Uma relação entre polinômios ortogonais e funções positivas definidas

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VI EIBPOA
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- ⋆ Polinômios ortogonais e Funções positivas definidas

- Polinômios de Legendre
  - ✓ Definições e propriedades

- Polinômios de Gegenbauer
  - ✓ Propriedades e suas "convergências"

- Polinômios no disco e funções positivas definidas
  - ✓ Definições, propriedades, exemplos

- Funções positivas definidas
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Notações, definições

- $x_{(d)} = x = (x_1, \ldots, x_d)$ elemento do espaço $\mathbb{R}^d$,
  ✓ identificamos $x_{(d-1)} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ com $x_{(d-1)} = (x_1, \ldots, x_{d-1}, 0) \in \mathbb{R}^q$
- $d\omega_{d-1}$ = o elemento de superfície usual sobre a esfera unitária S^{d-1} de $\mathbb{R}^d$
- $\sigma_{d-1} = \omega_{d-1}(S^{d-1})$

- $d = 2$:

$$\sigma_1 = \int_{S^1} d\omega_1 = \int_0^{2\pi} (\cos^2 t + \operatorname{sen}^2 t)dt = \int_0^{2\pi} dt = 2\pi.$$

- $d \geq 3$: podemos escrever $\xi \in S^{d-1}$ na forma

$$\xi = \xi_{(d)} = t\varepsilon_d + \sqrt{1-t^2}\xi_{(d-1)}, \quad t = \varepsilon_d \cdot \xi \in [-1, 1], \quad \xi_{(d-1)} \in S^{d-2}.$$

$$d\omega_{d-1}(\xi_{(d)}) = (1-t^2)^{(d-3)/2} dt d\omega_{d-2}(\xi_{(d-1)}).$$

$$\sigma_{d-1} = \int_{S^{d-1}} d\omega_{d-1} = \int_{-1}^{1} (1-t^2)^{(d-3)/2} dt \int_{S^{d-2}} d\omega_{d-2}$$
$$= \sigma_{d-2} \int_{-1}^{1} (1-t^2)^{(d-3)/2} dt$$

$$\sigma_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad d \geq 3.$$

- $\mathcal{H}_n^d$ $(d \geq 2)$ = espaço de todos os polinômios homogêneos de grau n em d variáveis.
  $H_n(x_{(d)}) = \sum_{|\alpha|=n} a_\alpha x_{(d)}^\alpha, \; a_\alpha \in \mathbb{C}$

$$\dim(\mathcal{H}_n^d) = \frac{(d+n-1)!}{n!(d-1)!}.$$

- (Laplaciano) $\Delta_{(d)} := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$
- harmônicos homogêneos (ou harmônicos sólidos) = polinômios homogêneos H_n de grau n em d variáveis que satisfazem $\Delta_{(d)} H_n = 0$.
- $\mathcal{Y}_n^*(d)$ = espaço dos harmônicos homogêneos

$$\dim(\mathcal{Y}_n^*(d)) := N(d,n) = \begin{cases} 1 & , \; n = 0 \\ d & , \; n = 1 \\ \dfrac{(2n+d-2)(n+d-3)!}{n!(d-2)!} & , \; n \geq 2 \end{cases}$$

- Harmônico esférico: $Y_n(d,\xi)$ é a restrição de um harmônico homogêneo $H_n(x)$ a S^{d-1}:

$$H_n(r\xi) = r^n H_n(\xi) = r^n Y_n(d,\xi),\ \xi \in S^{d-1}$$

- $\mathcal{Y}_n^d$ = espaço dos harmônicos esféricos de ordem n em d variáveis

$$\dim(\mathcal{Y}_n^d) = N(d,n) = \dim(\mathcal{Y}_n^*(d))$$

1. $\mathcal{H}_n^d$, $\mathcal{Y}_n^*(d)$ são invariantes: $f \circ A \in X, \forall f \in X, A \in O(d)$
2. $\mathcal{Y}_n^d$ são invariantes e irredutíveis $\neq L_1 \oplus L_2,\ \ L_1 \perp L_2$ invariantes
3. Se $\mathcal{L} \subset C(S^{d-1})$ é invariante e irredutível, então $\mathcal{L} = \mathcal{Y}_n^d$ ou $\mathcal{L} \perp \mathcal{Y}_n^d$
4. polinômios harmônicos esféricos de ordem distintas são ortogonais:

$$\int_{S^{d-1}} Y_n(d,\xi)\overline{Y_m(d,\xi)}d\omega_{d-1}(\xi) = 0.$$

Polinômios de Legendre

$\star\, H_n \in \mathcal{H}_n^d$:

$$H_n(x_1,\ldots,x_d) = \sum_{k=0}^{n} x_d^k h_{n-k}(x_1,\ldots,x_{d-1}),\;\; h_{n-k} \in \mathcal{H}_{n-k}^{d-1}$$

$$\bullet \Delta_{(d)} = \Delta_{(d-1)} + \frac{\partial^2}{\partial x_d^2}$$

$$\star\, \Delta_{(d)} H_n(x_{(d)}) \overset{\bullet}{=} \sum_{k=0}^{n-2} x_d^k \Delta_{(d-1)} h_{n-k}(\tilde{x}) + \sum_{k=0}^{n-2} (k+2)(k+1) x_d^k h_{n-k-2}(\tilde{x})$$

$$\therefore\; \Delta_{(d)} H_n = 0 \overset{\sum x^k a_k = 0}{\Leftrightarrow} h_{n-k-2} = -\frac{\Delta_{(d-1)} h_{n-k}}{(k+2)(k+1)}, \quad k = 0,\ldots,n-2$$

$\therefore$ cada escolha para h_n e h_{n-1} determina indutivamente um único elemento de $\mathcal{Y}_n^*(d)$

Harmônico de Legendre: $L_n(d, \cdot) : \mathbb{R}^d \to \mathbb{R}$

(I) $L_n(d, x) \in \mathcal{Y}_n^*(d)$

(II) $L_n(d, Ax) = L_n(d, x), \quad A \in J(d, \varepsilon_d) := \{A \in O(d) : A\varepsilon_d = \varepsilon_d\}$

(III) $L_n(d, \varepsilon_d) = 1$

- (I) $\Longrightarrow L_n(d, x) \in \mathcal{H}_n^d$ $\boxed{L_n(d,x) = \sum_{k=0}^{n} x_d^k h_{n-k}(x_{(d-1)})}$
- (II) $\Longrightarrow h_j(A' x_{(d-1)}) = h_j(x_{(d-1)}), \forall A' \in O(d-1) \Longrightarrow \exists c_l;$

$$h_{n-k}(x_{(d-1)}) \overset{h_j\ homog.}{=} \begin{cases} c_l |x_{(d-1)}|^{2l}, & n-k = 2l \\ 0, & n-k = 2l+1 \end{cases}$$

$$L_n(d, x) = \sum_{l=0}^{[n/2]} c_l |x_{(d-1)}|^{2l} x_d^{n-2l}.$$

- (III) $\Longrightarrow c_0 = 1$ e (I) $\Longrightarrow \Delta_{(d)} L_n(d, \cdot) \equiv 0$

  - $h_{n-k-2} = -\dfrac{\Delta_{(d-1)} h_{n-k}}{(k+2)(k+1)}, \quad k = 0, \ldots, n-2$

$$L_n(d,x) = n!\Gamma\left(\frac{d-1}{2}\right)\sum_{l=0}^{[n/2]}\left(\frac{-1}{4}\right)^l \frac{|x_{(d-1)}|^{2l}x_d^{n-2l}}{l!(n-2l)!\Gamma(l+(d-1)/2)}$$

$$x = r\underbrace{\xi}_{\in S^{d-1}} = r(t\varepsilon_d + \sqrt{1-t^2}\underbrace{\xi_{(d-1)}}_{\in S^{d-2}}) = x_d\varepsilon_d + x_{(d-1)},$$

$$x_d = rt, \quad x_{(d-1)} = r\sqrt{1-t^2}\xi_{(d-1)}, \quad |x_{(d-1)}| = r\sqrt{1-t^2}$$

$$L_n(d,x) = L_n(d,r\xi) = r^n L_n(d,\xi) = r^n P_n(d,t) = r^n P_n(d,\varepsilon_d\cdot\xi)$$

Polinômio de Legendre : $P_n(d,\cdot) : [-1,1] \to \mathbb{R}$

$$P_n(d,t) := L_n(d,\xi), \quad t = \varepsilon_d\cdot\xi \in [-1,1],$$

$$P_n(d,t) := n!\Gamma\left(\frac{d-1}{2}\right)\sum_{l=0}^{[n/2]}\left(\frac{-1}{4}\right)^l \frac{(1-t^2)^l t^{n-2l}}{l!(n-2l)!\Gamma(l+(d-1)/2)}$$

- $P_0(d,t) = 1, \quad \forall t$
- $P_n(d,1) = 1, \quad \forall n$
- $P_n(d,-t) = (-1)^n P_n(d,t), \quad \forall t, n$

Lema

1. $H \in \mathcal{Y}_n^*(d)$ *invariante com relação a* $J(d,\varepsilon_d) \Longrightarrow H = cL_n(d,\cdot)$.
2. $\xi \in S^{d-1}$ *e* $Y_n(d,\cdot) \in \mathcal{Y}_n^d$ *invariante com relação a* $J(d,\xi) \Longrightarrow$

$$Y_n(d,\eta) = Y_n(d,\xi)P_n(d,\xi\cdot\eta), \quad \eta \in S^{d-1}$$

$$H_n(x) = \sum_{l=0}^{[n/2]} c_l |x_{(d-1)}|^{2l} x_d^{n-2l}, \quad H_n(\varepsilon_d) = c_0 \text{ e } L_n(d,\varepsilon_d) = 1 \Longrightarrow H_n \equiv c_0 L_n$$

Prova. $\exists B \in O(d); \quad \xi = B\varepsilon_d \Longrightarrow$
$Y_n(d, B\cdot)$ é invariante com relação $J(d,\varepsilon_d) \Longrightarrow$
$r^n Y_n(d,B\cdot) \in \mathcal{Y}_n^*(d)$ invariante com relação $J(d,\varepsilon_d) \overset{Lem.ant.}{\Longrightarrow}$
$r^n Y_n(d,B\eta) = cL_n(d,r^n\eta), \forall r \geq 0 \Longrightarrow \; Y_n(d,B\eta) = cL_n(d,\eta)$
$\therefore Y_n(d,\eta) = cL_n(d,B^t\eta) = cP_n(d,\varepsilon_d\cdot B^t\eta) = cP_n(d,\xi\cdot\eta)$
$\overset{P_n(d,\xi\cdot\xi)=1}{\Longrightarrow} c = Y_n(d,\xi)$ ■

Teorema

(Teorema da Adição) Seja $\{Y_j^d; j = 1, \ldots, N(d, n)\}$ *uma base ortonormal de* $\mathcal{Y}_n^d$*:*

$$\int_{S^{d-1}} \overline{Y_j^d(\zeta)} Y_k^d(\zeta) d\omega_{d-1}(\zeta) = \delta_{jk}.$$

Então,

$$P_n(d, \xi \cdot \eta) = \frac{\sigma_{d-1}}{N(d, n)} \sum_{j=1}^{N(d,n)} \overline{Y_j^d(\xi)} Y_j^d(\eta), \quad \xi, \eta \in S^{d-1}.$$

Prova.

$$F(\xi, \eta) := \sum_{j=1}^{N(d,n)} \overline{Y_j^d(\xi)} Y_j^d(\eta), \quad \xi, \eta \in S^{d-1}$$

- $F(\xi, \cdot) \in \mathcal{Y}_n^d$ e $F(\cdot, \eta) \in \mathcal{Y}_n^d$

- $A \in O(d)$

$$F(A\xi, A\eta) = \sum_{j=1}^{N(d,n)} \overline{Y_j^d(A\xi)}\, Y_j^d(A\eta)$$

$$\overset{Y_j^d \circ A \in \mathcal{Y}_n^d}{=} \sum_{j=1}^{N(d,n)} \sum_{k=1}^{N(d,n)} \overline{u_{jk}(A)}\, \overline{Y_k^d(\xi)} \sum_{m=1}^{N(d,n)} u_{jm}(A)\, Y_m^d(\eta)$$

$$= \sum_{k=1}^{N(d,n)} \sum_{m=1}^{N(d,n)} \overline{Y_k^d(\xi)}\, Y_m^d(\eta) \sum_{j=1}^{N(d,n)} \overline{u_{jk}(A)}\, u_{jm}(A)$$

$$\sum_{j=1}^{N(d,n)} \overline{u_{jk}(A)}\, u_{jm}(A) =$$

$$= \sum_{j=1}^{N(d,n)} \sum_{l=1}^{N(d,n)} \overline{u_{jk}(A)}\; u_{lm}(A) \int_{S^{d-1}} \overline{Y_j^d(\xi)}\, Y_l^d(\xi) d\omega_{d-1}(\xi)$$

$$= \int_{S^{d-1}} \left(\sum_{j=1}^{N(d,n)} \overline{u_{jk}(A)}\, \overline{Y_j^d(\xi)} \sum_{l=1}^{N(d,n)} u_{lm}(A) Y_l^d(\xi) \right) d\omega_{d-1}(\xi)$$

$$= \int_{S^{d-1}} \overline{Y_k^d(A\xi)}\, Y_m^d(A\xi) d\omega_{d-1}(\xi) = \delta_{km}$$

$$\therefore F(A\xi, A\eta) = \sum_{k=1}^{N(d,n)} \sum_{m=1}^{N(d,n)} \overline{Y_k^d(\xi)}\, Y_m^d(\eta) \sum_{j=1}^{N(d,n)} \overline{u_{jk}(A)}\, u_{jm}(A)$$

$$= \sum_{k=1}^{N(d,n)} \overline{Y_k^d(\xi)}\, Y_k^d(\eta) = F(\xi, \eta)$$

$$F(A\xi, A\eta) = F(\xi,\eta), \quad A \in O(d), \xi,\eta \in S^{d-1}$$

$$\left.\begin{array}{l} \bullet\ F(\xi,\cdot) \in \mathcal{Y}_n^d, \text{ é invariante com relação a } J(d,\xi) \\ \bullet\ F(\cdot,\eta) \in \mathcal{Y}_n^d, \text{ é invariante com relação a } J(d,\eta) \end{array}\right\} \Longrightarrow$$

$$\begin{cases} F(\xi,\eta) = F(\xi,\xi)P_n(d,\xi\cdot\eta) \\ F(\xi,\eta) = F(\eta,\eta)P_n(d,\xi\cdot\eta) \end{cases}$$

$$\therefore \begin{cases} F(\xi,\xi) = F(\eta,\eta), \quad \forall \xi,\eta \in S^{d-1} \\ P_n(d,\xi\cdot\eta) = \dfrac{1}{F(\xi,\xi)}F(\xi,\eta) = \dfrac{1}{F(\xi,\xi)}\sum_{j=1}^{N(d,n)} \overline{Y_j^d(\xi)}Y_j^d(\eta) \end{cases}$$

$$F(\xi,\xi)\sigma_{d-1} = \int_{S^{d-1}} F(\xi,\xi)\,d\omega_{d-1}(\xi)$$

$$= \sum_{j=1}^{N(d,n)} \underbrace{\int_{S^{d-1}} \overline{Y_j^d(\xi)}Y_j^d(\xi)\,d\omega_{d-1}(\xi)}_{1} = N(d,n)$$

■

A Fórmula de Adição para $d = 2$ diz que:

$$P_n(2, \xi \cdot \eta) = \frac{2\pi}{2} \sum_{j=1}^{2} \overline{Y_{j,n}^2(\xi)} Y_{j,n}^2(\eta), \quad \xi, \eta \in S^1.$$

Escrevendo $\xi = (\cos\theta, \sin\theta)$ e $\eta = (\cos\phi, \sin\phi)$, uma base ortonormal de $\mathcal{Y}_n(2)$ é

$$Y_{1,n}^2(\xi) = \frac{1}{\sqrt{\pi}} \cos(n\theta), \quad Y_{2,n}^2(\xi) = \frac{1}{\sqrt{\pi}} \sin(n\theta)$$

$$\frac{1}{\pi} P_n(2, \cos(\theta - \phi)) = \sum_{j=1}^{2} \overline{Y_{j,n}^2(\xi)} Y_{j,n}^2(\eta)$$

$$= \frac{1}{\pi} [\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi)] = \frac{1}{\pi} \cos(n(\theta - \phi)), \quad \xi, \eta \in S^1$$

$$P_n(2, t) = \cos(n \arccos t), \quad t \in [-1, 1].$$

Fórmula de Rodrigues

Lema

(Fórmula de Rodrigues) Para todo n,

$$P_n(d,t) = (-1)^n R_n(d)(1-t^2)^{(3-q)/2} \frac{d^n}{dt^n}(1-t^2)^{n+(d-3)/2},$$

onde $R_n(d)$ é a constante de Rodrigues dada por

$$R_n(d) := \left(\frac{1}{2}\right)^n \frac{\Gamma((d-1)/2)}{\Gamma(n+(d-1)/2)}.$$

Representação integral de Laplace

Teorema

Para $d \geq 3$, $n \geq 0$ e $t \in [-1, 1]$,

$$P_n(d, t) = \frac{\sigma_{d-3}}{\sigma_{d-2}} \int_{-1}^{1} (t + is\sqrt{1 - t^2})^n (1 - s^2)^{(d-4)/2} ds.$$

Prova. $\eta = (\eta_1, \ldots, \eta_{d-1}) \in S^{d-2}$:

$$x \mapsto (x_d + ix_{(d-1)} \cdot \eta)^n \quad \text{pol. harm. homg. de grau n}$$

$$\bullet\, L_n(x) := \frac{1}{\sigma_{d-2}} \int_{S^{d-2}} (x_d + ix_{(d-1)} \cdot \eta)^n d\omega_{d-2}(\eta)$$

$\star$ $L_n \in \mathcal{Y}_n^*(d)$

$\star$ $L_n(\varepsilon_d) \overset{def}{=} 1$

$\star$ $L_n(Ax) = L_n(x), \quad A \in J(d, \varepsilon_d)$:

$$\boxed{x_d = rt, x_{(d-1)} = r\sqrt{1-t^2}\xi_{(d-1)}}$$

$Ax = x_d\varepsilon_d + Bx_{(d-1)}$, $B \in O(d-1)$

$$\begin{aligned} L_n(Ax) &= \frac{1}{\sigma_{d-2}}\int_{S^{d-2}}(x_d + ix_{(d-1)}\cdot B^t\eta)^n d\omega_{d-2}(\eta) \\ &= \frac{1}{\sigma_{d-2}}\int_{S^{d-2}}(x_d + ix_{(d-1)}\cdot\zeta)^n d\omega_{d-2}(\zeta) = L_n(x) \end{aligned}$$

$\therefore L_n$ é o harmônico de Legendre de grau n em d variáveis.

$$\begin{aligned} \therefore P_n(d,t) &= \frac{1}{\sigma_{d-2}}\int_{S^{d-2}}(t + i\xi_{(d-1)}\cdot\eta\sqrt{1-t^2})^n d\omega_{d-2}(\eta) \\ &\overset{\star}{=} \frac{\sigma_{d-3}}{\sigma_{d-2}}\int_{-1}^{1}(t + is\sqrt{1-t^2})^n(1-s^2)^{(d-4)/2}ds \end{aligned}$$

$$\star\begin{cases} \xi_{(d-1)} = (0,\dots,0,1) \\ d\omega_{d-2}(\eta) = (1-s^2)^{(d-4)/2}ds\,d\omega_{d-3}(\eta_{(d-2)}) \\ s = \eta\cdot\varepsilon_{(d-1)} \end{cases}$$

$\blacksquare$

Relação de recorrência

Lema

Para $d \geq 2$ e $n \geq 1$,

$$(1-t^2)P_n'(d,t) = n[P_{n-1}(d,t) - tP_n(d,t)].$$

Prova. Para $d \geq 3$:

$$(1-t^2)P_n'(d,t) = C_d(1-t^2)\frac{d}{dt}\left[\int_{-1}^{1}(t+is\sqrt{1-t^2})^n(1-s^2)^{(d-4)/2}ds\right]$$

$$(1-t^2)\frac{d}{dt}(t+is\sqrt{1-t^2}) = 1 - t(t+is\sqrt{1-t^2}).$$

O caso $d = 2$ pode ser obtido diferenciando-se diretamente:

$$P_n(2,t) = \cos(n \arccos t).$$

■

$$P_n(d,t) = a_n^0(d)t^n + \text{ termos de graus menores que } n,$$

$$a_n^0(d) = \frac{\Gamma(d-1)}{\Gamma(d/2)} 2^{n-1} \frac{\Gamma(n+(d-2)/2)}{\Gamma(n+d-2)}.$$

Lema

Para $d \geq 2$, temos

$$P_n'(d,t) = \frac{n(n+d-2)}{(d-1)} P_{n-1}(d+2,t).$$

Prova.

$$\frac{a_n^0(d)}{a_{n-1}^0(d+2)} = \frac{n+d-2}{d-1} \iff (d-1)a_n^0(d) = (n+d-2)a_{n-1}^0(d+2)$$

$$\text{grau}((d-1)P_n'(d,t) - n(n+d-2)P_{n-1}(d+2,t)) \leq n-2$$

Para $k = 0, \ldots, n-2$:

$$\int_{-1}^{1} P_n'(d,t) \underbrace{P_k(d+2,t)(1-t^2)^{(d-1)/2}} dt =$$

$$- \int_{-1}^{1} \left[\underbrace{(1-t^2)P_k'(d+2,t) - (d-1)tP_k(d+2,t)}_{grau \leq n-1} \right] P_n(d,t)(1-t^2)^{\frac{d-3}{2}} dt$$

$$= 0$$

$$\therefore (d-1)P_n'(d,t) - n(n+d-2)P_{n-1}(d+2,t)$$

tem grau $\leq n-2$ e é ortogonal a todos os polinômios de grau $\leq n-2$

$$\therefore (d-1)P_n'(d,t) - n(n+d-2)P_{n-1}(d+2,t) = 0$$

■

Limitação

Lema

Para $d \geq 2$ os polinômios de Legendre $P_n(d,\cdot)$ satisfazem

$$|P_n(d,t)| \leq 1, \quad n \geq 0, \; t \in [-1,1].$$

Prova. Para $d = 2$: $P_n(2,t) = \cos(n \arccos t)$.
Para $d \geq 3$, pela representação integral de Laplace:

$$|P_n(d,t)| \leq \frac{\sigma_{d-3}}{\sigma_{d-2}} \int_{-1}^{1} \underbrace{|t + is\sqrt{1-t^2}|^n}_{=(1-(1-s^2)(1-t^2))^{n/2} \leq 1} (1-s^2)^{(d-4)/2} ds$$

$$\leq \frac{\sigma_{d-3}}{\sigma_{d-2}} \int_{-1}^{1} (1-s^2)^{(d-4)/2} ds \overset{\star}{=} 1$$

$$\star \sigma_{d-1} = \int_{S^{d-1}} d\omega_{d-1} = \sigma_{d-2} \int_{-1}^{1} (1-t^2)^{(d-3)/2} dt.$$

Referências

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