EXEMPLO 1. Verifique que $\lim\limits_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$.

Solução

Fazendo $x = -(t + 1)$, $t > 0$, vem

$$\left(1 + \frac{1}{x}\right)^x = \left(1 - \frac{1}{1+t}\right)^{-t-1} = \left(1 + \frac{1}{t}\right)^t \frac{t+1}{t}.$$

Para $x \to -\infty$, $t \to +\infty$, assim

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{t \to +\infty} \left(1 + \frac{1}{t}\right)^t \frac{t+1}{t} = e. \quad ■$$

EXEMPLO 2. Verifique que

a) $\lim\limits_{h \to 0^+} (1 + h)^{\frac{1}{h}} = e$ *b*) $\lim\limits_{h \to 0^-} (1 + h)^{\frac{1}{h}} = e$.

Solução

a) Fazendo $h = \frac{1}{x}$ $(h \to 0^+ \Rightarrow x \to +\infty)$ vem

$$\lim_{h \to 0^+} (1 + h)^{\frac{1}{h}} = \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

b) Faça você.

Segue do Exemplo 2 que

$$\boxed{\lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e.}$$

■

EXEMPLO 3. Mostre que $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$.

Solução

Fazendo $u = e^h - 1$ ou $h = \ln(1 + u)$ vem

$$\frac{e^h - 1}{h} = \frac{u}{\ln(1+u)} = \frac{1}{\ln(1+u)^{\frac{1}{u}}}$$

$(h \to 0 \Rightarrow u \to 0)$; assim

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{u \to 0} \frac{1}{\ln(1+u)^{\frac{1}{u}}} = \frac{1}{\ln e} = 1. \quad ■$$

Exercícios 6.3

1. Calcule.

a) $\lim\limits_{x \to +\infty} \left(1 + \frac{2}{x}\right)^x$ e^2

b) $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x+2}$ e

c) $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{2x}\right)^x$ $e^{1/2}$

d) $\lim\limits_{x \to +\infty} \left(1 + \frac{2}{x}\right)^{x+1}$ e^2

e) $\lim\limits_{x \to +\infty} \left(\frac{x+2}{x+1}\right)^x$ e

f) $\lim\limits_{x \to 0} (1 + 2x)^x$ 1

g) $\lim\limits_{x \to 0} (1 + 2x)^{\frac{1}{x}}$ e^2

h) $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{2x}$ e^2

2. Seja $a > 0$, $a \neq 1$. Mostre que

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a.$$

3. Calcule.

a) $\lim\limits_{x \to 0} \frac{e^{2x} - 1}{x}$ 2

b) $\lim\limits_{x \to 0} \frac{e^{x^2} - 1}{x}$ 0

c) $\lim\limits_{x \to 0} \frac{5^x - 1}{x}$ $\ln 5$

d) $\lim\limits_{x \to 0^+} \frac{3^x - 1}{x^2}$ ∞