in \mathcal{P}(\mathbb{S}^\infty, L)$ with respect to $x \in \,]-1, 1[$ given in Proposition 3.2, and the Rodrigues formula for the Gegenbauer polynomials. This is our first main result.

Theorem 1.1. *Let $f \in \mathcal{P}(\mathbb{S}^\infty, L)$ and let $\varphi_{n,d}, n \geq 0, d < \infty$, denote the d-Schoenberg functions associated with f.*

For each $n = 0, 1, \ldots$

$$\lim_{d\to\infty} \varphi_{n,d}(u) = \frac{1}{n!}\frac{\partial^n f(0,u)}{\partial x^n}$$

uniformly for u in compact subsets of L. The sequence of functions

$$u \mapsto \frac{1}{n!}\frac{\partial^n f(0,u)}{\partial x^n}, \quad n \geq 0,$$

belongs to $\mathcal{P}(L)$ and gives the coefficient sequence $(\varphi_{n,\infty})$ in Eq. (2) when $d = \infty$.

The proof of Theorem 1.1 will be given in Section 3.

The main purpose of this paper is to achieve a similar result for the complex Hilbert sphere, but since this is considerably more technical, the real case will serve as an introduction to the complex case, which we shall introduce now.

Schoenberg's theorems have been extended to complex spheres Ω_{2q}, where $q = 1, 2, \ldots$ or $q = \infty$. The case $q = \infty$ was settled by Christensen and Ressel [7] in 1982 and the case $q < \infty$ was done by Menegatto and Peron [15] in 2001. The authors obtain uniformly convergent expansions

$$f(z) = \sum_{m,n=0}^{\infty} \varphi_{m,n}^{(q-2)} R_{m,n}^{q-2}(z), \quad z \in \overline{\mathbb{D}} \tag{4}$$

for certain classes $\mathcal{P}(\Omega_{2q})$ of continuous functions $f : \overline{\mathbb{D}} \to \mathbb{C}$, where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \overline{\mathbb{D}} = \{z \in \mathbb{D} : |z| \leq 1\}. \tag{5}$$

Here, $(\varphi_{m,n}^{(q-2)})_{m,n\geq 0}$ is a double sequence of non-negative numbers satisfying

$$\sum_{m,n=0}^{\infty} \varphi_{m,n}^{(q-2)} < \infty,$$

and $R^{q-2}_{m,n}$ are special cases of the disc polynomials $R^{\alpha}_{m,n}, \alpha > -1$, see (22), while

$$R^{\infty}_{m,n}(z) = z^m \overline{z}^n.$$

See [13] for examples of functions from the class $\mathcal{P}(\Omega_{2q})$.

Note that while Schoenberg's Theorem for $\mathcal{P}(\mathbb{S}^\infty)$ was obtained by a limit procedure from his theorem for $\mathcal{P}(\mathbb{S}^d)$ by letting d tend to infinity, this has not been the case for complex spheres. The result for $\mathcal{P}(\Omega_\infty)$ was obtained long before the case $\mathcal{P}(\Omega_{2q})$ with finite q was settled. See Remark 4.7 for an open question related to this.

The expansions (4) have been extended in [4] for $q < \infty$ to classes $\mathcal{P}(\Omega_{2q}, L)$ of continuous functions $f : \overline{\mathbb{D}} \times L \to \mathbb{C}$, where L as before is an arbitrary locally compact group. Functions $f \in \mathcal{P}(\Omega_{2q}, L)$ have a uniformly convergent expansion

$$f(z,u) = \sum_{m,n=0}^{\infty} \varphi^{(q-2)}_{m,n}(u) R^{q-2}_{m,n}(z), \quad z \in \overline{\mathbb{D}}, u \in L. \tag{6}$$

Like the case of real spheres the coefficient functions $\varphi^{(q-2)}_{m,n}$ in (6) belong to $\mathcal{P}(L)$. In [4] it is pointed out that similar expansions hold for all compact Gelfand pairs, thus giving a unified treatment of the expansion questions related to real and complex finite-dimensional spheres.

The sets $\mathcal{P}(\Omega_{2q}, L)$ are decreasing:

$$\mathcal{P}(\Omega_{2(q+1)}, L) \subseteq \mathcal{P}(\Omega_{2q}, L), \quad \mathcal{P}(\Omega_\infty, L) = \bigcap_{q=2}^{\infty} \mathcal{P}(\Omega_{2q}, L).$$

In this paper we shall settle the case $q = \infty$ and prove an analogue of Theorem 1.1. It is based on a sharpening of the differentiability properties of $f \in \mathcal{P}(\Omega_\infty, L)$ given in Proposition 4.5, and a Rodrigues formula for the disc polynomials given in [21]. It is our second main result.

Theorem 1.2. *Let $f \in \mathcal{P}(\Omega_\infty, L)$ and for $q \geq 2$ let $(\varphi^{(q-2)}_{m,n})_{m,n\geq 0}$ denote the double sequence from $\mathcal{P}(L)$ such that (6) holds. For $m,n \geq 0$ we have*

$$\lim_{q\to\infty} \varphi^{(q-2)}_{m,n}(u) = \frac{1}{m!n!} \frac{\partial^{m+n}}{\partial \overline{z}^n \partial z^m} f(0,u)$$

uniformly for u in compact subsets of L. The functions

$$\varphi_{m,n}(u) := \frac{1}{m!n!} \frac{\partial^{m+n}}{\partial \overline{z}^n \partial z^m} f(0,u)$$

belong to $\mathcal{P}(L)$, and we have the representation

$$f(z,u) = \sum_{m,n=0}^{\infty} \varphi_{m,n}(u) z^m \overline{z}^n, \quad z \in \overline{\mathbb{D}}, u \in L, \tag{7}$$

where

$$\sum_{m,n=0}^{\infty} \varphi_{m,n}(e_L) < \infty. \tag{8}$$

The series in (7) *is uniformly convergent on* $\overline{\mathbb{D}} \times L$.

Theorem 1.2 settles the difficult "only if"-part of the following representation theorem for $\mathcal{P}(\Omega_\infty, L)$.

Theorem 1.3. *Let* $f : \overline{\mathbb{D}} \times L \to \mathbb{C}$ *be a continuous function. Then* $f \in \mathcal{P}(\Omega_\infty, L)$ *if and only if* f *has a representation*

$$f(z,u) = \sum_{m,n=0}^{\infty} \varphi_{m,n}(u) z^m \overline{z}^n, \quad z \in \overline{D}, u \in L, \tag{9}$$

where $\varphi_{m,n} \in \mathcal{P}(L)$ *satisfy* $\sum_{m,n} \varphi_{m,n}(e_L) < \infty$.

The proofs of Theorem 1.2 and 1.3 will be given in Section 4.

2 Background material

In his seminal paper [19] Schoenberg introduced and characterized positive definite functions on spheres. The d-dimensional unit sphere of $\mathbb{R}^{d+1}$ is given as

$$\mathbb{S}^d = \left\{ x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : \sum_{k=1}^{d+1} x_k^2 = 1 \right\}, \ d \geq 1.$$

For vectors ξ, η belonging to $\mathbb{S}^d$, the scalar product $\xi \cdot \eta$ belongs to $[-1,1]$. By $\mathcal{P}(\mathbb{S}^d)$ we denote the set of continuous functions $f : [-1,1] \to \mathbb{R}$ such that the kernel $(\xi, \eta) \mapsto f(\xi \cdot \eta)$ is positive definite on $\mathbb{S}^d$ in the sense that for any $n \in \mathbb{N}$, arbitrary $\xi_1, \ldots, \xi_n \in \mathbb{S}^d$ and $c_1, \ldots, c_n \in \mathbb{R}$ one has

$$\sum_{j,k=1}^{n} f(\xi_j \cdot \xi_k) c_j c_k \geq 0, \tag{10}$$

i.e., the symmetric matrix $[f(\xi_j \cdot \xi_k)]_{j,k=1}^n$ is positive semidefinite.

In a general setting we recall that for an arbitrary non-empty set X, a kernel on X is a function $K : X^2 \to \mathbb{C}$. It is called a positive definite kernel, if for any $n \in \mathbb{N}$, arbitrary points $x_1, \ldots, x_n \in X$ and numbers $c_1, \ldots, c_n \in \mathbb{C}$ one has

$$\sum_{j,k=1}^{n} K(x_j, x_k) c_j \overline{c_k} \geq 0,$$

i.e., the matrix $[K(x_j, x_k)]_{j,k=1}^n$ is Hermitian and positive semidefinite. For a treatment of these concepts see e.g. [3]. A positive definite kernel on $\mathbb{S}^d$ of the form $f(\xi \cdot \eta)$ is automatically real-valued by symmetry of the scalar product.

Let L denote an arbitrary locally compact group written multiplicatively and with neutral element e_L. By $\mathcal{P}(L)$ we denote the set of continuous positive definite functions $f : L \to \mathbb{C}$, i.e., the continuous functions f for which the kernel $(u, v) \mapsto f(u^{-1}v)$ is positive definite on L. This class of functions is very important in the theory of unitary representations of L on Hilbert spaces, see [8], [18].

Schoenberg's characterization of the class $\mathcal{P}(\mathbb{S}^d)$ is a special case of the Bochner-Godement Theorem for Gelfand pairs, see [4] and the references therein. In another direction it has been extended in [5] to the class $\mathcal{P}(\mathbb{S}^d, L)$ of continuous functions $f : [-1, 1] \times L \to \mathbb{C}$ such that the kernel $((\xi, u), (\eta, v)) \mapsto f(\xi \cdot \eta, u^{-1}v)$ is positive definite on $\mathbb{S}^d \times L$.

Their result is reported here for a self-contained exposition.

Theorem 2.1. (Theorem 3.3 in [5]) *Let $d \in \mathbb{N}$ and let $f : [-1, 1] \times L \to \mathbb{C}$ be a continuous function. Then f belongs to $\mathcal{P}(\mathbb{S}^d, L)$ if and only if there exists a sequence of functions $(\varphi_{n,d})_{n\geq 0}$ from $\mathcal{P}(L)$ with $\sum_n \varphi_{n,d}(e_L) < \infty$ such that*

$$f(x, u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d, x), \quad x \in [-1, 1],\ u \in L. \tag{11}$$

The above expansion is uniformly convergent for $(x, u) \in [-1, 1] \times L$, and we have

$$\varphi_{n,d}(u) = \frac{N_n(d)\sigma_{d-1}}{\sigma_d} \int_{-1}^{1} f(x, u) c_n(d, x)(1 - x^2)^{d/2-1}\, dx. \tag{12}$$

Here we have used the notation

$$c_n(d, x) = C_n^{(\lambda)}(x)/C_n^{(\lambda)}(1), \quad \lambda = (d - 1)/2,\ d \geq 2, \tag{13}$$

for the ultraspherical polynomials $c_n(d, x)$ as normalized Gegenbauer polynomials $C_n^{(\lambda)}(x)$ for the parameter $\lambda = (d-1)/2$, while $c_n(1, x) = T_n(x)$ are the Chebyshev polynomials, cf. [2, p. 302], [5]. For later use we recall that

$$C_n^{(\lambda)}(1) = \frac{(2\lambda)_n}{n!}, \quad \lambda > 0. \tag{14}$$

The symbol $(a)_n$ refers to the Pochhammer symbol:

$$(a)_n = a(a + 1) \dots (a + n - 1),\ n \geq 1, \quad (a)_0 = 1.$$

The constant σ_d denotes the total mass of the surface measure ω_d on $\mathbb{S}^d$

$$\sigma_d = \omega_d(\mathbb{S}^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d + 1)/2)}. \tag{15}$$

Note that

$$\frac{\sigma_{d-1}}{\sigma_d}\int_{-1}^{1}(1-x^2)^{d/2-1}\,dx=1.$$

Finally, $N_n(d)$ is the dimension of a space of spherical harmonics, cf. [5, Eq. (11)] or [16, p. 4], and is given by

$$N_n(d)=\frac{(d)_{n-1}}{n!}(d+2n-1),\ n\geq 1,\quad N_0(d)=1. \tag{16}$$

Schoenberg's Theorem for $\mathcal{P}(\mathbb{S}^d)$ is the special case of the previous theorem, where the group $L=\{e_L\}$ is trivial. The functions in $\mathcal{P}(L)$ are then just non-negative constants.

If we restrict the vectors $\xi_1,\ldots,\xi_n\in\mathbb{S}^d$ to lie on the subsphere $\mathbb{S}^{d-1}$, identified with the equator of $\mathbb{S}^d$, we see that $\mathcal{P}(\mathbb{S}^d,L)\subseteq\mathcal{P}(\mathbb{S}^{d-1},L)$.

We also consider

$$\mathcal{P}(\mathbb{S}^\infty,L):=\bigcap_{d=1}^{\infty}\mathcal{P}(\mathbb{S}^d,L), \tag{17}$$

which is the set of continuous functions $f:[-1,1]\times L\to\mathbb{C}$ such that the kernel

$$((\xi,u),(\eta,v))\mapsto f(\xi\cdot\eta,u^{-1}v) \tag{18}$$

is positive definite on $\mathbb{S}^d\times L$ for all $d\in\mathbb{N}$. We note in passing that the notation $\mathcal{P}(\mathbb{S}^\infty,L)$ suggests an intrinsic definition using the real Hilbert sphere

$$\mathbb{S}^\infty=\left\{(x_k)_{k\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}:\sum_{k=1}^{\infty}x_k^2=1\right\},$$

which is the unit sphere in the Hilbert sequence space $\ell_2(\mathbb{R})$ of square summable real sequences. The intrinsic definition of $\mathcal{P}(\mathbb{S}^\infty,L)$ is as the set of continuous functions $f:[-1,1]\times L\to\mathbb{C}$ such that the kernel (18) is positive definite on $\mathbb{S}^\infty\times L$. This identification is made explicit in [5].

The following holds:

Theorem 2.2. (Theorem 3.10 in [5]) *Let L denote a locally compact group and let $f:[-1,1]\times L\to\mathbb{C}$ be a continuous function. Then f belongs to $\mathcal{P}(\mathbb{S}^\infty,L)$ if and only if there exists a sequence $(\varphi_{n,\infty})_{n\geq 0}$ from $\mathcal{P}(L)$ with $\sum_n\varphi_{n,\infty}(e_L)<\infty$ such that*

$$f(x,u)=\sum_{n=0}^{\infty}\varphi_{n,\infty}(u)x^n,\quad (x,u)\in[-1,1]\times L. \tag{19}$$

The above expansion is uniformly convergent for $(x,u)\in[-1,1]\times L$.

Ziegel [23] discovered that $f\in\mathcal{P}(\mathbb{S}^d)$ is continuously differentiable of order $[(d-1)/2]$ on $]-1,1[$ and this was extended to $\mathcal{P}(\mathbb{S}^d,L)$ in [5].

For $f \in \mathcal{P}(\mathbb{S}^\infty, L)$ we know that $f(\cdot, u) \in C^\infty(]-1,1[)$ for each $u \in L$ and from (19) we get

$$\varphi_{n,\infty}(u) = \frac{1}{n!}\frac{\partial^n f(0,u)}{\partial x^n}, \quad u \in L, n \geq 0.$$

We are now going to explain similar results for complex spheres.

The complex unit sphere of (real) dimension $2q-1$ is given by

$$\Omega_{2q} = \left\{ z = (z_1, \ldots, z_q) \in \mathbb{C}^q : ||z||^2 = \sum_{k=1}^{q} |z_k|^2 = 1 \right\}, \qquad q \geq 1,$$

and $\mathbb{C}^q$ is equipped with the Hermitian scalar product

$$z \cdot w = \sum_{k=1}^{q} z_k \overline{w_k}, \quad z, w \in \mathbb{C}^q.$$

Similarly Ω_∞ denotes the unit sphere in the complex Hilbert space $\ell_2(\mathbb{C})$ with the usual Hermitian scalar product $z \cdot w$ for $z, w \in \ell_2(\mathbb{C})$.

For vectors $\xi, \eta \in \Omega_{2q}, q = 1, 2, \ldots, \infty$, the Hermitian scalar product $\xi \cdot \eta$ belongs to the closed unit disc $\overline{\mathbb{D}}$ defined in (5). For $2 \leq q \leq \infty$ we have

$$\{\xi \cdot \eta \ : \ \xi, \eta \in \Omega_{2q}\} = \overline{\mathbb{D}},$$

while for $q = 1$

$$\xi \cdot \eta = \xi\overline{\eta} = \xi\eta^{-1}, \quad \xi, \eta \in \mathbb{C}, \ |\xi| = |\eta| = 1,$$

hence

$$\{\xi \cdot \eta \ : \ \xi, \eta \in \Omega_2\} = \{z \in \mathbb{C} \ : \ |z| = 1\} =: \mathbb{T}.$$

By $\mathcal{P}(\Omega_{2q}), q \geq 2$, we denote the set of continuous functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ such that the kernel $(\xi, \eta) \mapsto f(\xi \cdot \eta)$ is positive definite on Ω_{2q}. For a locally compact group L we denote by $\mathcal{P}(\Omega_{2q}, L)$ the set of continuous functions $f : \overline{\mathbb{D}} \times L \to \mathbb{C}$ such that the kernel $((\xi, u), (\eta, v)) \mapsto f(\xi \cdot \eta, u^{-1}v)$ is positive definite on $\Omega_{2q} \times L$. When $L = \{e_L\}$ is trivial, then $\mathcal{P}(\Omega_{2q}, L)$ can be identified with $\mathcal{P}(\Omega_{2q})$.

For $q = 1$ we define $\mathcal{P}(\Omega_2) = \mathcal{P}(\mathbb{T})$ as the set of continuous positive definite functions $f : \mathbb{T} \to \mathbb{C}$ on the circle group $\mathbb{T}$. They are characterized as the functions

$$f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}, \quad e^{i\theta} \in \mathbb{T},$$

where $(a_n)_{n \in \mathbb{Z}}$ is a sequence of non-negative numbers satisfying $\sum_n a_n < \infty$. This is Bochner's Theorem for the compact abelian group $\mathbb{T}$. Similarly $\mathcal{P}(\Omega_2, L) =$

$\mathcal{P}(\mathbb{T}\times L)$ is the set of continuous positive definite functions $f:\mathbb{T}\times L\to\mathbb{C}$. They are characterized as the functions

$$f(e^{i\theta},u)=\sum_{n\in\mathbb{Z}}a_n(u)e^{in\theta},\quad e^{i\theta}\in\mathbb{T},u\in L,$$

where $(a_n(u))_{n\in\mathbb{Z}}$ is a sequence of functions from $\mathcal{P}(L)$ satisfying $\sum_n a_n(e_L)<\infty$. This is a special case of Theorem 3.4 in [4]. See also Corollary 3.5 in [4].

Note that Ω_{2q} as a set is equal to $\mathbb{S}^{2q-1}$, if $\mathbb{C}^q$ is identified with $\mathbb{R}^{2q}$. However, $\mathcal{P}(\mathbb{S}^{2q-1},L)$ and $\mathcal{P}(\Omega_{2q},L)$ are different since the first consists of functions $f:[-1,1]\times L\to\mathbb{C}$ and the second of consists of functions $f:\overline{\mathbb{D}}\times L\to\mathbb{C}$. See [15, Section 5] for relations between these classes when $L=\{e_L\}$.

In the following we shall always assume that $q\geq 2$.

In [4] the authors proved the following result, which extends a result by Menegatto and Peron [15, Theorem 4.2] for the case $\mathcal{P}(\Omega_{2q})$.

Theorem 2.3. (Theorem 6.1 in [4]) *Let $2\leq q<\infty$ and let $f:\overline{\mathbb{D}}\times L\to\mathbb{C}$ be a continuous function. Then f belongs to $\mathcal{P}(\Omega_{2q},L)$ if and only if there exists a double sequence of functions $(\varphi_{m,n}^{(q-2)})_{m,n\geq 0}$ from $\mathcal{P}(L)$ with*

$$\sum_{m,n=0}^{\infty}\varphi_{m,n}^{(q-2)}(e_L)<\infty$$

such that

$$f(z,u)=\sum_{m,n=0}^{\infty}\varphi_{m,n}^{(q-2)}(u)R_{m,n}^{q-2}(z),\quad z\in\overline{\mathbb{D}},u\in L. \tag{20}$$

The above expansion is uniformly convergent on $\overline{\mathbb{D}}\times L$, and for $u\in L$ we have

$$\varphi_{m,n}^{(q-2)}(u)=N(q;m,n)\frac{q-1}{\pi}\int_0^1\int_0^{2\pi}f(re^{i\theta},u)\overline{R_{m,n}^{q-2}(re^{i\theta})}r(1-r^2)^{q-2}\,d\theta\,dr. \tag{21}$$

Here

$$N(q;m,n)=\frac{m+n+q-1}{q-1}\binom{m+q-2}{m}\binom{n+q-2}{n}$$

is the dimension of a certain finite-dimensional space, see [12, Eq. (3.12)],[20, p. 295].

The functions $R_{m,n}^{q-2}(z)$ belong to the class of disc polynomials given in [12, Eq. (3.15)] for $\alpha>-1$ as

$$R_{m,n}^{\alpha}(re^{i\theta})=r^{|m-n|}e^{i(m-n)\theta}R_{\min(m,n)}^{(\alpha,|m-n|)}(2r^2-1),\quad 0\leq r\leq 1,\ 0\leq\theta<2\pi, \tag{22}$$

where

$$R_k^{(\alpha,\beta)}(x)=P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1),\quad \alpha,\beta>-1,\ k\in\mathbb{N}_0 \tag{23}$$

are normalized Jacobi polynomials $P_k^{(\alpha,\beta)}$, cf. [2, p. 99]. The case $\alpha = 0$ was considered by Zernike and Brinkman in [22].

See [21] for other expressions and properties of the disc polynomials.

Like for the case of real spheres we have

$$\mathcal{P}(\Omega_{2(q+1)}, L) \subseteq \mathcal{P}(\Omega_{2q}, L),$$

and we consider the set

$$\mathcal{P}(\Omega_\infty, L) := \bigcap_{q=2}^{\infty} \mathcal{P}(\Omega_{2q}, L), \tag{24}$$

which can be identified with the set of continuous functions $f : \overline{\mathbb{D}} \times L \to \mathbb{C}$ such that the kernel $((\xi, u), (\eta, v)) \mapsto f(\xi \cdot \eta, u^{-1}v)$ is positive definite on $\Omega_\infty \times L$.

3 Proofs in the case of the real Hilbert sphere

We need the following sharpening of Proposition 3.8 in [5], which is inspired by results of Ziegel [23].

For functions $F : [-1, 1] \times L \to \mathbb{C}$ we denote

$$||F|| = \sup\{|F(x, u)| : x \in [-1, 1], u \in L\} \leq \infty.$$

Note that if $f \in \mathcal{P}(\mathbb{S}^d, L)$ then $||f|| = f(1, e_L) < \infty$.

Proposition 3.1. *Let $d \in \mathbb{N}$ and suppose that $f \in \mathcal{P}(\mathbb{S}^{d+2}, L)$. Then $f(\cdot, u)$ is continuously differentiable with respect to x in $]-1, 1[$ and $(1-x^2)\partial f/\partial x$ extends to a continuous function on $[-1, 1] \times L$ such that*

$$(1 - x^2)\frac{\partial f(x, u)}{\partial x} = f_1(x, u) - f_2(x, u), \quad (x, u) \in [-1, 1] \times L \tag{25}$$

for functions $f_i \in \mathcal{P}(\mathbb{S}^d, L)$ satisfying

$$||f_i|| \leq d||f||, \qquad i = 1, 2. \tag{26}$$

Proof. Let us first assume $d \geq 2$. By the proof of Proposition 3.8 in [5] we have (25) for $(x, u) \in]-1, 1[\times L$, where

$$f_1(x, u) = d \sum_{n=0}^{\infty} \frac{(2n + d - 1)(n + 1)}{(2n + d + 1)(n + d - 1)} \varphi_{n+1,d}(u) c_n(d, x)$$

and

$$f_2(x, u) = d \sum_{n=2}^{\infty} \frac{n - 1}{n + d - 1} \varphi_{n-1,d+2}(u) c_n(d, x).$$

These formulas show that $f_1, f_2 \in \mathcal{P}(\mathbb{S}^d, L)$ and that

$$\begin{aligned}
||f_1|| &= f_1(1, e_L) \\
&= d\sum_{n=0}^{\infty} \frac{(2n+d-1)(n+1)}{(2n+d+1)(n+d-1)}\varphi_{n+1,d}(e_L) \\
&\leq d\sum_{n=0}^{\infty} \varphi_{n+1,d}(e_L) = d\sum_{n=1}^{\infty} \varphi_{n,d}(e_L) \leq d||f||,
\end{aligned}$$

and

$$||f_2|| = f_2(1, e_L) = d\sum_{n=2}^{\infty} \frac{n-1}{n-1+d}\varphi_{n-1,d+2}(e_L) \leq d\sum_{n=1}^{\infty} \varphi_{n,d+2}(e_L) \leq d||f||.$$

This also shows that the left-hand side of Eq. (25) extends to a continuous function on $[-1, 1] \times L$.

For $d = 1$, Eq. (25) holds again, now with

$$f_1(x, u) = \frac{1}{2}\varphi_{1,1}(u)c_0(1, x) + \sum_{n=1}^{\infty} \varphi_{n+1,1}(u)c_n(1, x)$$

and

$$f_2(x, u) = \sum_{n=2}^{\infty} \frac{n-1}{n}\varphi_{n-1,3}(u)c_n(1, x).$$

This shows that (26) holds also in this case. □

Let $\mathcal{E}_d$ denote the subspace of continuous functions $F : [-1, 1] \times L \to \mathbb{C}$ spanned by functions of the form $p(x)f(x, u)$, where p is a polynomial with complex coefficients and $f \in \mathcal{P}(\mathbb{S}^d, L)$. Clearly $\mathcal{E}_{d+1} \subseteq \mathcal{E}_d$. By Proposition 3.1 we see that $(1 - x^2)\partial/\partial x$ maps $\mathcal{E}_{d+2}$ into $\mathcal{E}_d$.

Proposition 3.2. *Let $d, n \in \mathbb{N}$ and assume that $f \in \mathcal{P}(\mathbb{S}^{d+2n}, L)$. Then $f(\cdot, u) \in C^n(]-1, 1[)$ for $u \in L$ and for $k \leq n$ we have*

$$(1 - x^2)^k \frac{\partial^k f(x, u)}{\partial x^k} \in \mathcal{E}_{d+2(n-k)}. \tag{27}$$

In particular the function in (27) has a continuous extension to $[-1, 1] \times L$.

Proof. It follows by Proposition 3.1 that $f(\cdot, u) \in C^n(]-1, 1[)$ for $u \in L$.

We prove (27) by induction in k, and it certainly holds for $k = 1$ by Proposition 3.1.

Suppose (27) holds for $k < n$. Then the function in (27) is differentiable for $-1 < x < 1$ and differentiation and multiplication with $1 - x^2$ shows that

$$-2kx(1 - x^2)^k \frac{\partial^k f(x, u)}{\partial x^k} + (1 - x^2)^{k+1} \frac{\partial^{k+1} f(x, u)}{\partial x^{k+1}} \in \mathcal{E}_{d+2(n-k-1)}.$$

Using

$$2kx(1-x^2)^k \frac{\partial^k f(x,u)}{\partial x^k} \in \mathcal{E}_{d+2(n-k)} \subseteq \mathcal{E}_{d+2(n-k-1)},$$

we see that

$$(1-x^2)^{k+1} \frac{\partial^{k+1} f(x,u)}{\partial x^{k+1}} \in \mathcal{E}_{d+2(n-k-1)}.$$

□

In the next proposition we prove the weak convergence of a certain net $(\tau_\lambda)_{\lambda>-1}$ of measures introduced below. This convergence is decisive for the proof of our main Theorem 1.1.

For $\lambda > -1$ define the probability measure τ_λ on $[-1,1]$ by

$$\tau_\lambda = B(\lambda+1, 1/2)^{-1}(1-x^2)^\lambda \, dx, \tag{28}$$

where B is the Beta-function given by $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

The set $C([-1,1])$ of continuous functions $f : [-1,1] \to \mathbb{C}$ is a Banach space under the uniform norm $||f||_\infty = \sup_{x\in[-1,1]} |f(x)|$.

Proposition 3.3. *Let $\mathcal{F} \subset C([-1,1])$ be a set of continuous functions on $[-1,1]$ such that*

(i) *$\mathcal{F}$ is bounded, i.e., $\sup_{f\in\mathcal{F}} ||f||_\infty < \infty$,*

(ii) *$\mathcal{F}$ is equicontinuous at $x = 0$, i.e., for every $\varepsilon > 0$ there exists $0 < \delta < 1$ such that $|f(x) - f(0)| \le \varepsilon$ for all $f \in \mathcal{F}$ and all real x with $|x| \le \delta$.*

Then $\lim_{\lambda\to\infty} \int f \, d\tau_\lambda = f(0)$, uniformly for $f \in \mathcal{F}$.

In particular, $\lim_{\lambda\to\infty} \tau_\lambda = \delta_0$ weakly, where δ_0 denotes the Dirac measure concentrated at 0.

Proof. For any $0 < \delta < 1$ and $f \in \mathcal{F}$ we have

$$\int f \, d\tau_\lambda - f(0) = \int_{-\delta}^{\delta} \Big(f(x) - f(0)\Big) \, d\tau_\lambda(x) + \int_{\delta\le|x|\le1} \Big(f(x) - f(0)\Big) \, d\tau_\lambda(x).$$

Using $|f(x) - f(0)| \le 2||f||_\infty$, we get for $\lambda > 0$

$$\begin{aligned} \left| \int f \, d\tau_\lambda - f(0) \right| &\le \sup_{|x|\le\delta} \Big| f(x) - f(0) \Big| + \frac{2||f||_\infty}{B(\lambda+1,1/2)} \int_{\delta\le|x|\le1} (1-x^2)^\lambda \, dx \\ &\le \sup_{|x|\le\delta} \Big| f(x) - f(0) \Big| + \frac{4||f||_\infty(1-\delta)}{B(\lambda+1,1/2)} (1-\delta^2)^\lambda. \end{aligned}$$

For given $\varepsilon > 0$ we first choose $0 < \delta < 1$ so that by (ii)

$$|f(x) - f(0)| \le \varepsilon/2, \quad \text{for all } |x| \le \delta, \ f \in \mathcal{F}.$$

By Stirling's formula

$$B(\lambda+1,1/2)^{-1} \sim \pi^{-1/2}\lambda^{1/2}, \quad \lambda \to \infty,$$

and $\lim_{\lambda\to\infty} \lambda^{1/2}(1-\delta^2)^\lambda = 0$. Therefore, and using (i),

$$\sup_{f\in\mathcal{F}} ||f||_\infty \frac{4(1-\delta)}{B(\lambda+1,1/2)}(1-\delta^2)^\lambda < \varepsilon/2$$

for $\lambda \geq \Lambda_0$, where Λ_0 is sufficiently large. This shows that

$$\sup_{f\in\mathcal{F}} \left| \int f \, d\tau_\lambda - f(0) \right| \leq \varepsilon, \quad \lambda \geq \Lambda_0.$$

$\square$

Proof of Theorem 1.1:

It is known that the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ satisfy the Rodrigues formula, cf. [2, Eq. (6.6.14)]

$$C_n^{(\lambda)}(x) = \frac{(-2)^n(\lambda)_n}{n!(n+2\lambda)_n}(1-x^2)^{1/2-\lambda}\frac{d^n}{dx^n}(1-x^2)^{n+\lambda-1/2}.$$

For the normalized ultraspherical polynomials $c_n(d,x)$ given by (13), the Rodrigues formula reads

$$c_n(d,x) = \frac{(-1)^n}{2^n(d/2)_n}(1-x^2)^{1-d/2}\frac{d^n}{dx^n}(1-x^2)^{n+d/2-1}. \tag{29}$$

Inserting this in Eq. (12) we get

$$\varphi_{n,d}(u) = \frac{N_n(d)\sigma_{d-1}}{\sigma_d}\frac{(-1)^n}{2^n(d/2)_n}\int_{-1}^1 f(x,u)\frac{d^n}{dx^n}(1-x^2)^{n+d/2-1}\,dx.$$

We now make use of n integrations by parts to get

$$\varphi_{n,d}(u) = \frac{N_n(d)\sigma_{d-1}}{\sigma_d}\frac{1}{2^n(d/2)_n}\int_{-1}^1 \frac{\partial^n f(x,u)}{\partial x^n}(1-x^2)^{n+d/2-1}\,dx,$$

because the boundary terms

$$\frac{\partial^k f(x,u)}{\partial x^k}\frac{d^{n-k-1}}{dx^{n-k-1}}(1-x^2)^{n+d/2-1}, \; k = 0,1,\ldots,n-1$$

vanish for $x = \pm 1$ by Proposition 3.2. In fact,

$$\frac{d^{n-k-1}}{dx^{n-k-1}}(1-x^2)^{n+d/2-1} = (1-x^2)^{k+d/2}R_k(x)$$

for some polynomial $R_k(x)$ and

$$(1-x^2)^k \frac{\partial^k f(x,u)}{\partial x^k}\,,$$

has finite values while $(1-x^2)^{d/2}R_k(x)$ vanishes for $x=\pm 1$.

Using the measure (28) with $\lambda = d/2-1$, we find

$$\varphi_{n,d}(u) = \frac{N_n(d)}{2^n(d/2)_n}\int_{-1}^{1}(1-x^2)^n\frac{\partial^n f(x,u)}{\partial x^n}\,d\tau_{d/2-1}(x),$$

and we note that

$$\lim_{d\to\infty}\frac{N_n(d)}{2^n(d/2)_n} = \lim_{d\to\infty}\frac{1}{n!}\frac{(d)_{n-1}(d+2n-1)}{2^n(d/2)_n} = \frac{1}{n!}.$$

By Proposition 3.3 we then get that

$$\lim_{d\to\infty}\varphi_{n,d}(u) = \frac{1}{n!}\left[(1-x^2)^n\frac{\partial^n f(x,u)}{\partial x^n}\right]_{x=0} = \frac{1}{n!}\frac{\partial^n f(0,u)}{\partial x^n}. \tag{30}$$

Given a compact set K in L the family

$$\mathcal{F} := \left\{x\mapsto (1-x^2)^n\frac{\partial^n f(x,u)}{\partial x^n} : u\in K\right\}$$

satisfies the conditions of Proposition 3.3, so the convergence in (30) is uniform for u in compact subsets of L.

This also implies that the function

$$u\mapsto \frac{1}{n!}\frac{\partial^n f(0,u)}{\partial x^n}$$

belongs to $\mathcal{P}(L)$ and is the coefficient $\varphi_{n,\infty}(u)$ of the power series in (19). $\square$

4 Proofs in the case of the complex Hilbert sphere

Let us first consider a function $f\in\mathcal{P}(\Omega_{2q},L)$, where $2\le q\le\infty$. Then we know that

$$\overline{f(z,u)} = f(\overline{z},u^{-1}),\quad |f(z,u)|\le f(1,e),\quad z\in\overline{\mathbb{D}}, u\in L.$$

To f and to elements $u_1,\dots,u_n\in L$ and numbers $c_1,\dots,c_n\in\mathbb{C}$ we define a new function $F:\overline{\mathbb{D}}\to\mathbb{C}$ by

$$F(z) = \sum_{j,k=1}^{n} f(z,u_j^{-1}u_k)c_j\overline{c_k}. \tag{31}$$

It is easy to see that $F(\overline{z}) = \overline{F(z)}$, but in fact, this follows from the more general result inspired by [10] and which can be stated as:

Lemma 4.1. *For any f in $\mathcal{P}(\Omega_{2q}, L)$, the function F in* (31) *belongs to $\mathcal{P}(\Omega_{2q})$.*

Proof. Let $\xi_1, \ldots, \xi_m \in \Omega_{2q}$ and $d_1, \ldots, d_m \in \mathbb{C}$ be arbitrary. We shall prove that $S \geq 0$, where

$$S := \sum_{\mu,\nu=1}^{m} F(\xi_\mu \cdot \xi_\nu) d_\mu \overline{d_\nu}.$$

However,

$$S = \sum_{\mu,\nu=1}^{m} \sum_{j,k=1}^{n} f(\xi_\mu \cdot \xi_\nu, u_j^{-1} u_k) c_j \overline{c_k} d_\mu \overline{d_\nu} \geq 0,$$

because it is "a sum" belonging to the list of mn elements from $\Omega_{2q} \times L$

$$(\xi_1, u_1), \ldots, (\xi_1, u_n), (\xi_2, u_1), \ldots, (\xi_2, u_n), \ldots, (\xi_m, u_1), \ldots, (\xi_m, u_n)$$

together with the list of scalars

$$d_1 c_1, \ldots, d_1 c_n, d_2 c_1, \ldots, d_2 c_n, \ldots, d_m c_1, \ldots, d_m c_n.$$

□

As preparation for the proof of Theorem 1.2 we shall discuss smoothness of functions from $\mathcal{P}(\Omega_{2q}, L)$.

The smoothness results of Ziegel [23] for functions in $\mathcal{P}(\mathbb{S}^d)$ have been extended to functions in $\mathcal{P}(\Omega_{2q})$ in a paper by Menegatto, see [14]. This extension required new ideas, while a further extension to functions in $\mathcal{P}(\Omega_{2q}, L)$ follows the same lines as in [14], so we shall just give the results with a few indications.

For $f \in \mathcal{P}(\Omega_{2(q+1)}, L) \subseteq \mathcal{P}(\Omega_{2q}, L)$ we have the expansions, cf. Theorem 2.3,

$$f(z,u) = \sum_{m,n=0}^{\infty} \varphi_{m,n}^{(q-1)}(u) R_{m,n}^{q-1}(z) = \sum_{m,n=0}^{\infty} \varphi_{m,n}^{(q-2)}(u) R_{m,n}^{q-2}(z), \quad z \in \overline{\mathbb{D}}, u \in L,$$

and the coefficient functions are related in the following way:

Proposition 4.2. *Let $2 \leq q < \infty$. If $f \in \mathcal{P}(\Omega_{2(q+1)}, L)$, then for $m, n \geq 0$ and $u \in L$*

$$\varphi_{m,n}^{(q-1)}(u) = \frac{(m+q-1)(n+q-1)}{(q-1)(m+n+q-1)} \varphi_{m,n}^{(q-2)}(u) - \frac{(m+1)(n+1)}{(q-1)(m+n+q+1)} \varphi_{m+1,n+1}^{(q-2)}(u). \tag{32}$$

In particular,

$$\varphi_{m,n}^{(q-2)}(e_L) \geq \frac{(m+1)(n+1)(m+n+q-1)}{(m+q-1)(n+q-1)(m+n+q+1)} \varphi_{m+1,n+1}^{(q-2)}(e_L). \tag{33}$$

Proof. For the first part we can use the same technique as in the proof of [14, Proposition 4.1]. The second part follows from the fact that $\varphi_{m,n}^{(q-1)}(e_L) \geq 0$ and from the first part. □

Proposition 4.3. *Let $2 \le q < \infty$. If $f \in \mathcal{P}(\Omega_{2(q+1)}, L)$, then for each $u \in L$ fixed*

$$\lim_{M,N\to\infty} \sum_{n=1}^{N+1} \frac{M(n+q-2)}{M+n+q-2} \varphi^{(q-2)}_{M,n-1}(u) = 0$$

and

$$\lim_{M,N\to\infty} \sum_{m=1}^{M-1} \frac{m(N+q-1)}{m+N+q-1} \varphi^{(q-2)}_{m,N}(u) = 0.$$

Both limits are uniform with respect to $u \in L$.

Proof. Define

$$A_{M,N} := \sum_{n=1}^{N+1} \frac{M(n+q-2)}{M+n+q-2} \varphi^{(q-2)}_{M,n-1}(u).$$

Then

$$|A_{M,N}| \le \sum_{n=1}^{\infty} \frac{M(n+q-2)}{M+n+q-2} |\varphi^{(q-2)}_{M,n-1}(u)| \le \sum_{n=1}^{\infty} \frac{M(n+q-2)}{M+n+q-2} \varphi^{(q-2)}_{M,n-1}(e_L).$$

Define

$$c_M := \sum_{n=1}^{\infty} \frac{n+q-2}{M+n+q-2} \varphi^{(q-2)}_{M,n-1}(e_L), \quad M = 1, 2, \dots.$$

We have $0 \le c_M < \infty$ and

$$\sum_{M=1}^{\infty} c_M = \sum_{M=1}^{\infty} \sum_{n=1}^{\infty} \frac{n+q-2}{M+n+q-2} \varphi^{(q-2)}_{M,n-1}(e_L) < \infty,$$

because

$$\frac{n+q-2}{M+n+q-2} \le 1, \quad \sum_{m,n=0}^{\infty} \varphi^{(q-2)}_{m,n}(e_L) < \infty,$$

and then we can use Lemma 3.2 in [14].

Since $0 \le |A_{M,N}| \le M c_M$ for all M, N, we have

$$\lim_{M,N\to\infty} A_{M,N} = 0$$

provided $\lim_{M\to\infty} M c_M = 0$. To see this we get from (33) with $m = M$ and n replaced by $n-1$

$$
\begin{aligned}
c_M &\geq \sum_{n=1}^{\infty} \frac{(M+1)n}{(M+q-1)(M+n+q)} \varphi^{q-2}_{M+1,n}(e_L) \\
&= \frac{M+1}{M+q-1} \sum_{n=1}^{\infty} \frac{n+q-2}{M+1+n+q-2} \left(1 - \frac{q-1}{n+q-2}\right) \varphi^{q-2}_{M+1,n-1}(e_L) \\
&= \frac{M+1}{M+q-1} c_{M+1} - \frac{M+1}{M+q-1} \sum_{n=1}^{\infty} \frac{q-1}{M+n+q-1} \varphi^{q-2}_{M+1,n-1}(e_L) \\
&\geq \frac{M+1}{M+q-1} c_{M+1} - \frac{(M+1)(q-1)}{(M+q-1)^2} \sum_{n=1}^{\infty} \varphi^{q-2}_{M+1,n-1}(e_L).
\end{aligned}
$$

Now, $\lim_{M\to\infty} Mc_M = 0$ follows as in [14]. $\square$

In analogy with Theorem 1.1 in [14] we have:

Proposition 4.4. *Let $2 \leq q < \infty$ and assume that $f \in \mathcal{P}(\Omega_{2(q+1)}, L)$. Then $f(\cdot, u)$ is differentiable with respect to z and $\overline{z}$ in $\mathbb{D}$ and there exist functions $f_i \in \mathcal{P}(\Omega_{2q}, L), i = 1, 2, 3, 4$ such that for $(z, u) \in \mathbb{D} \times L$*

$$
(1-|z|^2)\frac{\partial f(z,u)}{\partial z} = f_1(z,u) - f_2(z,u) \tag{34}
$$

$$
(1-|z|^2)\frac{\partial f(z,u)}{\partial \overline{z}} = f_3(z,u) - f_4(z,u). \tag{35}
$$

In particular, the two functions to the left in (34) and in (35) have continuous extensions to $\overline{\mathbb{D}} \times L$.

Let $\mathcal{G}_{2q}$ denote the subspace of continuous functions $F : \overline{\mathbb{D}} \times L \to \mathbb{C}$ spanned by functions of the form $p(z, \overline{z})f(z, u)$, where p is a polynomial in z and $\overline{z}$ with complex coefficients and $f \in \mathcal{P}(\Omega_{2q}, L)$. Clearly $\mathcal{G}_{2(q+1)} \subseteq \mathcal{G}_{2q}$.

By Proposition 4.4 we see that $(1-|z|^2)\partial/\partial z$ and $(1-|z|^2)\partial/\partial\overline{z}$ maps $\mathcal{G}_{2(q+1)}$ into $\mathcal{G}_{2q}$.

Proposition 4.5. *Let $2 \leq q < \infty$ and assume that $f \in \mathcal{P}(\Omega_{2(q+n)}, L)$ for $n \geq 1$. Then $f(\cdot, u)$ is n times differentiable with respect to z and $\overline{z}$ in $\mathbb{D}$ and for $r+s \leq n$*

$$
(1-|z|^2)^{r+s}\frac{\partial^{r+s} f(z,u)}{\partial z^r \partial \overline{z}^s} \in \mathcal{G}_{2(q+n-r-s)}. \tag{36}
$$

In particular, the function in (36) has a continuous extension to $\overline{\mathbb{D}} \times L$.

Proof. We prove (36) by induction in $r + s$.

It certainly holds for $r+s=1$ by Proposition 4.4. Assume that it holds for $r+s<n$. Differentiating the function in (36) with respect to z and multiplication with $1-|z|^2$ shows that

$$(1-|z|^2)^{r+s+1}\frac{\partial^{r+s+1}f(z,u)}{\partial z^{r+1}\partial\overline{z}^s} - (r+s)\overline{z}(1-|z|^2)^{r+s}\frac{\partial^{r+s}f(z,u)}{\partial z^r\partial\overline{z}^s}$$

belongs to $\mathcal{G}_{2(q+n-r-s-1)}$, and since

$$(r+s)\overline{z}(1-|z|^2)^{r+s}\frac{\partial^{r+s}f(z,u)}{\partial z^r\partial\overline{z}^s} \in \mathcal{G}_{2(q+n-r-s)} \subseteq \mathcal{G}_{2(q+n-r-s-1)},$$

we see that

$$(1-|z|^2)^{r+s+1}\frac{\partial^{r+s+1}f(z,u)}{\partial z^{r+1}\partial\overline{z}^s} \in \mathcal{G}_{2(q+n-r-s-1)}.$$

Differentiating the function in (36) with respect to $\overline{z}$ and multiplying with $(1-|z|^2)$ gives that

$$(1-|z|^2)^{r+s+1}\frac{\partial^{r+s+1}f(z,u)}{\partial z^{r}\partial\overline{z}^{s+1}} \in \mathcal{G}_{2(q+n-r-s-1)}.$$

□

In the next proposition we prove the weak convergence of a certain net $(\nu_\alpha)_{\alpha>-1}$ of measures introduced below. This convergence is decisive for the proof of Theorem 1.2.

Let $\nu_\alpha, \alpha>-1$, denote the probability measure on $\overline{\mathbb{D}}$ given by

$$\nu_\alpha = \frac{\alpha+1}{\pi}(1-x^2-y^2)^\alpha\,dx\,dy, \quad x^2+y^2<1, \tag{37}$$

and in polar coordinates the expression is

$$\nu_\alpha = \frac{\alpha+1}{\pi}(1-r^2)^\alpha r\,dr\,d\theta, \quad 0\le r<1, 0\le\theta<2\pi.$$

The set $C(\overline{\mathbb{D}})$ of continuous functions $f:\overline{\mathbb{D}}\to\mathbb{C}$ is a Banach space under the uniform norm $||f||_\infty = \sup_{z\in\overline{\mathbb{D}}}|f(z)|$.

Proposition 4.6. *Let $\mathcal{F}\subset C(\overline{\mathbb{D}})$ be a set of continuous functions on $\overline{\mathbb{D}}$ such that*

(i) *$\mathcal{F}$ is bounded, i.e., $\sup_{f\in\mathcal{F}}||f||_\infty<\infty$,*

(ii) *$\mathcal{F}$ is equicontinuous at $z=0$, i.e., for every $\varepsilon>0$ there exists $0<\delta<1$ such that $|f(z)-f(0)|\le\varepsilon$ for all $f\in\mathcal{F}$ and all complex z with $|z|\le\delta$.*

Then $\lim_{\alpha\to\infty}\int f\,d\nu_\alpha = f(0)$, uniformly for $f\in\mathcal{F}$.
In particular, $\lim_{\alpha\to\infty}\nu_\alpha=\delta_0$ weakly.

Proof. For any $0 < \delta < 1$ and $f \in \mathcal{F}$ we have

$$\int f \, d\nu_\alpha - f(0) = \frac{\alpha+1}{\pi} \int_0^{2\pi} \left(\int_0^\delta + \int_\delta^1 (f(re^{i\theta}) - f(0)) r(1-r^2)^\alpha \, dr \right) d\theta,$$

hence

$$\begin{aligned} \left| \int f \, d\nu_\alpha - f(0) \right| &\leq \sup_{|z|\leq\delta} |f(z) - f(0)| + 2||f||_\infty \frac{\alpha+1}{\pi} \int_0^{2\pi} \int_\delta^1 r(1-r^2)^\alpha \, dr \, d\theta \\ &= \sup_{|z|\leq\delta} |f(z) - f(0)| + 2||f||_\infty (1-\delta^2)^{\alpha+1}. \end{aligned}$$

For given $\varepsilon > 0$, we first choose $\delta > 0$ so small that the first term is smaller than $\varepsilon/2$.

With this δ, the second term tends to zero as $\alpha \to \infty$, hence $\leq \varepsilon/2$ for α sufficiently large. $\square$

Proof of Theorem 1.2:

If $f \in \mathcal{P}(\Omega_\infty, L)$, then for every $2 \leq q < \infty$

$$f(z,u) = \sum_{m,n=0}^{\infty} \varphi_{m,n}^{(q-2)}(u) R_{m,n}^{q-2}(z), \quad (z,u) \in \overline{\mathbb{D}} \times L, \tag{38}$$

where $\varphi_{m,n}^{(q-2)} \in \mathcal{P}(L)$ satisfy

$$\sum_{m,n=0}^{\infty} \varphi_{m,n}^{(q-2)}(e_L) < \infty,$$

and

$$\varphi_{m,n}^{(q-2)}(u) = N(q;m,n) \int_{\overline{\mathbb{D}}} f(z,u) \overline{R_{m,n}^{q-2}(z)} d\nu_{q-2}(z) \tag{39}$$

by Theorem 2.3 and (37).

There is a formula of Rodrigues type for the disc polynomials, see [21, Eq. (2.6)]:

$$(1-|z|^2)^{q-2} R_{m,n}^{q-2}(z) = \frac{(-1)^{m+n}(q-2)!}{(m+n+q-2)!} \frac{\partial^{m+n}}{\partial \overline{z}^m \partial z^n} (1-|z|^2)^{m+n+q-2}. \tag{40}$$

Thus, using $\overline{R_{m,n}^{q-2}(z)} = R_{n,m}^{q-2}(z)$,

$$\varphi_{m,n}^{(q-2)}(u) = \\ \frac{q-1}{\pi} \frac{(-1)^{m+n}(q-2)!}{(m+n+q-2)!} N(q;m,n) \int_{\mathbb{D}} f(z,u) \frac{\partial^{m+n}}{\partial \overline{z}^n \partial z^m} (1-|z|^2)^{m+n+q-2} dxdy.$$

Denote by I the integral in the previous equation. Now, note that

$$\begin{aligned} I &= \int_{\mathbb{D}} f(z,u)\frac{\partial}{\partial z}\left[\frac{\partial^{m+n-1}}{\partial \overline{z}^n \partial z^{m-1}}(1-|z|^2)^{m+n+q-2}\right] dxdy \\ &= \int_{\mathbb{D}} \frac{\partial}{\partial z}\left[f(z,u)\frac{\partial^{m+n-1}}{\partial \overline{z}^n \partial z^{m-1}}(1-|z|^2)^{m+n+q-2}\right] dxdy \\ &- \int_{\mathbb{D}} \frac{\partial}{\partial z} f(z,u)\frac{\partial^{m+n-1}}{\partial \overline{z}^n \partial z^{m-1}}(1-|z|^2)^{m+n+q-2} dxdy. \end{aligned}$$

By Green's Theorem,

$$\int_{\mathbb{D}} \frac{\partial g}{\partial \overline{z}}(z)dxdy = -\frac{i}{2}\int_{\partial \mathbb{D}} g(z)dz \tag{41}$$

and

$$\int_{\mathbb{D}} \frac{\partial g}{\partial z}(z)dxdy = \frac{i}{2}\int_{\partial \mathbb{D}} g(z)d\overline{z} \tag{42}$$

for a continuously differentiable function g on $\overline{\mathbb{D}}$. Using (42) we get

$$\begin{aligned} I = \frac{i}{2}&\int_{\partial\mathbb{D}} f(z,u)\frac{\partial^{m+n-1}}{\partial \overline{z}^n \partial z^{m-1}}(1-|z|^2)^{m+n+q-2} d\overline{z} \\ &- \int_{\mathbb{D}} \frac{\partial}{\partial z} f(z,u)\frac{\partial^{m+n-1}}{\partial \overline{z}^n \partial z^{m-1}}(1-|z|^2)^{m+n+q-2} dxdy. \end{aligned}$$

Since

$$\frac{\partial^{m+n-1}}{\partial \overline{z}^n \partial z^{m-1}}(1-|z|^2)^{m+n+q-2}$$

is the product of a polynomial in z and $\overline{z}$ by $(1-|z|^2)^{q-1}$, we have

$$f(z,u)\frac{\partial^{m+n-1}}{\partial \overline{z}^n \partial z^{m-1}}(1-|z|^2)^{m+n+q-2} = 0, \quad z \in \partial\overline{\mathbb{D}}.$$

Therefore

$$I = -\int_{\mathbb{D}} \frac{\partial}{\partial z} f(z,u)\frac{\partial^{m+n-1}}{\partial \overline{z}^n \partial z^{m-1}}(1-|z|^2)^{m+n+q-2} dxdy,$$

and similarly by (41)

$$I = -\int_{\mathbb{D}} \frac{\partial}{\partial \overline{z}} f(z,u)\frac{\partial^{m+n-1}}{\partial \overline{z}^{n-1} \partial z^{m}}(1-|z|^2)^{m+n+q-2} dxdy.$$

We now make further integrations by parts, a total of m integrations with respect to z and n with respect to $\overline{z}$. We need the following terms to vanish on the boundary of $\mathbb{D}$

$$\frac{\partial^l f(z,u)}{\partial z^l}\frac{\partial^{m+n-l-1}}{\partial \overline{z}^n \partial z^{m-l-1}}(1-|z|^2)^{m+n+q-2}, \quad l = 1,2,\ldots,m-1, \tag{43}$$

and
$$\frac{\partial^{k+m} f(z,u)}{\partial \overline{z}^k \partial z^m} \frac{\partial^{n-k-1}}{\partial \overline{z}^{n-k-1}} (1-|z|^2)^{m+n+q-2}, \quad k = 0, 1, \ldots, n-1. \tag{44}$$
This is true because
$$\frac{\partial^{r+s}}{\partial \overline{z}^r \partial z^s} (1-|z|^2)^N = p(z, \overline{z})(1-|z|^2)^{N-r-s}, \quad r+s \leq N$$
for a polynomial p in $z, \overline{z}$. Therefore the terms in (43), (44) are of the form $(1-|z|^2)^{q-1} F(z,u)$, where F is continuous on $\overline{\mathbb{D}} \times L$ by Proposition 4.5. The terms then vanish on the boundary of $\mathbb{D}$ because $q \geq 2$.

We obtain
$$\begin{aligned}
&\varphi_{m,n}^{(q-2)}(u) = \\
&\quad \frac{q-1}{\pi} \frac{(q-2)!}{(m+n+q-2)!} N(q;m,n) \int_{\mathbb{D}} \frac{\partial^{m+n}}{\partial \overline{z}^n \partial z^m} f(z,u)(1-|z|^2)^{m+n+q-2} dxdy \\
&= N(q;m,n) \frac{(q-2)!}{(m+n+q-2)!} \int_{\mathbb{D}} \frac{\partial^{m+n}}{\partial \overline{z}^n \partial z^m} f(z,u)(1-|z|^2)^{m+n} d\nu_{q-2}(z).
\end{aligned}$$
We have
$$\begin{aligned}
N(q;m,n) \frac{(q-2)!}{(m+n+q-2)!} &= \\
&\frac{1}{m!n!} \frac{q-1+m+n}{q-1} \frac{(q-2+m)!}{(q-2)!} \frac{(q-2+n)!}{(q-2+m+n)!}.
\end{aligned}$$
Using
$$(a+n)! = a!(a+1)_n,$$
we find for $a = q-2$
$$N(q;m,n) \frac{(q-2)!}{(m+n+q-2)!} = \frac{1}{m!n!} \frac{q-1+m+n}{q-1} \frac{(q-1)_m (q-1)_n}{(q-1)_{m+n}},$$
and then
$$\lim_{q \to \infty} \left[N(q;m,n) \frac{(q-2)!}{(m+n+q-2)!} \right] = \frac{1}{m!n!}.$$

The function
$$h(z,u) := (1-|z|^2)^{m+n} \frac{\partial^{m+n}}{\partial \overline{z}^n \partial z^m} f(z,u)$$
is continuous on $\overline{\mathbb{D}} \times L$ by Proposition 4.5, and therefore the family $\mathcal{F}$ of the functions $h(\cdot, u) \in C(\overline{\mathbb{D}})$, where u belongs to a compact subset of L, is bounded and equicontinuous at $z = 0$.

By Proposition 4.6 it follows that
$$\lim_{q \to \infty} \varphi_{m,n}^{(q-2)}(u) = \frac{1}{m!n!} \frac{\partial^{m+n}}{\partial \overline{z}^n \partial z^m} f(0,u),$$

uniformly for u in compact subsets of L. It follows that

$$\varphi_{m,n}(u) := \frac{1}{m!n!}\frac{\partial^{m+n}}{\partial\overline{z}^n\partial z^m}f(0,u)$$

belongs to $\mathcal{P}(L)$ for all $m,n \geq 0$.

We still have to establish (7) and (8).

For $f \in \mathcal{P}(\Omega_\infty, L)$ we get by Lemma 4.1 that F of (31) belongs to $\mathcal{P}(\Omega_\infty)$. Using a theorem due to Christensen and Ressel, see [7], it can be written as

$$F(z) = \sum_{m,n=0}^{\infty} a_{m,n} z^m \overline{z}^n,$$

where $a_{m,n} \geq 0$ are uniquely determined by F and satisfy $\sum a_{m,n} < \infty$. In fact

$$a_{m,n} = \frac{1}{m!n!}\frac{\partial^{m+n}F}{\partial\overline{z}^n\partial z^m}(0). \tag{45}$$

We now use the special case of (31) with $n = 2$, $u_1 = e_L$, $u_2 = u$, $c_1 = 1$, $c_2 = c$, so $F = F_{u,c}$ takes the form

$$F_{u,c}(z) = f(z, e_L)(1 + |c|^2) + f(z, u)\overline{c} + f(z, u^{-1})c. \tag{46}$$

For all $u \in L, c \in \mathbb{C}$ we have a representation

$$F_{u,c}(z) = \sum_{m,n=0}^{\infty} a_{m,n}(u,c) z^m \overline{z}^n, \quad z \in \overline{\mathbb{D}},$$

where

$$a_{m,n}(u,c) \geq 0, \qquad \sum_{m,n=0}^{\infty} a_{m,n}(u,c) < \infty. \tag{47}$$

Letting $c = 1, -1, i$ we obtain

$$\begin{aligned}
F_{u,1}(z) &= 2f(z, e_L) + f(z, u) + f(z, u^{-1}) = \sum_{m,n=0}^{\infty} a_{m,n}(u, 1) z^m \overline{z}^n, \\
F_{u,-1}(z) &= 2f(z, e_L) - f(z, u) - f(z, u^{-1}) = \sum_{m,n=0}^{\infty} a_{m,n}(u, -1) z^m \overline{z}^n, \\
F_{u,i}(z) &= 2f(z, e_L) - if(z, u) + if(z, u^{-1}) = \sum_{m,n=0}^{\infty} a_{m,n}(u, i) z^m \overline{z}^n.
\end{aligned}$$

This gives

$$\frac{1-i}{4}F_{u,1}(z) - \frac{1+i}{4}F_{u,-1}(z) + \frac{i}{2}F_{u,i}(z) = f(z,u) = \sum_{m,n=0}^{\infty} \widetilde{\varphi}_{m,n}(u) z^m \overline{z}^n,$$

where

$$\widetilde{\varphi}_{m,n}(u) := \frac{1-i}{4}a_{m,n}(u,1) - \frac{1+i}{4}a_{m,n}(u,-1) + \frac{i}{2}a_{m,n}(u,i).$$

By (47) we get

$$\sum_{m,n=0}^{\infty} |\widetilde{\varphi}_{m,n}(u)| < \infty, \quad u \in L,$$

hence

$$\widetilde{\varphi}_{m,n}(u) = \frac{1}{m!n!}\frac{\partial^{m+n}}{\partial \overline{z}^n \partial z^m} f(0,u) = \varphi_{m,n}(u).$$

This shows that (7) and (8) hold because $|\widetilde{\varphi}_{m,n}(e_L)| = \varphi_{m,n}(e_L)$. □

Proof of Theorem 1.3:

The difficult "only if"-part of the proof is contained in Theorem 1.2.

For the "if"-part we note that it is easy to see that $(\xi, \eta) \mapsto \xi \cdot \eta$ is a positive definite kernel on Ω_∞. By the Schur product theorem for positive definite kernels, cf. [3, Theorem 3.1.12], we see that $z^m \overline{z}^n$ belongs to $\mathcal{P}(\Omega_\infty)$ for $m, n \geq 0$. It is therefore elementary that any function of the form (7) with $\varphi_{m,n} \in \mathcal{P}(L)$ satisfying

$$\sum_{m,n=0}^{\infty} \varphi_{m,n}(e_L) < \infty,$$

belongs to $\mathcal{P}(\Omega_\infty, L)$. □

Remark 4.7. It is known and easy to see that the disc polynomials $R^{\alpha}_{m,n}$ have the following limit property

$$\lim_{\alpha \to \infty} R^{\alpha}_{m,n}(z) = z^m \overline{z}^n, \quad z \in \mathbb{D} \tag{48}$$

for each $m, n \geq 0$ fixed, cf. [21, (2.12)].

This is the analogue of the following limit result for the normalized Gegenbauer polynomials

$$\lim_{\lambda \to \infty} C_n^{(\lambda)}(x)/C_n^{(\lambda)}(1) = x^n, \quad -1 < x < 1$$

for each $n \geq 0$. Schoenberg [19, p. 103] proved that this convergence is uniform in $n \geq 0$ for fixed x, and this was the clue to his proof of the representation theorem for $\mathcal{P}(\mathbb{S}^\infty)$, cf. [19, Theorem 2].

A proof of the theorem of Christensen and Ressel or the more general Theorem 1.3 can be given following the ideas of Schoenberg provided that one can prove that the convergence in (48) is uniform in $m, n \geq 0$ for each fixed $z \in \mathbb{D}$. It is not difficult to prove that the convergence is uniform in n for each fixed m, but we have not been able to prove the uniformity in m and n. This is equivalent to the following property of the normalized Jacobi polynomials cf. (23)

$$\lim_{\alpha\to\infty}((1+x)/2)^{\beta/2}R_n^{(\alpha,\beta)}(x) = ((1+x)/2)^{n+\beta/2}, \quad -1 < x < 1$$

uniformly in $n, \beta \in \mathbb{N}_0$. Unfortunately, this question is open and does not seem to be related to the recent deep result in [11, Theorem 1.1] about the Jacobi polynomials.

Acknowledgements

The first two named authors want to thank the Department of Mathematics at Universidad Técnica Federico Santa Maria for hospitality during their visit.

Funding: The travel of the first author to Chile was supported by A. Collstrop's Foundation. The second author was partially supported by São Paulo Research Foundation (FAPESP) [grant numbers 2016/03015-7, 2014/25796-5 and 2016/09906-0].

The authors want to thank two independent referees for useful comments.

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