Orthogonal expansions related to compact Gelfand pairs

Christian Berg, Ana P. Peron and Emilio Porcu

September 9, 2017

Abstract

For a locally compact group G, let $\mathcal{P}(G)$ denote the set of continuous positive definite functions $f : G \to \mathbb{C}$. Given a compact Gelfand pair (G, K) and a locally compact group L, we characterize the class $\mathcal{P}_K^{\sharp}(G, L)$ of functions $f \in \mathcal{P}(G \times L)$ which are bi-invariant in the G-variable with respect to K. The functions of this class are the functions having a uniformly convergent expansion $\sum_{\varphi \in Z} B(\varphi)(u)\varphi(x)$ for $x \in G, u \in L$, where the sum is over the space Z of positive definite spherical functions $\varphi : G \to \mathbb{C}$ for the Gelfand pair, and $(B(\varphi))_{\varphi \in Z}$ is a family of continuous positive definite functions on L such that $\sum_{\varphi \in Z} B(\varphi)(e_L) < \infty$. Here e_L is the neutral element of the group L. For a compact Abelian group G considered as a Gelfand pair (G, K) with trivial $K = \{e_G\}$, we obtain a characterization of $\mathcal{P}(G \times L)$ in terms of Fourier expansions on the dual group $\widehat{G}$.

The result is described in detail for the case of the Gelfand pairs $(O(d+1), O(d))$ and $(U(q), U(q-1))$ as well as for the product of these Gelfand pairs.

The result generalizes recent theorems of Berg-Porcu (2016) and Guella-Menegatto (2016).

2010 MSC: 43A35, 43A85, 43A90, 33C45,33C55

Keywords: Gelfand pairs, Positive definite functions, Spherical functions, Spherical harmonics for real an complex spheres.

1 Introduction

In [4] Berg and Porcu found an extension of Schoenberg's Theorem from [15] about positive definite functions on spheres in Euclidean spaces. Schoenberg's Theorem has played an important role in statistics because covariance kernels for isotropic random fields on spheres are modelled via the class $\mathcal{P}(\mathbb{S}^d)$ of continuous functions $f : [-1, 1] \to \mathbb{R}$, which are positive definite in the sense introduced by Schoenberg: For $n \in \mathbb{N}, \xi_1, \ldots, \xi_n \in \mathbb{S}^d$, the matrix $[f(\xi_j \cdot \xi_k)_{j,k=1}^n]$ is positive

semidefinite. Here $\mathbb{S}^d$ denotes the d-dimensional unit sphere in Euclidean space $\mathbb{R}^{d+1}$.

The extension of Schoenberg's theorems in [4] was motivated by the need in statistics to consider data which depend both on the position on the earth and on the time, i.e., random fields defined on the product $\mathbb{S}^d \times \mathbb{R}$. Assuming isotropy for the position and stationarity in time, one is lead to consider the class $\mathcal{P}(\mathbb{S}^d, \mathbb{R})$ of continuous functions $f : [-1, 1] \times \mathbb{R} \to \mathbb{C}$ such that the kernel $((\xi, u), (\eta, v)) \mapsto f(\xi \cdot \eta, v - u)$ is positive definite on the space $\mathbb{S}^d \times \mathbb{R}$.

We recall that for an arbitrary non-empty set X, a kernel on X is a function $k : X^2 \to \mathbb{C}$, and it is called positive definite if for any $n \in \mathbb{N}$, any finite collection of points $x_1, \ldots, x_n \in X$ and numbers $c_1, \ldots, c_n \in \mathbb{C}$ one has

$$\sum_{j,k=1}^{n} k(x_j, x_k) c_j \overline{c_k} \geq 0,$$

i.e., the matrix $[k(x_j, x_k)_{j,k=1}^{n}]$ is Hermitian and positive semidefinite. By $\mathcal{P}(X^2)$ we denote the class of positive definite kernels on X. The theory of positive definite kernels is treated in [3].

It turns out that in the characterization of the class $\mathcal{P}(\mathbb{S}^d, \mathbb{R})$, the additive group $\mathbb{R}$ can be replaced by an arbitrary locally compact group L. We recall that a continuous function $\varphi : L \to \mathbb{C}$ is called positive definite on L, in symbols $\varphi \in \mathcal{P}(L)$, if the kernel $(x, y) \mapsto f(x^{-1}y)$ is positive definite on L.

The neutral element in the group L is denoted e_L.

The following theorem holds:

Theorem 1.1. (Theorem 3.3 in [4]) *Let $d \in \mathbb{N}$ and let $f : [-1, 1] \times L \to \mathbb{C}$ be a continuous function. Then f belongs to $\mathcal{P}(\mathbb{S}^d, L)$ in the sense that the kernel*

$$(\xi, u), (\eta, v)) \mapsto f(\xi \cdot \eta, u^{-1}v)$$

is positive definite on $\mathbb{S}^d \times L$, if and only if there exists a sequence $(\varphi_{n,d})_{n \geq 0}$ of functions from $\mathcal{P}(L)$ with $\sum \varphi_{n,d}(e_L) < \infty$ such that

$$f(x, u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d, x), \quad x \in [-1, 1], \ u \in L. \tag{1}$$

The above expansion is uniformly convergent for $(x, u) \in [-1, 1] \times L$, and we have

$$\varphi_{n,d}(u) = \frac{N_n(d)\sigma_{d-1}}{\sigma_d} \int_{-1}^{1} f(x, u) c_n(d, x)(1 - x^2)^{d/2-1} \, dx. \tag{2}$$

Here we have used the notation

$$c_n(d, x) = C_n^{(\lambda)}(x)/C_n^{(\lambda)}(1), \quad \lambda = (d - 1)/2 \tag{3}$$

for the ultraspherical polynomials $c_n(d, x)$ as normalized Gegenbauer polynomials $C_n^{(\lambda)}(x)$ for the parameter $\lambda = (d-1)/2$, cf. [1]. The constants σ_d and $N_n(d)$ are defined in Section 5.

Schoenberg's Theorem for $\mathcal{P}(\mathbb{S}^d)$ is the special case of the previous theorem, where the group $L = \{e_L\}$ is trivial. The functions in $\mathcal{P}(L)$ are then just non-negative constants.

In [10] Menegatto and Peron proved a theorem for the complex unit sphere Ω_{2q} in $\mathbb{C}^q$, see Section 6, analogous to Schoenberg's Theorem for $\mathcal{P}(\mathbb{S}^d)$. In an attempt to extend their result by taking product with an arbitrary locally compact group L similar to the extension of Schoenberg's Theorem in [4], we realized that the real and complex spheres are homogeneous spaces associated with certain groups and subgroups of respectively orthogonal and unitary matrices, and these matrix groups form compact Gelfand pairs. Therefore, Schoenberg's Theorem for real spheres $\mathbb{S}^d$ and Menegatto-Peron's result for complex spheres can be viewed as special cases of the Bochner-Godement Theorem for Gelfand pairs, see [6]. It turns out to be possible to extend the Bochner-Godement theorem for compact Gelfand pairs by taking product with an arbitrary locally compact group L, and it is the main purpose of this paper to give a proof of that.

We refer to Section 2 for the necessary background about Gelfand pairs, but we shall briefly introduce the setting of our main result, Theorem 1.2 below. Let G and K be compact groups with K a subgroup of G such that (G, K) is a compact Gelfand pair. The dual space Z of positive definite spherical functions $\varphi : G \to \mathbb{C}$ is an orthogonal basis for the Hilbert space $L_K^2(G)^\sharp$ of square integrable functions on G, which are bi-invariant with respect to K.

To a compact Gelfand pair (G, K) and an arbitrary locally compact group L we shall characterize the set $\mathcal{P}_K^\sharp(G, L)$ of continuous positive definite functions $f : G \times L \to \mathbb{C}$, which are bi-invariant in the G-variable with respect to K. The normalized Haar measure on G is denoted ω_G.

Theorem 1.2. *Let (G, K) denote a compact Gelfand pair, let L be a locally compact group and let $f : G \times L \to \mathbb{C}$ be a continuous function. Then f belongs to $\mathcal{P}_K^\sharp(G, L)$ if and only if there exists a function $B : Z \to \mathcal{P}(L)$ satisfying $\sum_{\varphi \in Z} B(\varphi)(e_L) < \infty$ such that*

$$f(x, u) = \sum_{\varphi \in Z} B(\varphi)(u)\varphi(x), \quad x \in G, \ u \in L. \tag{4}$$

The above expansion is uniformly convergent for $(x, u) \in G \times L$, and we have

$$B(\varphi)(u) = \delta(\varphi) \int_G f(x, u)\overline{\varphi(x)} \, d\omega_G(x) \quad u \in L. \tag{5}$$

This theorem is stated again as Theorem 3.3 in Section 3. The dimension $\delta(\varphi)$ is defined in (16).

In Section 2 we outline the general theory of Gelfand pairs and specialize the theory to compact Gelfand pairs.

In Section 3 we formulate our results about the class $\mathcal{P}_K^\sharp(G, L)$ and prove the results in Section 4. A compact Abelian group G can be considered as a compact Gelfand pair with $K = \{e_G\}$. Applying Theorem 3.3 to this Gelfand pair leads to an expansion result for functions in $\mathcal{P}(G \times L)$, see Theorem 3.4 and Corollary 3.5.

In Section 5 and 6 we treat the special cases of the real and complex spheres.

In Section 7 we use that the product of compact Gelfand pairs is again a compact Gelfand pair. This makes it possible to obtain a recent result about positive definite functions on products of real spheres: Theorem 2.9 in [8]. We deduce an analogous theorem for products of complex spheres.

In [7] Guella and Menegatto have extended the result of [4] about expansions

$$\sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d, x), \quad x \in [-1, 1], u \in L,$$

with coefficient functions $\varphi_{n,d}$ belonging to $\mathcal{P}(L)$ to expansions, where the coefficient functions are positive definite kernels k_n on an arbitrary set X, i.e., the coefficient functions k_n belong to $\mathcal{P}(X^2)$.

We show in Section 8 that such a generalization is also possible in the framework of compact Gelfand pairs.

2 Harmonic analysis on Gelfand pairs

For the convenience of the reader we start with some expository material related to positive definite functions and Gelfand pairs. The main references are [5],[14],[16],[17],[18].

Let G denote a locally compact group with neutral element e_G. The set $\mathcal{P}(G)$ of continuous positive definite functions on G, defined in the Introduction, is important in representation theory, harmonic analysis and especially in probability theory when $G = \mathbb{R}^n$. These functions are treated in many books, see e.g. [5, p. 255] and [14, p.14]. It is known that any $\varphi \in \mathcal{P}(G)$ satisfies $\varphi(x^{-1}) = \overline{\varphi(x)}$ and $|\varphi(x)| \leq \varphi(e_G)$ for $x \in G$.

For a compact subgroup K of G we call a function $\varphi : G \to \mathbb{C}$ bi-invariant with respect to K if

$$\varphi(kxl) = \varphi(x), \quad x \in G, k, l \in K. \tag{6}$$

For a set A of functions on G we denote by $A_K^\sharp$ the set of functions from A which are bi-invariant with respect to K. In particular $C_K^\sharp(G)_c$ denotes the set of continuous complex-valued functions on G with compact support and bi-invariant with respect to K. It is easy to see that for $f, g \in C_K^\sharp(G)_c$ the convolution

$$f * g(x) = \int_G f(y) g(y^{-1}x) \, d\omega_G(y), \quad x \in G, \tag{7}$$

4

where ω_G denotes a left Haar measure on G, is again a bi-invariant function on G, and $C_K^\sharp(G)_c$ becomes a subalgebra of the group algebra $L^1(G)$.

We say that (G, K) is a *Gelfand pair* if $C_K^\sharp(G)_c$ is commutative, cf. [16, p. 75] or [18, Part 3]. The latter contains many equivalent conditions for (G, K) to be a Gelfand pair.

For a Gelfand pair the group G is necessarily unimodular, cf. [2],[16, p. 75]. A *spherical function* for (G, K) is a continuous function $\varphi : G \to \mathbb{C}$ satisfying

$$\int_K \varphi(xky)\, d\omega_K(k) = \varphi(x)\varphi(y), \quad x, y \in G; \quad \varphi(e_G) = 1, \tag{8}$$

where ω_K is Haar measure on K normalized to $\omega_K(K) = 1$. A spherical function is necessarily bi-invariant with respect to K. In fact, from (8) with $y = e_G$ we get

$$\int_K \varphi(xk)\, d\omega_K(k) = \varphi(x), \quad x \in G,$$

and the right invariance under K follows. Putting $x = e_G$ in (8), we similarly get the left invariance under K.

The compact subgroup K determines an equivalence relation $\sim$ in G defined by $x \sim y$ if and only if $x = kyl$ for some $k, l \in K$. The equivalence classes are the compact sets $KxK, x \in G$, which are called double cosets. The set of double cosets is denoted $K\backslash G/K$ and it is a locally compact space in the quotient topology, cf. [11]. Functions on $K\backslash G/K$ can be identified with functions on G which are bi-invariant with respect to K.

The *dual space* of a Gelfand pair (G, K) is the set Z of positive definite spherical functions. It is a locally compact space in the topology inherited from $C(G)$, which carries the topology of uniform convergence on compact subsets of G, cf. [16, p. 83]. Let $M_b(Z)$ denote the set of positive finite Radon measures on Z.

The Fourier transform of a function $f \in L_K^1(G)^\sharp$ is the function $\widehat{f} : Z \to \mathbb{C}$ defined by

$$\widehat{f}(\varphi) = \int_G f(x)\overline{\varphi(x)}\, d\omega_G(x), \quad \varphi \in Z. \tag{9}$$

It is a continuous function on Z vanishing at infinity, in symbols $\widehat{f} \in C_0(Z)$.

The set $\mathcal{P}_K^\sharp(G)$ of continuous positive definite bi-invariant functions on G is characterized by a Bochner type theorem due to Godement [6].

Theorem 2.1. *(Bochner-Godement) For any $\mu \in M_b(Z)$ the function*

$$f(x) = \int_Z \varphi(x)\, d\mu(\varphi), \quad x \in G \tag{10}$$

belongs to $\mathcal{P}_K^\sharp(G)$, and any function from this set has the form (10) *for a uniquely determined $\mu \in M_b(Z)$.*

The *Plancherel measure* ν on Z is a positive Radon measure depending on the normalization of ω_G such that the following holds:

Theorem 2.2. *(Plancherel-Godement) For any $f \in C_K^\sharp(G)_c$*

$$\int_G |f(x)|^2 \, d\omega_G(x) = \int_Z |\widehat{f}(\varphi)|^2 \, d\nu(\varphi). \tag{11}$$

The mapping $f \mapsto \widehat{f}$ extends uniquely to an isometric isomorphism of $L_K^2(G)^\sharp$ onto $L^2(Z, \nu)$.

We finally need the following *inversion theorem*, cf. [16, p. 84].

Theorem 2.3. *For $f \in \mathcal{P}_K^\sharp(G) \cap L^1(G)$ we have $\widehat{f} \in L^1(Z, \nu)$ and the unique measure $\mu \in M_b(Z)$ such that (10) holds has density $\widehat{f}$ with respect to the Plancherel measure ν, i.e.,*

$$f(x) = \int_Z \varphi(x)\widehat{f}(\varphi) \, d\nu(\varphi), \quad x \in G. \tag{12}$$

We now specialize to compact Gelfand pairs (G, K), meaning that G is assumed to be compact. We shall always normalize the Haar measure of G so that $\omega_G(G) = 1$. The following result is well-known in the context of the Peter-Weyl Theorem, but for the convenience of the reader we give a direct proof based on the commutativity of the convolution.

Proposition 2.4. *Let (G, K) denote a compact Gelfand pair. Two different functions $\varphi, \psi \in Z$ are orthogonal on G, i.e.,*

$$\int_G \varphi(x)\overline{\psi(x)} \, d\omega_G(x) = 0. \tag{13}$$

Proof. Defining

$$a = \int_G \varphi(x)\overline{\psi(x)} \, d\omega_G(x),$$

we get for $y \in G$

$$a\varphi(y) = \int_G \varphi(x)\varphi(y)\overline{\psi(x)} \, d\omega_G(x) = \int_G \int_K \varphi(xky) \, d\omega_K(k)\overline{\psi(x)} \, d\omega_G(x)$$

$$= \int_K \int_G \varphi(xky)\overline{\psi(x)} \, d\omega_G(x) d\omega_K(k) = \int_K \int_G \varphi(xy)\overline{\psi(xk^{-1})} \, d\omega_G(x) d\omega_K(k)$$

$$= \int_G \varphi(xy)\overline{\psi(x)} d\omega_G(x) = \int_G \psi(x)\varphi(x^{-1}y) d\omega_G(x) = \psi * \varphi(y),$$

where we first inserted (8) for $\varphi \in Z$ and changed the order of integration. For the fourth equality sign we replaced x by xk^{-1} and used invariance of integration.

Next we used that ψ is right invariant under K so the inner integral is independent of k. Finally, we used that $\overline{\psi(x)} = \psi(x^{-1})$ and replaced x by x^{-1} in the integration.

Similarly we get

$$a\psi(y) = \int_G \varphi(x)\psi(x^{-1})\psi(y)d\omega_G(x) = \int_G \varphi(x)\int_K \psi(x^{-1}ky)d\omega_K(k)d\omega_G(x)$$
$$= \int_K \int_G \varphi(kx)\psi(x^{-1}y)d\omega_G(x)d\omega_K(k) = \varphi * \psi(y).$$

Using that convolution is commutative we get $a(\varphi(y) - \psi(y)) = 0$ for all $y \in G$, but since $\varphi \neq \psi$, we get $a = 0$. $\qquad\square$

It follows from Proposition 2.4 that the dual space Z of a compact Gelfand pair is discrete, and if G is metrizable then Z is a countable set.

Remark 2.5. Any spherical function for a compact Gelfand pair is automatically positive definite, so the dual space Z is the set of all spherical functions. For a proof see [16, p. 86].

The compact homogeneous space G/K of left cosets $\xi = xK, x \in G$ carries a unique probability measure $d\xi$, which is invariant under the G-action $G \times G/K \to G/K$ given by

$$(g, \xi) \to g\xi = (gx)K, \text{ if } \xi = xK, \quad g, x \in G. \tag{14}$$

Functions $F : G/K \to \mathbb{C}$ are in one-to-one correspondence with functions $f : G \to \mathbb{C}$ which are right invariant functions under K, i.e., $f(g) = f(gk)$ for all $g \in G, k \in K$. The correspondence is given by $F(gK) = f(g), g \in G$ and the following integral formula holds

$$\int_G f(x)\, d\omega_G(x) = \int_{G/K} F(g\xi)\, d\xi = \int_{G/K} F(\xi)\, d\xi, \quad g \in G. \tag{15}$$

For each $\varphi \in Z$ and $g \in G$ the function $x \mapsto \varphi(g^{-1}x)$ on G is right invariant under K, so it can be considered as a function $\varphi_g \in C(G/K)$. It is a classical fact that

$$H_\varphi := span\{\varphi_g \mid g \in G\} \tag{16}$$

is a finite dimensional subspace of $C(G/K)$. The dimension of H_φ is denoted $\delta(\varphi)$.

The spaces $H_\varphi, \varphi \in Z$, are mutually orthogonal in $L^2(G/K, d\xi)$ which they span.

It is known that the Plancherel measure is given by $\nu(\{\varphi\}) = \delta(\varphi)$ for $\varphi \in Z$. The spherical functions from Z form an orthogonal system, cf. Proposition 2.4.

More precisely we have:

$$\int_G \varphi(x)\overline{\psi(x)}\, d\omega_G(x) = \begin{cases} 0 & \text{if } \varphi \neq \psi, \\ 1/\delta(\varphi) & \text{if } \varphi = \psi. \end{cases} \tag{17}$$

The previous three theorems take the following form, which gives the expansion of functions in $L_K^2(G)^\sharp$ after the orthonormal basis $\sqrt{\delta(\varphi)}\varphi, \varphi \in Z$:

Theorem 2.6. *Let (G, K) be a compact Gelfand pair.*
(i) Each $f \in L_K^2(G)^\sharp$ has the orthogonal expansion

$$f(x) \sim \sum_{\varphi \in Z} \delta(\varphi)\widehat{f}(\varphi)\varphi(x),$$

which converges in $L^2(G)$, and Parseval's formula holds:

$$\int_G |f(x)|^2\, d\omega_G(x) = \sum_{\varphi \in Z} |<f, \sqrt{\delta(\varphi)}\varphi>|^2 = \sum_{\varphi \in Z} \delta(\varphi)|\widehat{f}(\varphi)|^2.$$

(ii) A function $f \in C_K^\sharp(G)$ belongs to $\mathcal{P}_K^\sharp(G)$ if and only if there exist a family $(B(\varphi))_{\varphi \in Z}$ of non-negative numbers satisfying $\sum_{\varphi \in Z} B(\varphi) < \infty$ such that

$$f(x) = \sum_{\varphi \in Z} B(\varphi)\varphi(x), \quad x \in G, \tag{18}$$

and $B(\varphi) = \delta(\varphi)\widehat{f}(\varphi)$. The series in (18) is uniformly convergent on G.

3 Main Results

We start with a formal definition of the main class to be discussed.

Let (G, K) denote a compact Gelfand pair, and let L denote an arbitrary locally compact group with neutral element e_L. We shall consider a subset of the set $\mathcal{P}(G \times L)$ of continuous positive definite functions on the locally compact group $G \times L$.

Definition 3.1. *The set of continuous positive definite functions $f : G \times L \to \mathbb{C}$ which are bi-invariant with respect to K in the first variable is denoted $\mathcal{P}_K^\sharp(G, L)$.*

The following proposition states some properties of $\mathcal{P}_K^\sharp(G, L)$ which are easily obtained. The proofs are left to the reader.

Proposition 3.2.   (i) *For $f_1, f_2 \in \mathcal{P}_K^\sharp(G, L)$ and $r \geq 0$ we have $rf_1, f_1 + f_2,$ $f_1 f_2 \in \mathcal{P}_K^\sharp(G, L)$.*

(ii) *For a net of functions $(f_i)_{i \in I}$ from $\mathcal{P}_K^\sharp(G, L)$ converging pointwise to a continuous function $f : G \times L \to \mathbb{C}$, we have $f \in \mathcal{P}_K^\sharp(G, L)$.*

(iii) *For $f \in \mathcal{P}_K^\sharp(G, L)$ we have $f(\cdot, e_L) \in \mathcal{P}_K^\sharp(G)$ and $f(e_G, \cdot) \in \mathcal{P}(L)$.*

(iv) *For $f \in \mathcal{P}_K^\sharp(G)$ and $g \in \mathcal{P}(L)$ we have $f \otimes g \in \mathcal{P}_K^\sharp(G, L)$, where $f \otimes g(x, u) := f(x)g(u)$ for $(x, u) \in G \times L$. In particular we have $f \otimes 1_L \in \mathcal{P}_K^\sharp(G, L)$ and $f \mapsto f \otimes 1_L$ is an embedding of $\mathcal{P}_K^\sharp(G)$ into $\mathcal{P}_K^\sharp(G, L)$.*

Our first main theorem can be stated as follows. Note that a function $B : Z \to \mathcal{P}(L)$ can be considered as a family $(B(\varphi))_{\varphi \in Z}$ in $\mathcal{P}(L)$. In the applications Z is a countable set, so we can think of $(B(\varphi))_{\varphi \in Z}$ as a sequence from $\mathcal{P}(L)$.

Theorem 3.3. *Let (G, K) denote a compact Gelfand pair, let L be a locally compact group and let $f : G \times L \to \mathbb{C}$ be a continuous function. Then f belongs to $\mathcal{P}_K^\sharp(G, L)$ if and only if there exists a function $B : Z \to \mathcal{P}(L)$ satisfying $\sum_{\varphi \in Z} B(\varphi)(e_L) < \infty$ such that*

$$f(x, u) = \sum_{\varphi \in Z} B(\varphi)(u)\varphi(x), \quad x \in G, \ u \in L. \tag{19}$$

The above expansion is uniformly convergent for $(x, u) \in G \times L$, and we have

$$B(\varphi)(u) = \delta(\varphi) \int_G f(x, u)\overline{\varphi(x)} \, d\omega_G(x) = \delta(\varphi)\widehat{f(\cdot, u)}(\varphi), \quad u \in L. \tag{20}$$

When L denotes the group consisting just of the neutral element, Theorem 3.3 reduces to Bochner-Godement's Theorem.

We give the proof in the next section, but notice that the infinite series in Equation (19) is uniformly convergent by Weierstrass' M-test since $|B(\varphi)(u)| \leq B(\varphi)(e_L)$ for $u \in L$ and $|\varphi(x)| \leq 1$ for $x \in G$. It is also clear that any function given by (19) belongs to $\mathcal{P}_K^\sharp(G, L)$ because of Proposition 3.2 and the fact that by definition $\varphi \in \mathcal{P}_K^\sharp(G)$ for any $\varphi \in Z$. The main point in Theorem 3.3 is that all functions in $\mathcal{P}_K^\sharp(G, L)$ have an expansion (19).

If G is a compact Abelian group and $K = \{e_G\}$ is trivial, then (G, K) is a compact Gelfand pair. A spherical function φ is the same as a continuous homomorphism $\varphi : G \to \mathbb{T}$, where

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}, \tag{21}$$

i.e., φ is a continuous group character on G. This implies that the dual space Z of the Gelfand pair can be identified with the dual group $\widehat{G}$ of G. For the space defined in Equation (16) we have $H_\varphi = \mathbb{C}\varphi$ for $\varphi \in \widehat{G}$, hence $\delta(\varphi) = 1$.

Specializing Theorem 3.3 to this Gelfand pair leads to the following result:

Theorem 3.4. *Let G denote a compact Abelian group with dual group $\widehat{G}$, let L be a locally compact group and let $f : G \times L \to \mathbb{C}$ be a continuous function.*

Then f belongs to $\mathcal{P}(G \times L)$ if and only if there exists a function $B : \widehat{G} \to \mathcal{P}(L)$ satisfying $\sum_{\varphi \in \widehat{G}} B(\varphi)(e_L) < \infty$ such that

$$f(x, u) = \sum_{\varphi \in \widehat{G}} B(\varphi)(u)\varphi(x), \quad x \in G, \ u \in L. \tag{22}$$

The above expansion is uniformly convergent for $(x, u) \in G \times L$, and we have

$$B(\varphi)(u) = \int_G f(x, u)\overline{\varphi(x)} \, d\omega_G(x) = \widehat{f(\cdot, u)}(\varphi), \quad u \in L. \tag{23}$$

For the special case where $G = \mathbb{T}^N$ is the N-dimensional torus, we have $\widehat{G} = \mathbb{Z}^N$. As usual we identify functions on $\mathbb{T}^N$ with functions on $\mathbb{R}^N$, which are periodic with period 2π in each variable. This leads to the following result about Fourier series in N variables:

Corollary 3.5. *Let L be a locally compact group and let $f : \mathbb{R}^N \times L \to \mathbb{C}$ be a continuous function periodic with period 2π in the N real variables. Then f belongs to $\mathcal{P}(\mathbb{R}^N \times L)$ if and only if there exists a multisequence $(\varphi_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^N}$ from $\mathcal{P}(L)$ satisfying*

$$\sum_{\mathbf{n} \in \mathbb{Z}^N} \varphi_{\mathbf{n}}(e_L) < \infty$$

such that

$$f(\mathbf{x}, u) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \varphi_{\mathbf{n}}(u) \exp(i\,\mathbf{n} \cdot \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^N, \ u \in L. \tag{24}$$

The above expansion is uniformly convergent for $(\mathbf{x}, u) \in \mathbb{R}^N \times L$, and we have

$$\varphi_{\mathbf{n}}(u) = (2\pi)^{-N} \int_{[0, 2\pi]^N} f(\mathbf{x}, u) \exp(-i\,\mathbf{n} \cdot \mathbf{x}) \, d\mathbf{x}, \quad u \in L. \tag{25}$$

4   Proofs

Lemma 4.1. *Let H denote a locally compact group and let $C \subset H$ be a non-empty compact set. For any open neighbourhood U of the neutral element $e_H \in H$ there exists a partition of C in finitely many non-empty disjoint Borel sets, say $M_j, j = 1, \ldots, r$, with the property:*

For $j = 1, \ldots, r$ and for $x, y \in M_j$ we have $x^{-1}y \in U$.

Proof. Given an open neighbourhood U of $e_H \in H$, there exists a smaller open neighbourhood V of e_H such that $V^{-1}V \subset U$. By the definition of compactness, see [11, p. 164], there exists a finite covering of the compact set C by left translates $x_j V$. Defining $B_1 = x_1 V \cap C$ and

$$B_j = x_j V \cap C \setminus \bigcup_{k=1}^{j-1} x_k V, \ j \geq 2,$$

10

the non-empty sets among the B_j's will form a finite partition $M_1, \ldots, M_r$ of C such that each M_j is contained in a left translate of V, and therefore we have for any $x, y \in M_j$

$$x^{-1}y \in V^{-1}V \subset U.$$

$\square$

The following lemma is well-known, see [5, p.256], but for the convenience of the reader we give a self-contained proof.

Lemma 4.2. *Let H be a locally compact group. For a continuous function $f : H \to \mathbb{C}$ the following are equivalent:*

(i) $f \in \mathcal{P}(H)$.

(ii) *f is bounded and for any complex Radon measure μ on H of compact support we have*

$$\int_H \int_H f(x^{-1}y) \, d\mu(x) \, d\overline{\mu(y)} \geq 0. \tag{26}$$

Proof. "(i) $\Longrightarrow$ (ii)." Suppose first that (i) holds. As noticed, any positive definite function is bounded. For any discrete complex Radon measure of the form

$$\sigma = \sum_{j=1}^n \alpha_j \delta_{x_j},$$

where $x_1, \ldots x_n \in H$, $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, we have

$$\int_H \int_H f(x^{-1}y) \, d\sigma(x) \, d\overline{\sigma(y)} = \sum_{k,l=1}^n f(x_k^{-1}x_l) \alpha_k \overline{\alpha_l} \geq 0.$$

Let now μ denote an arbitrary complex Radon measure on H with compact support C, and let us consider the number

$$I := \int_H \int_H f(x^{-1}y) \, d\mu(x) \, d\overline{\mu(y)},$$

which is clearly real. We shall prove that $I \geq 0$, by showing that for any $\varepsilon > 0$ there exists $J \geq 0$ such that $|I - J| < \varepsilon ||\mu||^2$, where $||\mu||$ is the total variation of the complex measure μ, cf. [13].

First of all $f(x^{-1}y)$ is uniformly continuous on the compact set $C \times C$. Thus, for given $\varepsilon > 0$ there exists an open neighbourhood U of e_H such that for all pairs $(x, y), (\tilde{x}, \tilde{y}) \in C \times C$ satisfying $x^{-1}\tilde{x} \in U$, $y^{-1}\tilde{y} \in U$ we have

$$|f(x^{-1}y) - f(\tilde{x}^{-1}\tilde{y})| < \varepsilon.$$

Corresponding to U we choose a partition $M_j, j = 1, \ldots, r$ of C with the property of Lemma 4.1. In particular

$$|f(x^{-1}y) - f(\tilde{x}^{-1}\tilde{y})| < \varepsilon,$$

if

$$(x, y), (\tilde{x}, \tilde{y}) \in M_k \times M_l.$$

Next

$$I = \sum_{k,l=1}^{r} \int_{M_k} \int_{M_l} f(x^{-1}y) \, d\mu(x) \, d\overline{\mu(y)},$$

and if we choose an arbitrary point $x_j \in M_j$ and define $\alpha_j = \mu(M_j)$, $j = 1, \ldots, r$, then

$$J := \sum_{k,l=1}^{r} f(x_k^{-1}x_l)\alpha_k\overline{\alpha_l} \geq 0.$$

Furthermore,

$$I - J = \sum_{k,l=1}^{r} \int_{M_k} \int_{M_l} [f(x^{-1}y) - f(x_k^{-1}x_l)] \, d\mu(x) \, d\overline{\mu(y)},$$

and by the uniform continuity

$$|I - J| \leq \varepsilon \sum_{k,l=1}^{r} |\mu|(M_k)|\mu|(M_l) = \varepsilon ||\mu||^2,$$

where $|\mu|$ denotes the total variation measure.

The implication "(ii) $\implies$ (i)" is easy by specializing the measure μ to a complex discrete measure concentrated in finitely many points. $\qquad\square$

Proof of Theorem 3.3. Suppose that f belongs to $\mathcal{P}_K^\sharp(G, L)$ and let us consider the product measure $\mu := \omega_G \otimes \sigma$ on $G \times L$, where σ is an arbitrary complex Radon measure on L of compact support. By Lemma 4.2 for the locally compact group $H = G \times L$ applied to μ we get

$$\int_G \int_G \int_L \int_L f(x^{-1}y, u^{-1}v) \, d\omega_G(x) \, d\omega_G(y) \, d\sigma(u) \, d\overline{\sigma(v)} \geq 0. \tag{27}$$

The integral with respect to x, y can be simplified to

$$\int_G \int_G f(x^{-1}y, u^{-1}v) \, d\omega_G(x) \, d\omega_G(y) = \int_G f(y, u^{-1}v) \, d\omega_G(y)$$

by invariance of ω_G. Therefore Equation (27) amounts to

$$\int_G \int_L \int_L f(y, u^{-1}v) \, d\omega_G(y) \, d\sigma(u) \, d\overline{\sigma(v)} \geq 0. \tag{28}$$

For arbitrary $\varphi \in Z$ we next apply Equation (28) to the function $f(x, u)\overline{\varphi(x)}$, which belongs to $\mathcal{P}_K^\sharp(G, L)$ by Proposition 3.2. This gives

$$\int_G \int_L \int_L f(y, u^{-1}v)\overline{\varphi(y)} \, d\omega_G(y) \, d\sigma(u) \, d\overline{\sigma}(v) \geq 0. \tag{29}$$

The function $B(\varphi) : L \to \mathbb{C}$ defined by

$$B(\varphi)(u) := \delta(\varphi) \int_G f(y, u)\overline{\varphi(y)} \, d\omega_G(y) = \delta(\varphi)\widehat{f(\cdot, u)}(\varphi) \tag{30}$$

is clearly continuous and bounded, and it is positive definite on L, because Equation (29) holds for all complex Radon measures σ on L with compact support.

For each $u \in L$ the function $f(\cdot, u)$ belongs to $C_K^\sharp(G) \subset L_K^2(G)^\sharp$ and has the orthogonal expansion in spherical functions

$$f(y, u) = \sum_{\varphi \in Z} B(\varphi)(u)\varphi(y), \tag{31}$$

where the convergence is in $L^2(G)$ with respect to $y \in G$ by Theorem 2.6.

When $u = e_L$, then $f(\cdot, e_L) \in \mathcal{P}_K^\sharp(G)$, and by the Bochner-Godement Theorem we have

$$f(y, e_L) = \sum_{\varphi \in Z} B(\varphi)(e_L)\varphi(y), \quad y \in G.$$

Since $B(\varphi)(e_L) \geq 0$ with $\sum_{\varphi \in Z} B(\varphi)(e_L) < \infty$ and

$$|B(\varphi)(u)\varphi(y)| \leq B(\varphi)(e_L),$$

the M-test of Weierstrass shows that the series on the right-hand side of (31) converges uniformly to a continuous function $\tilde{f}(y, u)$ on $G \times L$. In particular, for each $u \in L$ the series (31) converges uniformly in $y \in G$ to $\tilde{f}(y, u)$, but this implies convergence in $L^2(G)$ and therefore $f(y, u) = \tilde{f}(y, u)$ for almost all $y \in G$. Since these functions are continuous, we have equality for all $y \in G$. $\qquad\square$

5 The sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$

For $n \geq 1$ let $O(n)$ denote the group of orthogonal $n \times n$ real matrices A and let $SO(n)$ denote the normal subgroup of those matrices $A \in O(n)$ with determinant 1. In the case $n = 1$ these groups are the simple multiplicative groups $\{\pm 1\}$ and $\{1\}$.

The groups $O(d+1)$ and $SO(d+1)$ both operate on the real unit sphere of dimension d

$$\mathbb{S}^d = \left\{ x \in \mathbb{R}^{d+1} \mid ||x||^2 = \sum_{k=1}^{d+1} x_k^2 = 1 \right\},$$

but while $O(d+1)$ operates transitively, this is the case for $SO(d+1)$ only for $d \geq 1$.

The surface measure of the sphere $\mathbb{S}^d$ is denoted ω_d, and it is of total mass

$$\sigma_d = \omega_d(\mathbb{S}^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}.$$

We use the notation $e_1, \ldots, e_{d+1}$ for the standard basis in $\mathbb{R}^{d+1}$. The fixed-point group of the matrices $A \in O(d+1)$ satisfying $Ae_1 = e_1$, is of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix},$$

where $\tilde{A} \in O(d)$, the zero in the upper right corner represents a zero row vector of length d, and the zero in the lower left corner represents a zero column vector of length d.

Let G denote the compact group $O(d+1)$. This shows that the fixed-point group K of e_1 is isomorphic to $O(d)$ and in the following identified with $O(d)$. The mapping $A \mapsto Ae_1$ of $G = O(d+1)$ onto $\mathbb{S}^d$ is constant on the left cosets $\xi = AK$, and hence induces a bijection of G/K onto $\mathbb{S}^d$, and it is a homeomorphism. The pair (G, K) is known to be a compact Gelfand pair, cf. [16],[18].

The mapping $A \mapsto Ae_1 \cdot e_1$ of G onto $[-1, 1]$ is constant on the double cosets and if $Ae_1 \cdot e_1 = Be_1 \cdot e_1$ for $A, B \in G$, then they belong to the same double coset. Therefore the space of double cosets $K\backslash G/K$ is homeomorphic to $[-1, 1]$. This shows that complex functions on G which are bi-invariant with respect to K, can be identified with functions $f : [-1, 1] \to \mathbb{C}$. In fact, for such a function, $A \mapsto f(Ae_1 \cdot e_1)$ is a bi-invariant function on G and all bi-invariant functions on G have this form. The bi-invariant functions depend only on the upper left corner a_{11} of $A \in O(d+1)$.

The image measure of Haar measure ω_G on $G = O(d+1)$ under the mapping $A \mapsto Ae_1$ of G onto $\mathbb{S}^d$ is the normalized surface measure ω_d/σ_d. The image measure of ω_d/σ_d under the mapping $\xi \mapsto \xi \cdot e_1$ of $\mathbb{S}^d$ onto $[-1, 1]$ is the probability measure on $[-1, 1]$ with density

$$(\sigma_{d-1}/\sigma_d) \left(1 - x^2\right)^{d/2-1}$$

with respect to Lebesgue measure, cf. [12].

The spherical functions are precisely the normalized ultraspherical polynomials $c_n(d, x)$ given by (3). As bi-invariant functions on $O(d+1)$ they are positive definite by Schoenberg's Theorem in [15].

Therefore the dual space Z of the compact Gelfand pair $(O(d+1), O(d))$ can be identified with $\mathbb{N}_0$ and we have

$$\delta(c_n(d, x)) = N_n(d) = \frac{(d)_{n-1}}{n!}(2n + d - 1), \ n \geq 1, \quad \delta(c_0(d, x)) = N_0(d) = 1.$$

14

For $f \in L^1_K(G)^\sharp$ considered as a function on $[-1, 1]$ we have

$$\widehat{f}(n) = (\sigma_{d-1}/\sigma_d) \int_{-1}^{1} f(x) c_n(d, x)(1 - x^2)^{d/2-1} \, dx, \quad n \in \mathbb{N}_0.$$

The space $\mathcal{P}^\sharp_K(G)$ can be identified with the space $\mathcal{P}(\mathbb{S}^d)$ from [4], and for an arbitrary locally compact group L the space $\mathcal{P}^\sharp_K(G, L)$ can be identified with $\mathcal{P}(\mathbb{S}^d, L)$. Theorem 3.3 of [4] is a special case of Theorem 3.3, and it can be formulated as follows:

Theorem 5.1. *Let $d \in \mathbb{N}$, let L be a locally compact group and let $f : [-1, 1] \times L \to \mathbb{C}$ be a continuous function. Then $A \mapsto f(Ae_1 \cdot e_1, u)$ belongs to $\mathcal{P}^\sharp_{O(d)}(O(d + 1), L)$ if and only if there exists a sequence $(\varphi_{n,d})_{n \geq 0}$ of functions from $\mathcal{P}(L)$ with $\sum \varphi_{n,d}(e_L) < \infty$ such that*

$$f(x, u) = \sum_{n=0}^{\infty} \varphi_{n,d}(u) c_n(d, x), \quad x \in [-1, 1], \ u \in L. \tag{32}$$

The above expansion is uniformly convergent for $(x, u) \in [-1, 1] \times L$, and we have

$$\varphi_{n,d}(u) = N_n(d)(\sigma_{d-1}/\sigma_d) \int_{-1}^{1} f(x, u) c_n(d, x)(1 - x^2)^{d/2-1} \, dx. \tag{33}$$

Remark 5.2. There is an apparent discrepancy between the symbol $\mathcal{P}(\mathbb{S}^d, L)$, which denotes a set of functions on $[-1, 1] \times L$, and $\mathcal{P}^\sharp_K(G, L)$, which denotes a set of functions on $G \times L$, but since these functions are bi-invariant with respect to K in the G-variable, these functions can be considered as functions on $(K \backslash G / K) \times L$. The double coset space $K \backslash G / K$ is homeomorphic to $[-1, 1]$ in case of the Gelfand pair $(O(d + 1), O(d))$, and therefore we think that the notation is reasonable.

Remark 5.3. In all the above one could also have considered the compact Gelfand pair $(SO(d + 1), SO(d))$ when $d \geq 2$ without any essential changes. For $d = 1$ we note that $SO(2)$ is commutative and isomorphic to $\mathbb{T}$, while $SO(1)$ is trivial. In this case Theorem 3.3 reduces to Corollary 3.5 for $N = 1$.

6 The complex sphere Ω_{2q} in $\mathbb{C}^q$

In the following we use the notation from [10].

For $q \geq 1$ let $U(q)$ denote the group of unitary $q \times q$ complex matrices A and let $SU(q)$ denote the normal subgroup of those matrices $A \in U(q)$ with determinant 1. In the case $q = 1$ these groups are the multiplicative groups $\mathbb{T}$ (defined in (21)) and $\{1\}$.

The groups $U(q)$ and $SU(q)$ both operate on the complex unit sphere of (real) dimension $2q - 1$

$$\Omega_{2q} = \left\{ z \in \mathbb{C}^q \mid ||z||^2 = \sum_{k=1}^{q} |z_k|^2 = 1 \right\},$$

but while $U(q)$ operates transitively, this is the case for $SU(q)$ only for $q \geq 2$.

Note that Ω_{2q} is equal to $\mathbb{S}^{2q-1}$ if $\mathbb{C}^q$ is identified with $\mathbb{R}^{2q}$.

We also use the notation $e_1, \ldots, e_q$ for the standard basis in $\mathbb{C}^q$.

In the following we always assume that $q \geq 2$.

The fixed-point group of the matrices $A \in U(q)$ satisfying $Ae_1 = e_1$, is of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix},$$

where $\tilde{A} \in U(q-1)$, the zero in the upper right corner represents a zero row vector of length $q - 1$, and the zero in the lower left corner represents a zero column vector of length $q - 1$.

Let G denote the compact group $U(q)$. This shows that the fixed-point group K of e_1 is isomorphic to $U(q-1)$ and in the following identified with $U(q-1)$. The mapping $A \mapsto Ae_1$ of $G = U(q)$ onto Ω_{2q} is constant on the left cosets $\xi = AK$ and hence induces a bijection of G/K onto Ω_{2q}. This bijection is a homeomorphism. The pair (G, K) is known to be a compact Gelfand pair, cf. [16],[18].

The bi-invariant functions on G with respect to K can be identified with functions $f : \overline{\mathbb{D}} \to \mathbb{C}$, where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the open unit disc and $\overline{\mathbb{D}}$ its closure. To see this notice that the mapping $A \mapsto Ae_1 \cdot e_1$ of G onto $\overline{\mathbb{D}}$ is constant on the double cosets and if $Ae_1 \cdot e_1 = Be_1 \cdot e_1$ for $A, B \in G$, then they belong to the same double coset. Therefore the space of double cosets $K\backslash G/K$ is homeomorphic to $\overline{\mathbb{D}}$.

The bi-invariant functions depend only on the upper left corner a_{11} of $A \in U(q)$.

The image measure of Haar measure ω_G on $G = U(q)$ under the mapping $A \mapsto Ae_1$ of G onto Ω_{2q} is the normalized surface measure $\omega_{2q-1}/\sigma_{2q-1}$ on Ω_{2q}. The image measure of $\omega_{2q-1}/\sigma_{2q-1}$ under the mapping $\xi \mapsto \xi \cdot e_1$ of Ω_{2q} onto $\overline{\mathbb{D}}$ is the probability measure on $\overline{\mathbb{D}}$ given in polar coordinates $z = re^{i\theta}, 0 \leq r \leq 1, 0 \leq \theta < 2\pi$ as

$$\frac{q-1}{\pi} r(1 - r^2)^{q-2} dr d\theta,$$

cf. formula (2.18) of [9].

The spherical functions are precisely the functions $R_{m,n}^{q-2}(z)$ given in [9],[10], and as bi-invariant functions on $U(q)$ they are positive definite. The functions $R_{m,n}^{q-2}(z)$ belong to the class of disc polynomials given in [9] for $\alpha > -1$ as

$$R_{m,n}^{\alpha}(re^{i\theta}) = r^{|m-n|} e^{i(m-n)\theta} R_{\min(m,n)}^{(\alpha,|m-n|)}(2r^2 - 1), \quad 0 \leq r \leq 1, \ 0 \leq \theta < 2\pi$$

16

and
$$R_k^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}(x)/P_k^{(\alpha,\beta)}(1), \quad \alpha, \beta > -1, \ k \in \mathbb{N}_0$$
are normalized Jacobi polynomials, cf. [1].

See [19] for other expressions for the disc polynomials.

The dual space Z of the compact Gelfand pair $(U(q), U(q-1))$ can be identified with $\mathbb{N}_0^2$ and we have

$$\delta(R_{m,n}^{q-2}(z)) = N(q; m, n) = \frac{m+n+q-1}{q-1}\binom{m+q-2}{q-2}\binom{n+q-2}{q-2}.$$

For $f \in L_K^1(G)^\sharp$ considered as a function on $\overline{\mathbb{D}}$ we have

$$\widehat{f}(m,n) = \frac{q-1}{\pi}\int_0^1\int_0^{2\pi} f(re^{i\theta})\overline{R_{m,n}^{q-2}(re^{i\theta})}r(1-r^2)^{q-2}d\theta dr.$$

The space $\mathcal{P}_K^\sharp(G)$ can be identified with the space of functions characterized in Theorem 4.2 in [10], and for an arbitrary locally compact group L the space $\mathcal{P}_K^\sharp(G, L)$ can be characterized as follows by Theorem 3.3:

Theorem 6.1. *Let $q \in \mathbb{N}, q \geq 2$, let L be a locally compact group and let $f : \overline{\mathbb{D}} \times L \to \mathbb{C}$ be a continuous function. Then $A \mapsto f(Ae_1 \cdot e_1, u)$ belongs to $\mathcal{P}_{U(q-1)}^\sharp(U(q), L)$ if and only if there exists a double sequence of functions $(\varphi_{m,n}^{q-2})_{m,n\geq0}$ from $\mathcal{P}(L)$ with*

$$\sum_{m,n\geq0}\varphi_{m,n}^{q-2}(e_L) < \infty$$

such that

$$f(z,u) = \sum_{m,n=0}^{\infty}\varphi_{m,n}^{q-2}(u)R_{m,n}^{q-2}(z), \quad z \in \overline{\mathbb{D}}, \ u \in L. \tag{34}$$

The above expansion is uniformly convergent on $\overline{\mathbb{D}} \times L$, and we have

$$\varphi_{m,n}^{q-2}(u) = N(q; m, n)\frac{q-1}{\pi}\int_0^1\int_0^{2\pi} f(re^{i\theta}, u)\overline{R_{m,n}^{q-2}(re^{i\theta})}r(1-r^2)^{q-2}\,dr\,d\theta. \tag{35}$$

Remark 6.2. In all the above one could also have considered the compact Gelfand pair $(SU(q), SU(q-1))$ when $q \geq 3$ without any essential changes. For $q = 2$ this is not a Gelfand pair since $SU(2)$ is a non-Abelian group and $SU(1)$ is trivial. Furthermore,

$$(z_1, z_2) \mapsto \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}$$

is a homeomorphism of Ω_4 onto $SU(2)$, and therefore Ω_4 can be given a group structure so it is isomorphic with $SU(2)$.

7 Products of Gelfand pairs

Let (G_1, K_1) and (G_2, K_2) be two Gelfand pairs with dual spaces Z_1 and Z_2. Then $(G_1 \times G_2, K_1 \times K_2)$ is a Gelfand pair and the dual space Z can be identified with the product space $Z_1 \times Z_2$.

In fact, if $\varphi_1 \in Z_1, \varphi_2 \in Z_2$, then $\varphi_1 \otimes \varphi_2 : G_1 \times G_2 \to \mathbb{C}$ defined by $\varphi_1 \otimes \varphi_2(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)$ is easily seen to belong to Z. Furthermore, it is not difficult to see that $(\varphi_1, \varphi_2) \mapsto \varphi_1 \otimes \varphi_2$ is a homeomorphism of $Z_1 \times Z_2$ onto Z.

If we consider the two compact Gelfand pairs $(O(d+1), O(d))$ and $(O(d'+1), O(d'))$ and apply Theorem 2.6 (ii) to their product, we get Theorem 2.9 of [8]. Another proof of this theorem was given in [4, Theorem 6.1].

If we consider the two compact Gelfand pairs $(U(q), U(q-1))$ and $(U(p), U(p-1))$ with $q, p \geq 2$ and apply Theorem 2.6 (ii) to their product, we get the following result about positive definite functions on products of complex spheres:

Theorem 7.1. *Let $f : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function and define $F : U(q) \times U(p) \to \mathbb{C}$ by*

$$F(A, B) = f(Ae_1 \cdot e_1, Be_1 \cdot e_1), \quad A \in U(q), B \in U(p).$$

Then F belongs to $\mathcal{P}^{\sharp}_{U(q-1) \times U(p-1)}(U(q) \times U(p))$ if and only if there exists a multi-sequence $c : \mathbb{N}_0^4 \to [0, \infty[$ with $\sum c(m, n, k, l) < \infty$ such that

$$f(z, w) = \sum_{(m,n,k,l) \in \mathbb{N}_0^4} c(m, n, k, l) R_{m,n}^{q-2}(z) R_{k,l}^{p-2}(w), \quad z, w \in \overline{\mathbb{D}}. \qquad (36)$$

The series (36) converges uniformly on $\overline{\mathbb{D}}^2$.

8 An extension from $\mathcal{P}(L)$ to $\mathcal{P}(X^2)$

In a recent paper [7] Guella and Menegatto showed how the expansions of functions in $\mathcal{P}(\mathbb{S}^d, L)$ with coefficient functions from $\mathcal{P}(L)$ can be extended to expansions with positive definite kernels on an arbitrary set as coefficient functions. We shall here show how this can be modified to the present framework, where the sphere is replaced by an arbitrary homogeneous space G/K associated with a compact Gelfand pair (G, K).

Let X denote an arbitrary non-empty set and recall that $\mathcal{P}(X^2)$ denotes the set of positive definite kernels on X.

To a function $f : G \times X^2 \to \mathbb{C}$ we consider the kernel on $G \times X$ given by

$$((x, u), (y, v)) \mapsto f(x^{-1}y, u, v), \quad x, y \in G, \ u, v \in X. \qquad (37)$$

Definition 8.1. *By $\mathcal{P}^{\sharp}_K(G, X^2)$ we shall denote the set of functions $f : G \times X^2 \to \mathbb{C}$ satisfying*

1. f is bi-invariant with respect to K in the first variable

2. $f(\cdot, u, v)$ is continuous on G for each $(u, v) \in X^2$

3. The kernel (37) is positive definite.

Theorem 2.3 in [7] can be extended in the following way:

Theorem 8.2. *Let (G, K) be a compact Gelfand pair. For a function $f : G \times X^2 \to \mathbb{C}$ the following are equivalent:*

(i) $f \in \mathcal{P}_K^\sharp(G, X^2)$

(ii) f has a series representation of the form

$$f(x, u, v) = \sum_{\varphi \in Z} B(\varphi)(u, v)\varphi(x), \quad x \in G, \ (u, v) \in X^2, \tag{38}$$

where $B(\varphi) \in \mathcal{P}(X^2)$ and $\sum_{\varphi \in Z} B(\varphi)(u, u) < \infty$ for each $u \in X$.

For $f \in \mathcal{P}_K^\sharp(G, X^2)$ we have

$$B(\varphi)(u, v) = \delta(\varphi) \int_G f(x, u, v)\overline{\varphi(x)} \, d\omega_G(x), \quad (u, v) \in X^2. \tag{39}$$

Proof. The proof is a simple modification of the proof of Theorem 2.3 in [7], so we shall be very brief.

"(i) $\implies$ (ii)."

Given $f \in \mathcal{P}_K^\sharp(G, X^2)$ and $u_1, \ldots, u_n \in X$, $c_1, \ldots, c_n \in \mathbb{C}$ we define

$$F(x) = \sum_{j,k=1}^n f(x, u_j, u_k)c_j\overline{c_k} \tag{40}$$

and observe that $F \in \mathcal{P}_K^\sharp(G)$, cf. Lemma 2.2 in [7]. By Theorem 2.6 (ii) we have the series representation

$$F(x) = \sum_{\varphi \in Z} B(\varphi, F)\varphi(x), \quad x \in G, \tag{41}$$

where

$$B(\varphi, F) = \delta(\varphi) \int_G F(x)\overline{\varphi(x)} \, d\omega_G(x) \geq 0$$

satisfy $\sum_{\varphi \in Z} B(\varphi, F) < \infty$. In particular for $n = 2, u_1 = u, u_2 = v \in X$ and $(c_1, c_2) = (1, 1), (1, -1), (1, i)$ we get that

$$
\begin{aligned}
F_1(x) &= f(x, u, u) + f(x, v, v) + f(x, u, v) + f(x, v, u) \\
F_2(x) &= f(x, u, u) + f(x, v, v) - f(x, u, v) - f(x, v, u) \\
F_3(x) &= f(x, u, u) + f(x, v, v) - if(x, u, v) + if(x, v, u)
\end{aligned}
$$

19

have convergent expansions like (41). Since $f(x, u, v)$ is a linear combination of F_1, F_2, F_3, there exists a uniquely determined kernel $B(\varphi) : X^2 \to \mathbb{C}$ such that

$$f(x, u, v) = \sum_{\varphi \in Z} B(\varphi)(u, v)\varphi(x), \quad x \in G, u, v \in X,$$

and the series is absolutely convergent. As in [7] we see that $B(\varphi) \in \mathcal{P}(X^2)$.

"(ii) $\Longrightarrow$ (i)."

If $B(\varphi) \in \mathcal{P}(X^2)$ satisfies $\sum_{\varphi \in Z} B(\varphi)(u, u) < \infty$ for each $u \in X$, then $\sum_{\varphi \in Z} |B(\varphi)(u, v)| < \infty$ for all $u, v \in X$ because

$$|B(\varphi)(u, v)| \le (B(\varphi)(u, u)B(\varphi)(v, v))^{1/2} \le (1/2)(B(\varphi)(u, u) + B(\varphi)(v, v)).$$

It is now easy to see that f defined by (38) satisfies (i). $\qquad \square$

Theorem 2.7 from [7] can also be generalized to the present framework:

Remark 8.3. Suppose in the setting of Theorem 8.2 that there exists a function $b : Z \to [0, \infty[$ with $\sum_{\varphi \in Z} b(\varphi) < \infty$ such that

$$B(\varphi)(u, u) \le b(\varphi), \quad u \in X,$$

then the expansion (38) is uniformly convergent for $x \in G, u, v \in X$.

References

[1] G. E. Andrews, R. Askey and R. Roy, Special Functions. Cambridge University Press 1999.

[2] C. Berg, Dirichlet forms on symmetric spaces, Ann. Institut Fourier, Grenoble **23** No. 1 (1973), 135–156.

[3] C. Berg, J. P. R. Christensen and P. Ressel, Harmonic analysis on semigroups. Theory of positive definite and related functions. Graduate Texts in Mathematics vol. **100**. Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[4] C. Berg and E. Porcu, From Schoenbeg Coefficients to Schoenberg Functions, Constr. Approx. **45 (2)** (2017), 217–241.

[5] J. Dixmier, Les C^*-algèbres et leurs représentations. Gauthier-Villars, Paris, 1964.

[6] R. Godement, Introduction aux travaux de A. Selberg, Séminaire Bourbaki **9** (1956-57), exposé 144.

[7] J. C. Guella, V. A. Menegatto, From Schoenberg coefficients to Schoenberg functions: a unifying framework. Manuscript, October 25, 2016.

[8] J. C. Guella, V. A. Menegatto and A. P. Peron, An extension of a theorem of Schoenberg to products of spheres. Banach J. Math. Anal. **10** no. 4 (2016), 671–685.

[9] T. H. Koornwinder, The addition formula for Jacobi polynomials II. The Laplace type integral representation and the product formula. Math. Centrum Amsterdam, Report TW133, 1972.

[10] V. A. Menegatto and A. P. Peron, Positive Definite Kernels on Complex Spheres. J. Math. Anal. Appl. **254** (2001), 219–232.

[11] J. R. Munkres, Topology, Second Edition. Prentice Hall, Inc. New Jersey, 2000.

[12] C. Müller, Spherical Harmonics. Lecture Notes in Mathematics. Springer-Verlag. Berlin, Heidelberg, New York, 1966.

[13] W. Rudin, Real and Complex Analysis. McGraw-Hill Book Company, Singapore, 1986.

[14] Z. Sasvári, Positive Definite and Definitizable Functions. Akademie Verlag, Berlin, 1994.

[15] I. J. Schoenberg, Positive definite functions on spheres, Duke Math. J. **9** (1942), 96–108.

[16] G. van Dijk, Introduction to harmonic analysis and generalized Gelfand pairs. De Gruyter Studies in Mathematics, 36. Walter de Gruyter & Co., Berlin, 2009. x+223 pp. ISBN: 978-3-11-022019-3.

[17] N. Ja. Vilenkin and A. U. Klimyk, Representations of Lie groups and special functions, Vol 2. Kluwer Academic Publishers, Dordrechts, The Netherlands, 1993.

[18] J. A. Wolf, Harmonic Analysis on Commutative spaces, Mathematical Surveys and Monographs Volume 142, American Mathematical Society, 2007.

[19] A. Wünsche, Generalized Zernike or disc polynomials. J. Comput. Appl. Math. **174** (2005), 135–163.

Christian Berg
Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, DK-2100, Denmark
e-mail: `berg@math.ku.dk`

Ana P. Peron
Departamento de Matemática, ICMC-USP-São Carlos
Caixa Postal 668, 13560-970 São Carlos SP, Brazil
e-mail: `apperon@icmc.usp.br`

Emilio Porcu
School of Mathematics and Statistics
Chair of Spatial Analytics Methods Center
Newcastle University, Newcastle upon Tyne, NE1 7RU, UK
e-mail: `emilio.porcu@newcastle.ac.uk`
    and
Department of Mathematics
Technical University Federico Santa Maria, Valparaíso, Chile