Onde encontramos fdp?

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Onde encontramos fdp?

⋆ Funções Definidas Positivas

- Núcleos Definidos Positivos
  - ✓ Definições, propriedades e exemplos
- Funções Definidas Positivas
  - ✓ Definições, exemplos e propriedades
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- Polinômios de Gegenbauer
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Núcleos Definidos Positivos

Dado um conjunto não vazio X, uma função $K : X \times X \to \mathbb{C}$, é dita ser um *núcleo definido positivo em X* (pd) quando

$$\sum_{\mu=1}^{n} \sum_{\nu=1}^{n} c_\mu \overline{c_\nu} K(x_\mu, x_\nu) \geq 0,$$

para todo $n \in \mathbb{N}$, $\{x_1, x_2, \ldots, x_n\} \subset X$ e $\{c_1, c_2, \ldots, c_n\} \subset \mathbb{C}$.

$(K(x_\mu, x_\nu))_{\mu,\nu=1}^{n}$ é hermitiana e não negativa definida.

Se a desigualdade acima é estrita quando pelo menos um dos c_μ é não nulo, então o núcleo é chamado *estritamente definido positivo em X* (spd).

Propriedades

$$\boxed{\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} c_\mu \overline{c_\nu} K(x_\mu, x_\nu) \geq 0}$$

1. K pd em $X \implies K(x,x) \geq 0, \ \forall x \in X$
2. K pd em $X \implies |K(x,y)|^2 \leq K(x,x)K(y,y), \ \forall x, y \in X$
3. K_1, K_2 pd em X, $a, b \geq 0 \implies$ são pd's em X:

$$\overline{K_1},$$

$$K_1 K_2,$$

$$aK_1 + bK_2$$

4. O limite pontual de uma sequência de núcleos pd é pd

Exemplos

$$\sum_{\mu=1}^n \sum_{\nu=1}^n c_\mu \overline{c_\nu} K(x_\mu, x_\nu) \geq 0$$

1. $K(x, y) = \cos(x - y)$ é pd em $\mathbb{R}$ go

$$\sum_{\mu,\nu=1}^n c_\mu \overline{c_\nu} \cos(x_\mu - x_\nu) = \sum_{\mu,\nu=1}^n c_\mu \overline{c_\nu} [\cos(x_\mu) \cos(x_\nu) + \sin(x_\mu) \sin(x_\nu)]$$

$$= \sum_{\mu=1}^n c_\mu \cos(x_\mu) \sum_{\nu=1}^n \overline{c_\nu} \cos(x_\nu) + \sum_{\mu=1}^n c_\mu \sin(x_\mu) \sum_{\nu=1}^n \overline{c_\nu} \sin(x_\nu)$$

$$= \left| \sum_{\mu=1}^n c_\mu \cos(x_\mu) \right|^2 + \left| \sum_{\mu=1}^n c_\mu \sin(x_\mu) \right|^2 \geq 0$$

2. Se $f : X \to \mathbb{C}$, então $K(x, y) = f(x)\overline{f(y)}$ é pd em X

$$\sum_{\mu=1}^n \sum_{\nu=1}^n c_\mu \overline{c_\nu} f(x_\mu) \overline{f(x_\nu)} = \sum_{\mu=1}^n c_\mu f(x_\mu) \sum_{\nu=1}^n \overline{c_\nu} \overline{f(x_\nu)} = \left| \sum_{\mu=1}^n c_\mu f(x_\mu) \right|^2 \geq 0$$

Funções Definidas Positivas

depende da estrutura do conjunto/espaço

- $X = \mathbb{R}$: $f : \mathbb{R} \to \mathbb{C}$ é uma FDP em $\mathbb{R}$ se:

$$K : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \ : \ K(x,y) = f(x-y) \quad \text{é pd em } \mathbb{R}$$

- $(X, \langle \cdot, \cdot \rangle)$ e.v.real (complexo): $f : \mathbb{R}\,(\mathbb{C}) \to \mathbb{C}$ é uma FDP em X se:

$$K : X \times X \to \mathbb{C} \ : \ K(x,y) = f(\langle x, y \rangle) \quad \text{é pd em } X$$

- $X = G$ grupo:

$f : G \to \mathbb{C}$ é uma FDP em G se:

$$K : G \times G \to \mathbb{C} \ : \ K(u,v) = f(u^{-1}v) \quad \text{é pd em } G$$

- $X = S^d := \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ $(d \geq 1)$:

$f : [-1, 1] \to \mathbb{C}$ é uma FDP em S^d se:

$$K : S^d \times S^d \to \mathbb{C} \;:\; K(x, y) = f(\underbrace{\cos(\Theta(x, y))}_{\langle x, y \rangle}) \quad \text{é pd em } S^d$$

$$\Theta(x, y) = \arccos(\langle x, y \rangle)$$

- $X = \Omega_{2q} := \{z \in \mathbb{C}^q : \|z\| = 1\}$ $(q \geq 2)$:

$f : \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\} \to \mathbb{C}$ é uma FDP em Ω_{2q} quando:

$$K : \Omega_{2q} \times \Omega_{2q} \to \mathbb{C} \;:\; K(x, y) = f(\langle x, y \rangle) \quad \text{é pd em } \Omega_{2q}$$

- $X = \Omega_2 := \{z \in \mathbb{C} : |z| = 1\}$: $f : \Omega_2 \to \mathbb{C}$ é uma FDP em Ω_2 quando:

$$K : \Omega_2 \times \Omega_2 \to \mathbb{C} \;:\; K(x, y) = f(\langle x, y \rangle) \quad \text{é pd em } \Omega_2$$

Propriedades gerais de fdp's

$$f, g \text{ fdp}'s \implies fg \text{ fdp}$$

$$f_n \text{ fdp}'s \implies f = \lim_{n\to\infty} f_n \text{ fdp}$$

$$f, g \text{ fdp}'s \implies af + bg \text{ fdp}, \qquad (a, b \geq 0)$$

Pergunta: Podem a e/ou b serem negativos e $af + bg$ ainda ser fdp?

Onde encontramos fdp?

como fdp's são em geral?

Exemplos de fdp's

- $f : \mathbb{R} \to \mathbb{R} : t \to \cos t$ é fdp em $\mathbb{R}$ go to kernel
- $g : [-1, 1] \to \mathbb{R} : x \to x$ é fdp em S^d, $d \geq 1$ return ▸ polinômio de Gegenbauer

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu g(\langle x_\mu, x_\nu \rangle) = \sum_{\mu,\nu=1}^{n} c_\mu c_\nu \langle x_\mu, x_\nu \rangle = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \langle c_\mu x_\mu, c_\nu x_\nu \rangle$$

$$= \left\langle \textstyle\sum_{\mu=1}^{n} c_\mu x_\mu, \sum_{\nu=1}^{n} c_\nu x_\nu \right\rangle = \left\| \textstyle\sum_{\mu=1}^{n} c_\mu x_\mu \right\|^2 \geq 0$$

- $h : \mathbb{D} \to \mathbb{C} : z \to z$ é fdp em Ω_{2q}, $q \geq 2$ return ▸ polinômio no disco

$$\textstyle\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} h(\langle z_\mu, z_\nu \rangle) \quad = \left\langle \displaystyle\sum_{\mu=1}^{n} c_\mu z_\mu, \sum_{\nu=1}^{n} c_\nu z_\nu \right\rangle = \left\| \sum_{\mu=1}^{n} c_\mu z_\mu \right\|^2 \geq 0$$

Polinômios de Gegenbauer

Sejam $\lambda > 0$ um número real e $n \in \mathbb{N}$. Definimos o *polinômio de Gegenbauer de grau n associado ao índice* λ por

$$P_n^\lambda(t) := \binom{n+2\lambda-1}{n} \frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}\Gamma(\lambda)} \int_{-1}^{1} (t + is\sqrt{1-t^2})^n (1-s^2)^{\lambda-1} ds$$

$$\binom{a}{n} := \frac{a(a-1)\ldots(a-(n-1))}{n!}, \quad a \in \mathbb{R}, \; n \in \mathbb{N}$$

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x \in \mathbb{R}_+$$

$$P_n^\lambda(1) = \binom{n+2\lambda-1}{n} \qquad P_1^1(t) = \frac{4\Gamma(3/2)}{\sqrt{\pi}}\, t$$

go to identity

- $\lambda = (d-1)/2$

$d \geq 2:$

$$P_n^{(d-1)/2}(t) = \binom{n+d-2}{n} n!\Gamma\left(\frac{d}{2}\right) \sum_{k=0}^{[n/2]} \frac{(t^2-1)^k t^{n-2k}}{4^k k!(n-2k)!\Gamma(k+d/2)}$$

$d = 1:$

$$P_n^0(t) = T_n(t) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (t^2-1)^k t^{n-2k} = \cos(n \arccos t)$$

- limitados:

$$|P_n^{(d-1)/2}(t)| \leq P_n^{(d-1)/2}(1), \quad t \in [-1,1], \quad n \geq 0$$

- pares/impares:

$$P_n^{(d-1)/2}(t) = (-1)^n P_n^{(d-1)/2}(-t)$$

- **Fórmula da Adição:**

$\mathcal{Y}_n^{d+1}$: pol. homog. que satisfazem eq. de Laplace: $\sum_{k=1}^{d+1} \frac{\partial^2 f}{\partial x_k^2} = 0$ rest. S^d

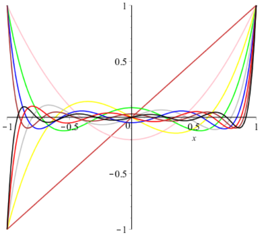
$\{Y_j; j = 1, \ldots, N(d+1,n)\}$ base ortonormal de $\mathcal{Y}_n^{d+1}$

$$P_n^{(d-1)/2}(\langle x,y\rangle) = \frac{\sigma_d}{N(d+1,n)} \sum_{j=1}^{N(d+1,n)} Y_j(x)\overline{Y_j(y)}, \quad x,y \in S^d.$$

$$c_n^d(x) := \frac{P_n^{(d-1)/2}(x)}{P_n^{(d-1)/2}(1)}, \quad x \in [-1,1].$$

$$\lim_{n\to\infty} c_n^d(x) = 0, \text{para cada } \ x \in (-1,1)$$

$d = 5$; $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$



Polinômios de Gegenbauer de grau n=1,2,3,4,5,6,7,8,9
associado ao indice 2

go Disc Polinomyals

Polinômios de Gegenbauer são FDP em S^d....

qualquer FDP em S^d está relacionada com Polinômios de Gegenbauer ??

Como são fdp's em S^d?

- $d \geq 1$

Teorema

([Sch42]) Uma função **contínua** $f : [-1,1] \to \mathbb{R}$ *é uma fdp em* S^d *see.*

$$f(x) = \sum_{n \geq 0} a_n^d c_n^d(x), \quad x \in [-1,1],$$

onde $a_n^d \geq 0,\ n \geq 0$, *e* $\sum_{n \geq 0} a_n^d < \infty$.

S^∞: esfera de Hilbert real:

$$S^\infty = \left\{ (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{k=1}^{\infty} x_k^2 = 1 \right\}$$

- f é fdp em $S^\infty \iff f$ é fdp em S^d, $\forall d \geq 1$

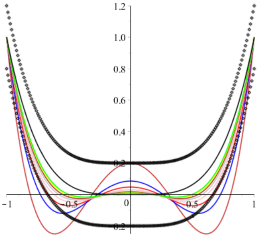
$$\lim_{d\to\infty} c_n^d(x) = x^n$$ é uniforme em x para cada n fixado graphic

Schoenberg : $\lim_{d\to\infty} c_n^d(x) = x^n$ uniforme em n para cada x fixado theorem

picture

Gráficos: Polinômios de Gegenbauer

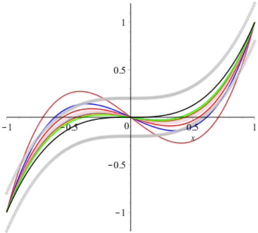
$n = 4$; $d = 1, 2, 3, 4, 5, 6, 7, 8$



Polinômios de Gegenbauer de grau $n = 4$ associado respectivamente a $d = 1,2,3,4,5,6,7,8$.

go Disc Polinomyals

$n = 3$; $d = 1, 2, 3, 4, 5, 6, 7, 8$



Polinômios de Gegenbauer de grau n=3 associado respectivamente a $d = 1,2,3,4,5,6,7,8$.

return

- $d = \infty$

Teorema

([Sch42]) Uma função contínua $f : [-1, 1] \to \mathbb{R}$ é uma fdp em S^∞ see.

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad x \in [-1, 1],$$

onde $a_n \geq 0$, $n \geq 0$, e $\sum_{n \geq 0} a_n < \infty$.

Schoenberg obtém a_n através de um "processo de diagonal de Cantor"

Berg & Porcu [2017] provam:

$$a_n = \lim_{d \to \infty} a_n^d, \quad \forall n \geq 0$$

▸ Ziegel, 2014

- Multiquadric functions

$$\psi(\theta) = \sigma^2 \left(\frac{1-p}{1-p\cos\theta} \right)^{\tau}, \quad 0 \leq \theta \leq \pi$$

$$(p \in (0,1), \ \tau \in \mathbb{N}, \ \sigma \in \mathbb{R}_+)$$

são fdp's em S^∞

$$a_n = \sigma^2 \binom{\tau + n - 1}{n} p^n (1-p)^{\tau}, \quad n = 0, 1, \ldots .$$

- Multiquadric functions:

$$\psi(\theta) = \sigma^2 \frac{(1-\delta)^{(d-1)}}{(1+\delta^2 - 2\delta\cos\theta)^{(d-1)/2}}, \quad 0 \le \theta \le \pi$$

$$(\delta \in (0,1),\ \sigma \in \mathbb{R}_+)$$

são fdp's em S^d $(d \geq 2)$

$$a_n^d = \sigma^2 (1-\delta)^{d-1} \binom{d+n-2}{n} \delta^n$$

- Sine Power functions: Admissible

$$\psi(\theta) = \sigma\left[1 - \left(\sin\frac{\theta}{2}\right)^{\alpha}\right], \quad 0 \leq \theta \leq 2\pi$$

$$(\alpha \in (0,2),\ \sigma \in \mathbb{R}_{+})$$

são fdp's em S^{∞}

$$a_n = -\frac{\sigma^2}{\sqrt{2}}\frac{1}{(n+1)!}\prod_{m=0}^{n}\left(m - \frac{\alpha}{2}\right), \quad n = 0, 1, \ldots$$

Polinômios no disco - Zernike generalizado

- $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ o disco unitário fechado.

Dados m e n inteiros não negativos, o *polinômio no disco* de grau $m+n$ em x e y associado a um número real $\alpha > -1$ é a função $R^{\alpha}_{m,n}$ dada por

$$R^{\alpha}_{m,n}(re^{i\theta}) := r^{|m-n|}e^{i(m-n)\theta}R^{(\alpha,|m-n|)}_{m\wedge n}(2r^2-1),$$

$z := re^{i\theta} = x + iy \in \overline{\mathbb{D}}$, $m \wedge n := \min\{m, n\}$ e

$$R^{(\alpha,|m-n|)}_{m\wedge n}(x) = \frac{P^{(\alpha,|m-n|)}_{m\wedge n}(x)}{P^{(\alpha,|m-n|)}_{m\wedge n}(1)}, \quad x \in [-1, 1]$$

polinômios de Jacobi de grau $m \wedge n$, associado aos índices α e $|m-n|$.

$$R^{\alpha}_{m,n}(z) = \begin{cases} R^{(\alpha,m-n)}_{n}(2z\bar{z}-1)z^{m-n}, & m \geq n \\ R^{(\alpha,n-m)}_{m}(2z\bar{z}-1)\bar{z}^{n-m}, & m \leq n \end{cases}$$

✓ $R^{\alpha}_{m,n}$ é um polinômio de grau m em z e de grau n em $\bar{z}$.

Propriedades

$\alpha > -1$:

1. $R^{\alpha}_{m,n}(1) = 1$;
2. $R^{\alpha}_{m,n}(\overline{z}) = \overline{R^{\alpha}_{m,n}(z)} = R^{\alpha}_{n,m}(z), \; z \in \overline{\mathbb{D}}$;
3. $R^{\alpha}_{m,n}(e^{i\varphi}z) = e^{i(m-n)\varphi} R^{\alpha}_{m,n}(z), \; \varphi \in [0, 2\pi), \; z \in \overline{\mathbb{D}}$;
4. $|R^{\alpha}_{m,n}(z)| \leq 1, \; z \in \overline{\mathbb{D}}, \; m, n \geq 0. \; (\alpha \geq 0)$

$$R^{\alpha}_{m,n}(re^{i\theta}) = r^{|m-n|} e^{i(m-n)\theta} R^{(\alpha,|m-n|)}_{m \wedge n}(2r^2 - 1), z = re^{i\theta}$$

Table 2

m, n	$\alpha = \frac{1}{2}$	$\alpha = 1$
0, 0	1	1
1, 0	z	z
0, 1	z^*	z^*
2, 0	z^2	z^2
1, 1	$\frac{1}{3}(5zz^* - 2)$	$\frac{1}{2}(3zz^* - 1)$
0, 2	z^{*2}	z^{*2}
3, 0	z^3	z^3
2, 1	$\frac{1}{3}(7z^2z^* - 4z)$	$2z^2z^* - z$
1, 2	$\frac{1}{3}(7zz^{*2} - 4z^*)$	$2zz^{*2} - z^*$
0, 3	z^{*3}	z^{*3}
4, 0	z^4	z^4
3, 1	$3z^3z^* - 2z^2$	$\frac{1}{2}(5z^3z^* - 3z^2)$
2, 2	$\frac{1}{15}(63z^2z^{*2} - 56zz^* + 8)$	$\frac{1}{3}(10z^2z^{*2} - 8zz^* + 1)$
1, 3	$3zz^{*3} - 2z^{*2}$	$\frac{1}{2}(5zz^{*3} - 3z^{*2})$
0, 4	z^{*4}	z^{*4}
5, 0	z^5	z^5
4, 1	$\frac{1}{3}(11z^4z^* - 8z^3)$	$3z^4z^* - 2z^3$
3, 2	$\frac{1}{5}(33z^3z^{*2} - 36z^2z^* + 8z)$	$5z^3z^{*2} - 5z^2z^* + z$
2, 3	$\frac{1}{5}(33z^2z^{*3} - 36zz^{*2} + 8z^*)$	$5z^2z^{*3} - 5zz^{*2} + z^*$
1, 4	$\frac{1}{3}(11zz^{*4} - 8z^{*3})$	$3zz^{*4} - 2z^{*3}$

FDP em Ω_{2q}, $q \geq 2$

$f(z) = R^{q-2}_{m,n}(z)$ é uma fdp em Ω_{2q}, $q \geq 2$

$\{Y^q_1, Y^q_2, \ldots, Y^q_{N(q;m,n)}\}$ uma base ortonormal de $H^q(m,n)$

$$R^{q-2}_{m,n}(\langle \xi, \zeta \rangle) = \frac{\omega_{2q}}{N(q;m,n)} \sum_{k=1}^{N(q;m,n)} Y^q_k(\xi)\overline{Y^q_k(\zeta)}, \quad \xi, \zeta \in \Omega_{2q}.$$

Teorema

[MP01] $f : \overline{\mathbb{D}} \to \mathbb{C}$ contínua é fdp em Ω_{2q}, $q \geq 2$, see.

$$f(z) = \sum_{m,n=0}^{\infty} a^{q-2}_{m,n} R^{q-2}_{m,n}(z), \quad z \in \overline{\mathbb{D}},$$

onde $a^{q-2}_{m,n} \geq 0$, $\forall m, n$, e $\sum a^{q-2}_{m,n} < \infty$.

FDP em Ω_{2q}, $q = 1$ ou $q = \infty$

Teorema

[MP01] $f : \Omega_2 \to \mathbb{C}$ *contínua é fdp em* Ω_2 *see.*

$$f(\xi) = \sum_{m \in \mathbb{Z}} a_m \xi^m, \quad \xi \in \Omega_2,$$

onde $a_m \geq 0$, $\forall m$, *e* $\sum a_m < \infty$.

Teorema

[CR82] $f : \overline{\mathbb{D}} \to \mathbb{C}$ *contínua é fdp em* Ω_∞, *see.*

$$f(z) = \sum_{m,n=0}^{\infty} a_{m,n} z^m \overline{z}^n, \quad z \in \overline{\mathbb{D}},$$

onde $a_{m,n} \geq 0$, $\forall m, n$, *e* $\sum a_{m,n}^{q-2} < \infty$.

Exemplos

Positive Definite Functions on Complex Spheres and their Walks through Dimensions 7

Theorem 2.1. *Let $f\colon \mathbb{D} \to \mathbb{C}$ be a continuous function. The following assertions are true:*

(1) *$f \in \Psi(\Omega_{2q})$ if, and only if,*

$$f(z) = \sum_{m,n\geq 0} a^{q-2}_{m,n} R^{q-2}_{m,n}(z), \qquad z \in \mathbb{D}, \tag{2.13}$$

where $\sum_{m,n\geq 0} a^{q-2}_{m,n} < \infty$ and $a^{q-2}_{m,n} \geq 0$ for all (m,n);

(2) *$f \in \Psi^+(\Omega_{2q})$ if, and only if, $f \in \Psi(\Omega_{2q})$ and*

$$\left\{m - n\colon a^{q-2}_{m,n} > 0,\ m,n \geq 0\right\} \cap (N\mathbb{Z} + j) \neq \varnothing, \tag{2.14}$$

for every $N \geq 1$, $j = 0, 1, \ldots, N-1$.

Note that the index $\alpha = q - 2$ of the disc polynomials is related to the sphere Ω_{2q} and consequently $\alpha + 1 = q - 1$ is related to Ω_{2q+2}.

The coefficients $a^{q-2}_{m,n}$ are the analogue of the d-Schoenberg coefficients a^d_k as in Daley and Porcu [12] and Ziegel [49], referring to the expansion of the members of the Schoenberg class Ψ_d. In analogy, we will call $a^{q-2}_{m,n}$ as $(2q)$-*complex Schoenberg coefficients*.

2.1 Families within the classes $\Psi(\Omega_{2q})$ and $\Psi^+(\Omega_{2q})$

It is well known that there exist many examples of functions in the class Ψ_d, some of them widely used in applications (see for example Gneiting [17] and Porcu et al. [28]).

In the literature it is also possible to find examples of functions that satisfy the conditions in Theorem 2.1, or those in Remark 1.4, and therefore they belong to the classes $\Psi(\Omega_{2q})$ and $\Psi^+(\Omega_{2q})$. Some of them, as well as their use in applications, appeared recently, probably originated by the work of Wünsche [46], that deals with disc polynomials: a fundamental tool for studying the functions in these classes. We give below a collection of such functions.

1. Disk Polynomials and related families. The product kernel (Boyd and Raychowdhury [10]),

$$f_{m,n}(z) = z^m \overline{z}^n = \sum_{j=0}^{\min\{m,n\}} c^j_{q,m,n} R^{q-2}_{m-j,n-j}(z), \qquad c^j_{q,m,n} \geq 0, \quad z \in \mathbb{D},$$

is an element of the class $\Psi(\Omega_{2q})$, for each $m,n \geq 0$.

2. Poisson–Szegő kernel and related families. An application of (2.9) and (2.10) shows that

$$f_r(z) := \frac{1}{\sigma_{2q}} \frac{(1-r^2)^q}{|1-rz|^{2q}} = \sum_{m,n\geq 0} \frac{h^{q-2}_{m,n}}{\sigma_{2q}} S^q_{m,n}(r) R^{q-2}_{m,n}(z), \qquad z \in \mathbb{D},$$

and hence it is a member of the class $\Psi(\Omega_{2q})$, for each $r \in [0,1)$.

3. Exponential function. The function (Menegatto et al. [27])

$$e^{z+\overline{z}} = \sum_{m+n=0}^{\infty} \frac{(m+1)_{q-2}(n+1)_{q-2}}{(q-2)!} \left(\sum_{j=0}^{\infty} \frac{1}{j!(m+n+q-1)_j} \right) R^{q-2}_{m,n}(z), \qquad z \in \mathbb{D},$$

belongs to the class $\Psi^+(\Omega_{2q})$.

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4. Aktaş, Taşdelen and Yavuz family. The function (Aktaş et al. [2])

$$f_t(z) := \frac{1}{R}\left(\frac{2}{1-t+R}\right)^{q-2} e^{(2tz)/(1+t+R)} = \sum_{m,n\geq 0} (q-1)_n \frac{t^{m+n}}{m!n!} R^{q-2}_{m+n,n}(z), \qquad z \in \mathbb{D},$$

where $R := \left(1 - 2\left(2|z|^2 - 1\right)t + t^2\right)^{1/2}$, is a member of $\Psi^+(\Omega_{2q})$, for each $t \in (0,1)$.

5. Horn family. Let r, R be positive integers such that $4r = (R-1)^2$. Horn's function H_4 is defined on p. 57 of Srivastava and Manocha [42] by

$$H_4(a,b;c,d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n} \frac{x^m y^n}{m!n!},$$

where $|x| < r$ and $|y| < R$. An application of Theorem 2.2 in Aktaş et al. [2] shows that

$$\begin{aligned} f_{t,s,b}(z) &:= \frac{1}{(1-s)^{q-1}} H_4\left(q-1, b; q-1, q-1; \frac{s(|z|^2-1)}{(1-s)^2}, \frac{t\overline{z}}{1-s}\right) \\ &= \sum_{m,n\geq 0} (q+n-1)_m (b)_n \frac{t^n s^m}{m!n!} R^{q-2}_{m,m+n}(z), \qquad z \in \mathbb{D}. \end{aligned}$$

Hence it is a member of $\Psi^+(\Omega_{2q})$, for each b, a positive integer, and t, s positive numbers satisfying

$$|s| < 1, \qquad \frac{|s|}{(1-s)^2} < r, \qquad \text{and} \qquad \frac{|t|}{1-s} < R.$$

6. Lauricella family. Let r_1, r_2 and r_3 be positive integers such that $r_1 r_2 = (1-r_2)(r_2-r_3)$. The Lauricella hypergeometric function of three variables F_{14} (Saran's notation F_F is also used (Saran [32])) is defined by (see p. 67 of Srivastava and Manocha [42])

$$F_{14}(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x_1,x_2,x_3) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}} \frac{x_1^m x_2^n x_3^p}{m!n!p!},$$

where $|x_1| < r_1$, $|x_2| < r_2$ and $|x_3| < r_3$. For $t, s \in \mathbb{R}$ such that $|s| < r_1$ and $|t| < r_2$, where $r_1 = r_2(1-r_2)$, define

$$f_{t,s,b}(z) := F_{14}\left(1,1,1,q-1,b,q-1;q-1,1,1;s\left(|z|^2-1\right),tz,s|z|^2\right), \qquad z \in \mathbb{D}.$$

From Theorem 2.3 in Aktaş et al. [2] we get

$$f_{t,s,b}(z) = \sum_{m,n\geq 0} (q-1)_n (b)_m \frac{t^m s^n}{m!n!} R^{q-2}_{m+n,n}(z), \qquad z \in \mathbb{D},$$

and hence, $f_{t,s,b}$ is a member of $\Psi^+(\Omega_{2q})$, for each b, a positive integer, and t, s positive numbers satisfying the relevant conditions above.

Some comments are in order. Lauricella functions are generalizations of the Gauss hypergeometric functions to multiple variables and were introduced by Lauricella in 1893. Recursion formulas and integral representation for Lauricella functions, including F_{14} (F_F), have been studied and can be found, for example, in Sahai and Verma [31] and Saran [33, 34]. In 1873, Schwarz [41] found a list of 15 cases where hypergeometric functions can be expressed algebraically. More precisely, Schwarz gave a list of parameters determining the cases where the hypergeometric differential equation has two independent solutions that are algebraic functions. Between 1989 and 2009 several researchers extended this list: to general one-variable hypergeometric functions ${}_{p+1}F_p$ (Beukers and Heckman [8]), the Appell–Lauricella functions F_1 and F_D (Beazley Cohen and Wolfart [5]), the Appell functions F_2 and F_4 (Kato [20, 21]), and the Horn function G_3 (Schipper [38]). In 2012, Bod [9] extended Schwarz' list to the four classes of Appell–Lauricella functions and the 14 complete Horn functions, including H_4.

Pesquisa

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