Uma relação entre polinômios ortogonais e funções positivas definidas

- Polinômios de Legendre
  - ✓ Definições e propriedades
- Polinômios de Gegenbauer
  - ✓ Propriedades e suas “convergências”
- Polinômios no disco e funções positivas definidas
  - ✓ Definições, propriedades, exemplos
- Funções positivas definidas
  - ✓ Alguns resultados e o Teorema de Schoenberg revisitado

Polinômios de Gegenbauer

Sejam $\lambda > 0$ um número real e $n \in \mathbb{N}$. Definimos o *polinômio de Gegenbauer de grau n associado ao índice λ* por

$$P_n^\lambda(t) := \binom{n+2\lambda-1}{n} \frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}\Gamma(\lambda)} \int_{-1}^{1} (t + is\sqrt{1-t^2})^n (1-s^2)^{\lambda-1} ds$$

$$\binom{a}{n} := \frac{a(a-1)\ldots(a-(n-1))}{n!}, \quad a \in \mathbb{R}, \ n \in \mathbb{N}.$$

$$P_n^\lambda(1) = \binom{n+2\lambda-1}{n}$$

$$P_n(d,t) = \frac{\sigma_{d-3}}{\sigma_{d-2}} \int_{-1}^{1} (t + is\sqrt{1-t^2})^n (1-s^2)^{(d-4)/2} ds \quad (d \geq 3)$$

$$\sigma_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad d \geq 3.$$

$$P_n^{\lambda}(t) = \binom{n+2\lambda-1}{n} \frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}\Gamma(\lambda)} \int_{-1}^{1} (t+is\sqrt{1-t^2})^n (1-s^2)^{\lambda-1} ds$$

$$P_n^{(d-1)/2}(t) = \binom{n+d-2}{n} P_n(d+1,t), \quad d \geq 2 \ (d \in \mathbb{N})$$

Pela definição do polinômio de Legendre,

$$P_n(d,t) := n!\Gamma\left(\frac{d-1}{2}\right)\sum_{l=0}^{[n/2]}\left(\frac{-1}{4}\right)^l \frac{(1-t^2)^l t^{n-2l}}{l!(n-2l)!\Gamma(l+(d-1)/2)}$$

$$P_n^{(d-1)/2}(t) = \binom{n+d-2}{n} P_n(d+1,t), \quad d \geq 2$$

$$P_n^{(d-1)/2}(t) = \binom{n+d-2}{n} n!\Gamma\left(\frac{d}{2}\right)\sum_{l=0}^{[n/2]}\left(\frac{-1}{4}\right)^l \frac{(1-t^2)^l t^{n-2l}}{l!(n-2l)!\Gamma(l+d/2)},$$

$(d \geq 2)$

$$P_n^0(t) := P_n(2,t) = \cos(n \arccos t)$$

Algumas propriedades

A relação entre os polinômios de Gegenbauer e de Legendre:

$$P_n^{(d-1)/2}(t) = \binom{n+d-2}{n} P_n(d+1,t), \quad P_n^0(t) = P_n(2,t) = \cos(n \arccos t)$$

nos permite obter as seguintes propriedades para os polinômios de Gegenbauer: para $d \geq 1$

✓ $|P_n^{(d-1)/2}(t)| \leq P_n^{(d-1)/2}(1), \quad t \in [-1,1], \quad n \geq 0,$

✓ $P_n^{(d-1)/2}(t) = (-1)^n P_n^{(d-1)/2}(-t).$

✓ **Teorema da Adição.** Se $\{Y_j; j = 1, \ldots, N(d+1,n)$, é uma base ortonormal de $\mathcal{Y}_n(d+1)$, então

$$P_n^{(d-1)/2}(x \cdot y) = \frac{\sigma_d}{N(d+1,n)} \sum_{j=1}^{N(d+1,n)} \overline{Y_j(x)} Y_j(y), \quad x, y \in S^d.$$

Forma alternativa - definição

Lema

$$\sum_{n=0}^{\infty} r^n P_n^{\lambda}(t) = \frac{1}{(1+r^2-2rt)^{\lambda}}, \quad r \in [0,1), \quad t \in [-1,1].$$

Prova.

$$\sum_{n=0}^{\infty} r^n P_n^{\lambda}(t) =$$

$$\boxed{\textstyle\sum_{n=0}^{\infty} \binom{n+2\lambda-1}{n} z^n = \frac{1}{(1-z)^{2\lambda}},\ |z|<1}$$

$$\frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}\Gamma(\lambda)} \int_{-1}^{1} \left(\sum_{n=0}^{\infty} \binom{n+2\lambda-1}{n} \overbrace{r^n (t+is\sqrt{1-t^2})^n}^{|r(t+is\sqrt{1-t^2})|<1} \right) (1-s^2)^{\lambda-1} ds$$

$$= \frac{\Gamma(\lambda+1/2)}{\sqrt{\pi}\Gamma(\lambda)} \int_{-1}^{1} \frac{(1-s^2)^{\lambda-1}}{(1-rt+irs\sqrt{1-t^2})^{2\lambda}} ds$$

$$s = \tanh u$$
$$1 - rt - ir\sqrt{1-t^2} = \sqrt{1+r^2-2rt}\,e^{-i\alpha},\ \alpha \in [0, \pi/2)$$

$$\int_{-1}^{1} \frac{(1-s^2)^{\lambda-1}}{(1-rt+irs\sqrt{1-t^2})^{2\lambda}}\,ds =$$

$$= \int_{-\infty}^{+\infty} \frac{(\cosh u)^{-2(\lambda-1)}(\cosh u)^{-2}}{(\sqrt{1+r^2-2rt}\cos\alpha - i\sqrt{1+r^2-2rt}\,\mathrm{sen}\,\alpha\tanh u)^{2\lambda}}\,du$$

$$= \frac{1}{(1+r^2-2rt)^{\lambda}} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{(\cosh(u-i\alpha))^{2\lambda}}\,du}_{<\infty,\ indep.\alpha}$$

$$= \frac{c}{(1+r^2-2rt)^{\lambda}}$$

$$\sum_{n=0}^{\infty} r^n P_n^{\lambda}(t) = \frac{C}{(1+r^2-2rt)^{\lambda}},\ \text{para alguma constante } C$$

$$t = 1 : \frac{C}{(1-r)^{2\lambda}} = \sum_{n=0}^{\infty} r^n \binom{n+2\lambda-1}{n} = \frac{1}{(1-r)^{2\lambda}} \Longrightarrow C = 1 \qquad \blacksquare$$

Gegenbauer normalizado

$$c_n(d,x) := \frac{P_n^{(d-1)/2}(x)}{P_n^{(d-1)/2}(1)}, \quad x \in [-1,1].$$

$$P_n^{(d-1)/2}(x) = \binom{n+d-2}{n} P_n(d+1,x), \quad P_n^0(x) = P_n(2,x) = \cos(n \arccos x)$$

$$c_n(d,x) = P_n(d+1,x), \quad d \geq 1.$$

$$\star\ c_n'(d,x) = \frac{n(n+d-1)}{d} c_{n-1}(d+1,x)$$

$$\star\ (1-x^2)c_n'(d,x) = n[c_{n-1}(d,x) - x c_n(d,x)]$$

Convergências...

Teorema

$$\lim_{n\to\infty} c_n(d,x) = 0, \, para \; cada \;\; x \in (-1,1)$$

Prova. $x \in (-1,1)$:

$$\begin{aligned}(1-x^2)\frac{(n+1)(n+d-1)}{d-1}c_n(d,x) &= (1-x^2)c'_{n+1}(d-1,x) \\ &= (n+1)[c_n(d-1,x) - xc_{n+1}(d-1,x)]\end{aligned}$$

$$c_n(d,x) = \frac{1}{(1-x^2)}\frac{d-1}{(n+d-1)}\left[c_n(d-1,x) - xc_{n+1}(d-1,x)\right]$$

$$|c_n(d,x)| < \frac{2}{(1-x^2)}\frac{d-1}{(n+d-1)}, \quad x \in (-1,1) \qquad \blacksquare$$

Teorema

$$\lim_{d\to\infty} c_n(d,x) = x^n,\ \textit{para cada}\, x \in [-1,1]\ \textit{e cada}\, n \geq 0$$

Prova.

$$c_n(d,x) = x^n + n! \sum_{l=1}^{[n/2]} \left(\frac{-1}{4}\right)^l \frac{(1-x^2)^l x^{n-2l}}{l!(n-2l)!} \frac{\Gamma(d/2)}{\Gamma(l+d/2)}$$

$$|c_n(d,x) - x^n| \leq n! \sum_{l=1}^{[n/2]} \frac{1}{4^l} \frac{1}{l!(n-2l)!} \underbrace{\frac{\Gamma(d/2)}{\Gamma(l+d/2)}}_{\to 0,\ d\to\infty}, \quad \forall x \in [-1,1]$$

■

$$\lim_{d\to\infty} c_n(d,x) = x^n,\ \text{uniformemente em } x \text{ para cada } n \text{ fixado}$$

Teorema

([Sch42]) Seja $x \in (-1,1)$. Se $\varepsilon > 0$, então existe $L = L(x,\varepsilon) > 0$ tal que se $d > L$,

$$|c_n(d,x) - x^n| < \varepsilon, \quad n = 0, 1, \ldots,$$

$\lim_{d\to\infty} c_n(d,x) = x^n$ *uniformemente em n, para cada x fixado.*

Prova.

$$P_n^{(d-1)/2}(x) = \binom{n+d-2}{n} \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma(d)} \int_{-1}^{1} (x + is\sqrt{1-x^2})^n (1-s^2)^{(d-3)/2} ds$$

$$c_n(d,\cos\theta) \underset{\theta\in(0,\pi)}{\overset{x=\cos\theta}{=}} \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \int_0^{\pi} (\cos\theta + is\sin\theta)^n (1-s^2)^{(d-3)/2} ds$$

$$\underset{ds=-\sin\varphi d\varphi}{\overset{s=\cos\varphi}{=}} \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \int_0^{\pi} (\cos\theta + i\sin\theta\cos\varphi)^n (\sin\varphi)^{d-2} d\varphi$$

$$\Delta_n^d := c_n(d, \cos\theta) - \cos^n\theta$$

$$\int_0^\pi (\sin\varphi)^{d-2} d\varphi = \frac{\sqrt{\pi}\,\Gamma((d-1)/2)}{\Gamma(d/2)}$$

$$c_n(d, \cos\theta) = \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d-1)/2)} \int_0^\pi (\cos\theta + i\sin\theta\cos\varphi)^n (\sin\varphi)^{d-2} d\varphi$$

$$\Delta_n^d = \frac{1}{\int_0^\pi (\sin\varphi)^{d-2} d\varphi} \left[\int_0^\pi \underbrace{(\cos\theta + i\sin\theta\cos\varphi)^n - \cos^n\theta} \right] (\sin\varphi)^{d-2} d\varphi$$

$$F_n(\theta, \varphi) := (\cos\theta + i\sin\theta\cos\varphi)^n - \cos^n\theta$$

$$|\Delta_n^d| \leq \frac{1}{\int_0^\pi (\sin\varphi)^{d-2} d\varphi} \int_0^\pi |F_n(\theta,\varphi)| (\sin\varphi)^{d-2} d\varphi$$

- $|F_n(\theta,\varphi)| \leq (\cos^2\theta + \sin^2\theta\cos^2\varphi)^{n/2} + |\cos\theta|^n \leq 2$

dado $\delta \in (0, \pi/2)$:

$$|\Delta_n^d| \le \frac{1}{\int_0^\pi (\sin\varphi)^{d-2} d\varphi} \left[\int_0^{\pi/2-\delta} \overbrace{|F_n(\theta,\varphi)|}^{\le 2} (\sin\varphi)^{d-2} d\varphi \right.$$

$$\left. + \int_{\pi/2-\delta}^{\pi/2+\delta} |F_n(\theta,\varphi)| (\sin\varphi)^{d-2} d\varphi + \int_{\pi/2+\delta}^{\pi} \underbrace{|F_n(\theta,\varphi)|}_{\le 2} (\sin\varphi)^{d-2} d\varphi \right]$$

$$\le \frac{1}{\int_0^\pi (\sin\varphi)^{d-2} d\varphi} \left[4 \int_0^{\pi/2-\delta} (\sin\varphi)^{d-2} d\varphi \right.$$

$$\left. + \int_{\pi/2-\delta}^{\pi/2+\delta} |F_n(\theta,\varphi)| (\sin\varphi)^{d-2} d\varphi \right]$$

$$|\Delta_n^d| \to 0, \quad d \to \infty ??$$

- $$I_1 := \frac{4\int_0^{\pi/2-\delta}(\sin\varphi)^{d-2}d\varphi}{\int_0^{\pi}(\sin\varphi)^{d-2}d\varphi} \longrightarrow 0, \quad d\to\infty$$

- $$I_2 := \frac{1}{\int_0^{\pi}(\sin\varphi)^{d-2}d\varphi}\int_{\pi/2-\delta}^{\pi/2+\delta}|F_n(\theta,\varphi)|(\sin\varphi)^{d-2}d\varphi$$

$$\frac{\int_{\pi/2-\delta}^{\pi/2+\delta}(\sin\varphi)^{d-2}d\varphi}{\int_0^{\pi}(\sin\varphi)^{d-2}d\varphi} \le 1$$

$$|F_n(\theta,\varphi)| \le (\cos^2\theta + \sin^2\theta\cos^2\varphi)^{n/2} + |\cos\theta|^n$$

$$\pi/2-\delta \le \varphi \le \pi/2+\delta \overset{\cos\ dec.}{\Longrightarrow}$$

$$-\sin\delta = \cos\left(\frac{\pi}{2}+\delta\right) < \cos\varphi < \cos\left(\frac{\pi}{2}-\delta\right) = \sin\delta$$

$$\therefore |F_n(\theta,\varphi)| \le (\cos^2\theta + \sin^2\theta\sin^2\delta)^{n/2} + |\cos\theta|^n$$

$$\therefore I_2 \le (\cos^2\theta + \sin^2\theta\sin^2\delta)^{n/2} + |\cos\theta|^n$$

$$I_2 \to 0??$$

$f(x) = \cos^2\theta + \sin^2\theta\sin^2 x$ é estritamente crescente em $[0,\pi/2]$ e $f(\pi/2) = 1$, podemos escolher $\delta \in (0,\pi/2)$, $\delta = \delta(\theta)$, tal que

$$\cos^2\theta + \sin^2\theta\sin^2\delta < 1.$$

Assim, para tal δ,

$$I_2 \longrightarrow 0, \quad n \to \infty.$$

$$|\Delta_n^d| \le I_1 + I_2 \to 0, \quad d \to \infty \quad \forall n??$$

$$I_1 \longrightarrow 0, \quad d \to \infty.$$

Dado $\varepsilon > 0$, existem $d_0 = d_0(\theta, \varepsilon)$ e $n_0 = n_0(\theta, \varepsilon)$ tais que

$$\star\ |\Delta_n^d| < \varepsilon, \quad d > d_0, \quad n > n_0$$

Por outro lado, para cada $n = 0, 1, \ldots, n_0$, temos que:

$$\lim_{d \to \infty} c_n(d, \cos\theta) = \cos^n \theta \iff |\Delta_n^d| \to 0,\ d \to \infty$$

isto implica que existem $\tilde{d}_0 = \tilde{d}_0(\varepsilon), \ldots, \tilde{d}_{n_0} = \tilde{d}_{n_0}(\varepsilon)$ tais que

$$\star\ |\Delta_n^d| < \varepsilon, \quad d > \tilde{d}_1 := \max\{\tilde{d}_0, \ldots, \tilde{d}_{n_0}\}, \quad 0 \le n \le n_0$$

tomando $L = L(\theta, \varepsilon) = \max\{d_0, d_1\}$, temos

$$|\Delta_n^d| < \varepsilon, \quad d > L, \quad n \geq 0$$

■

$$\lim_{d \to \infty} c_n(d, x) = x^n \begin{cases} \text{uniform. em } x \text{ para cada } n \text{ fixado} \\ \text{uniform. em } n \text{ para cada } x \text{ fixado} \end{cases}$$

Polinômios Ortogonais - Legendre e Gegenbauer revisitados

Seja μ uma função não decrescente de modo que as integrais

$$c_n = \int_a^b t^n d\mu(t) < \infty, \quad n \geq 0,$$

$$\{1, t, t^2, \ldots, t^n, \ldots\} \overset{l.i}{\subset} L((a, b), d\mu) \implies \text{existem polinômios}$$

$$p_0, p_1, \ldots, p_n, \ldots$$

unicamente determinados pelas condições:

(a) p_n é um polinômio de grau n cujo coeficiente de t^n é positivo;

(b) o sistema $\{p_n\}$ é ortonormal, isto é,

$$\int_a^b p_m(t)p_n(t)d\mu(t) = \delta_{m,n}, \quad m, n = 0, 1, \ldots$$

Os polinômios p_n são chamados de polinômios ortogonais em $[a, b]$ associados com a distribuição $d\mu$.

Caso especial:

$$d\mu = w(t)dt,$$

onde w é não negativa, mensurável no sentido de Lebesgue e $\int_a^b w(t)dt > 0$:

- p_n polinômios ortogonais em $[a, b]$ associados com a função peso w.

- $\begin{cases} \text{intervalo simétrico em relação à origem: } [-a, a] \\ w \text{ é uma função par} \end{cases}$

$$p_n(-t) = (-1)^n p_n(t).$$

Jacobi

Polinômios de Jacobi: $P_n^{(\alpha,\beta)}$ de grau n: são os polinômios ortogonais em $[-1, 1]$ associados à função peso

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}$$

A integrabilidade da função peso w é garantida quando α, β são números reais com $\alpha > -1$ e $\beta > -1$:

- $\displaystyle\int_{-1}^{1} P_n^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta} dt = h_{m,n}^{(\alpha,\beta)} \delta_{m,n}$

- $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \quad n \geq 0$

- $P_n^{(\alpha,\beta)}(t) = (-1)^n P_n^{(\beta,\alpha)}(-t)$

- Fórmula de Rodrigues:

$$(1-t)^{\alpha}(1+t)^{\beta}P_n^{(\alpha,\beta)}(t)=\frac{(-1)^n}{2^n n!}\frac{d^n}{dt^n}\left[(1-t)^{\alpha+n}(1+t)^{\beta+n}\right].$$

- Representação explicita dos polinômios de Jacobi é:

$$P_n^{(\alpha,\beta)}(t)=\sum_{k=0}^{n}\frac{(-1)^k(n+\alpha)!(n+\beta)!}{2^n k!(n-k)!(n+\beta-k)!(k+\alpha)!}(1-t)^k(1+t)^{n-k}$$

- $$\max_{x\in[-1,1]}|P_n^{(\alpha,\beta)}(x)|=\binom{n+\max\{\alpha,\beta\}}{n},\quad \max\{\alpha,\beta\}\geq -1/2$$

Legendre, Gegenbauer: casos especiais $\alpha = \beta$

- Polinômios de Legendre, P_n: quando $\alpha = \beta = 0$
- Polinômios de Tchebichef de primeira ordem, T_n: quando $\alpha = \beta = -1/2$:

$$T_n(t) = \cos(n \arccos t), \quad t \in [-1, 1]$$

- Polinômios de Gegenbauer, P_n^λ: quando $\alpha = \beta = \lambda - \frac{1}{2}$, $(\lambda > -1/2)$

- $P_n = P_n(3, \cdot)$
- $T_n = P_n^0 = P_n(2, \cdot)$
- P_n^λ foram discutidos anteriormente no caso $\lambda > 0$.

Referências

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