# Ladrilhamentos com triângulos e quadrados: origami e métodos de inflação

**LHF60**++
Celebrating the 60th Birthday of
Luiz Henrique de Figueiredo

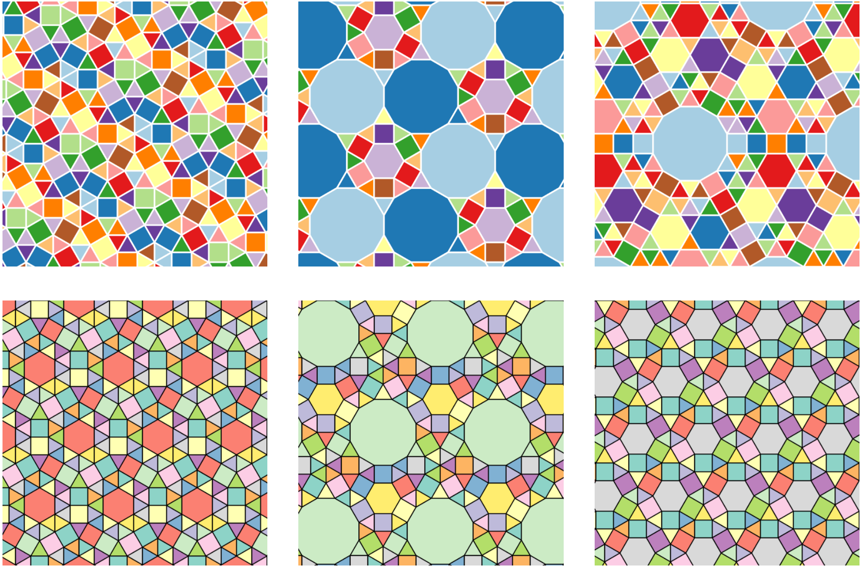
25/01/2023

José Ezequiel Soto Sánchez

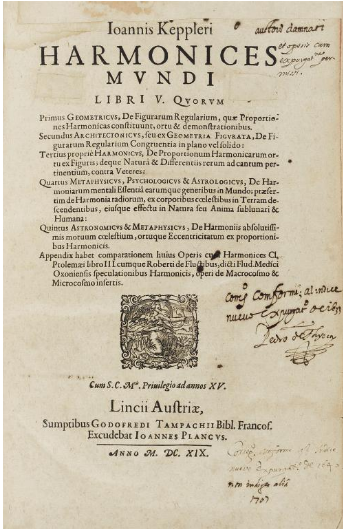
impa
Instituto de Matemática Pura e Aplicada

# Introdução

# Ladrilhamentos periódicos com polígonos regulares: rigidez & variedade



# O problema é antigo: Kepler fez o primeiro tratamento formal em 1619



Ioannis Keppleri

HARMONICES MVNDI

LIBRI V. QVORVM

Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas constituunt, ortu & demonstrationibus.

Secundus ARCHITECTONICVS, seu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel solido:

Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:

Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Essentiâ earumque generibus in Mundo; præsertim de Harmonia radiorum, ex corporibus cœlestibus in Terram descendentibus, eiusque effectu in Natura seu Anima sublunari & Humana:

Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis absolutissimis motuum cœlestium, ortuque Eccentricitatum ex proportionibus Harmonicis.

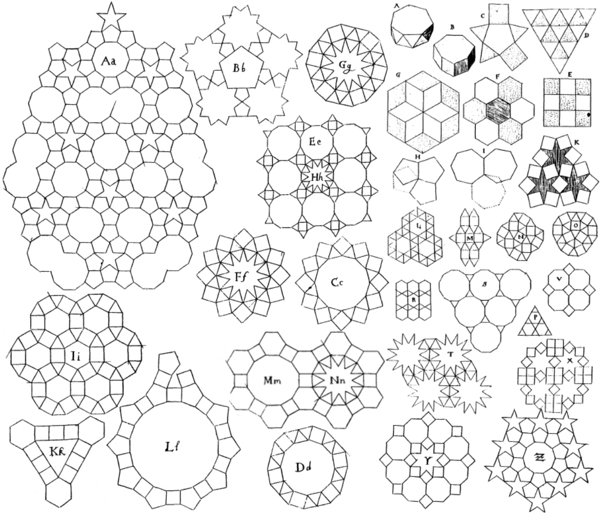
Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III. cumque Roberti de Fluctibus, dicti Flud. Medici Oxoniensis speculationibus Harmonicis, operi de Macrocosmo & Microcosmo insertis.

Cum S. C. M^tis. Priuilegio ad annos XV.

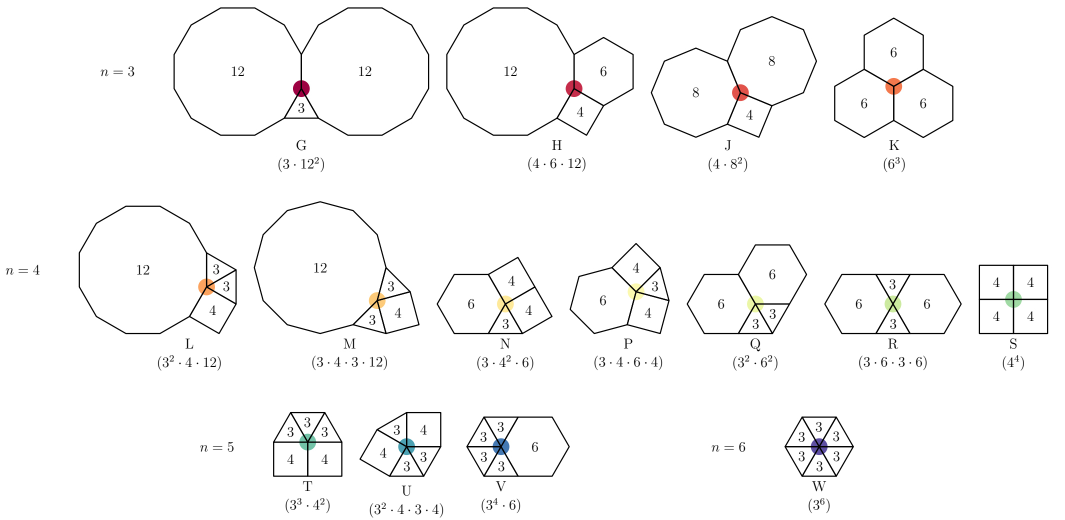
Lincii Austriæ,

Sumptibus GODOFREDI TAMPACHII Bibl. Francof.
Excudebat IOANNES PLANCVS.

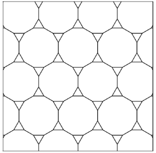
ANNO M. DC. XIX.



# Vértices Arquimedianos $\left(\sum_{i=1}^{n} \frac{k_i - 2}{k_i} = 2\right)$

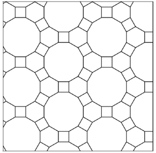


# Teselaciones clásicas: 1-uniformes



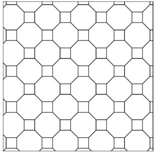
G

$(3 \cdot 12^2)$



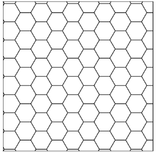
H

$(4 \cdot 6 \cdot 12)$



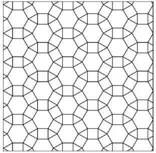
J

$(4 \cdot 8^2)$



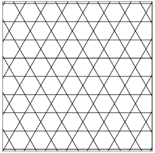
K

$(6^3)$



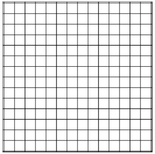
P

$(3 \cdot 4 \cdot 6 \cdot 4)$



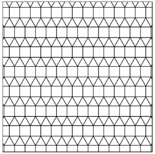
R

$(3 \cdot 6 \cdot 3 \cdot 6)$



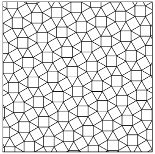
S

$(4^4)$



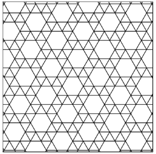
T

$(3^3 \cdot 4^2)$



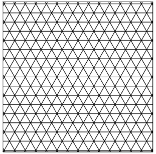
U

$(3^2 \cdot 4 \cdot 3 \cdot 4)$



V

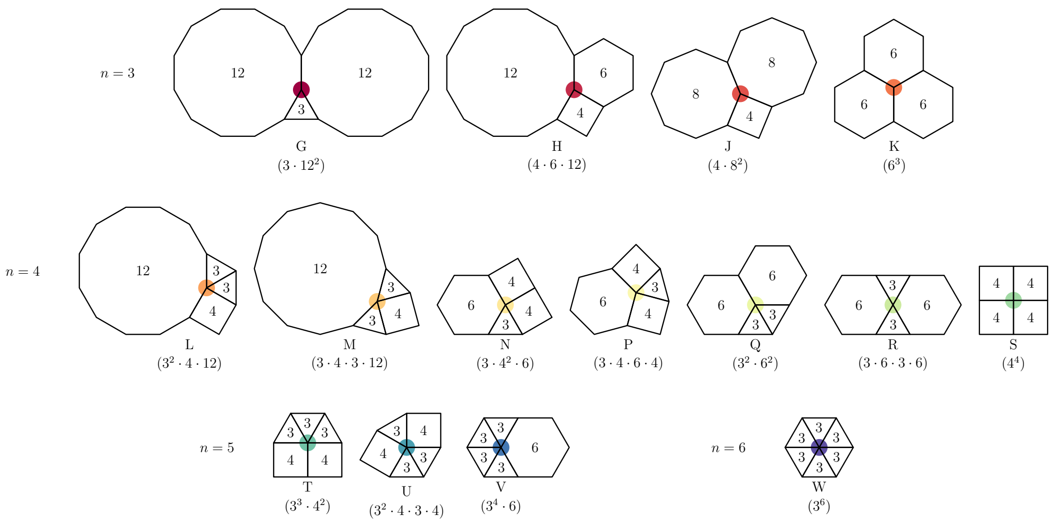
$(3^4 \cdot 6)$



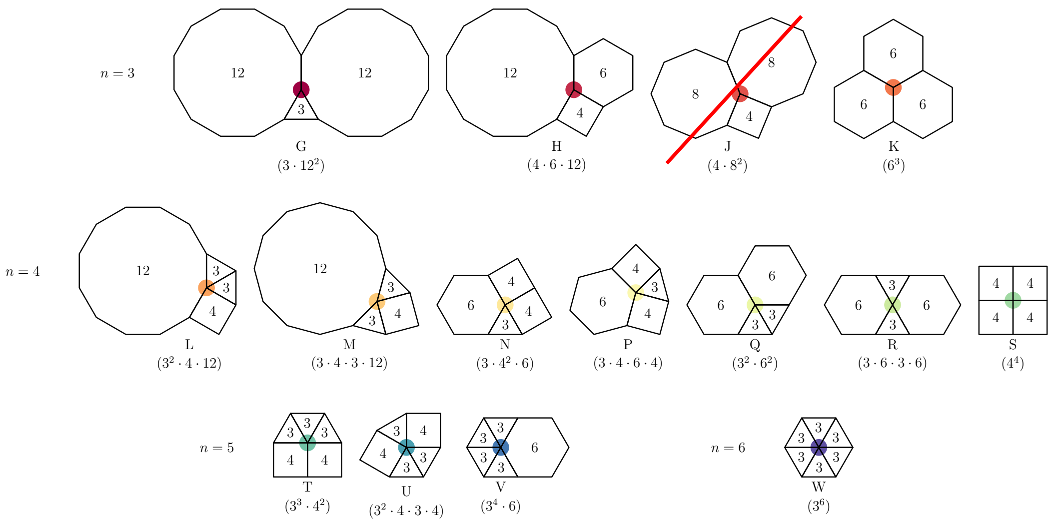
W

$(3^6)$

# Vértices Arquimedianos

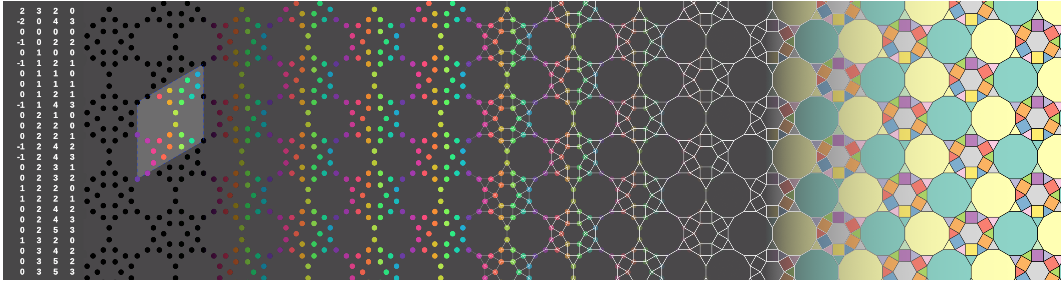


# Vértices Arquimedianos

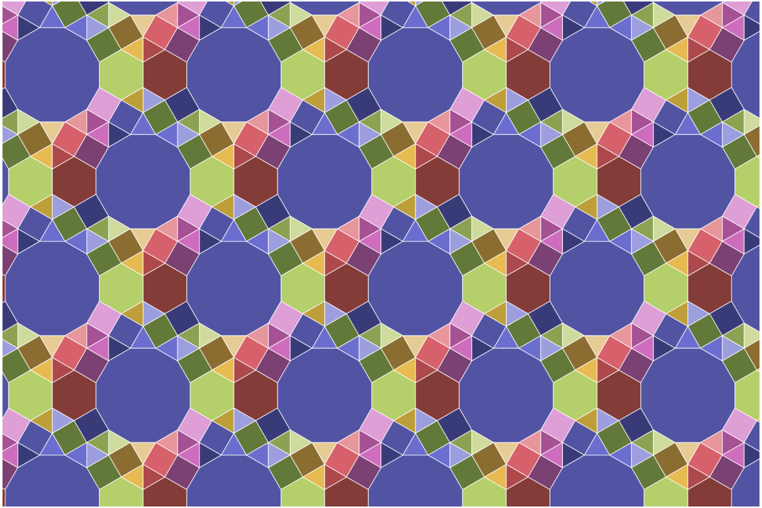


## *Representação inteira*

# Representação inteira: primeira etapa da tese



# Filosofia da representação



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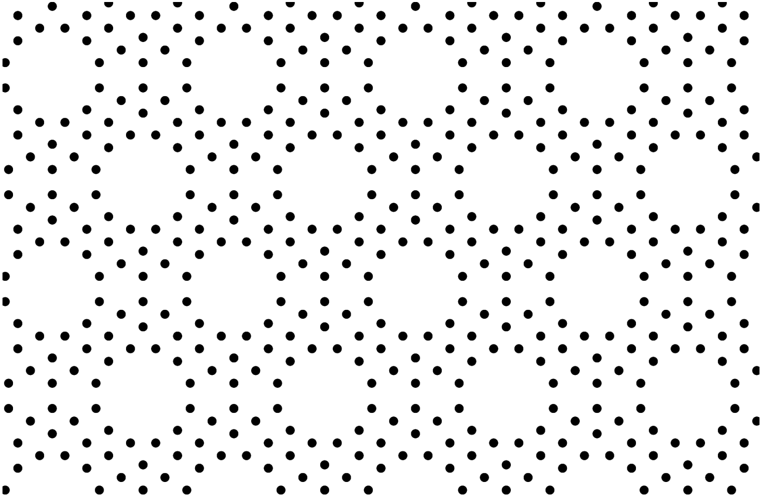
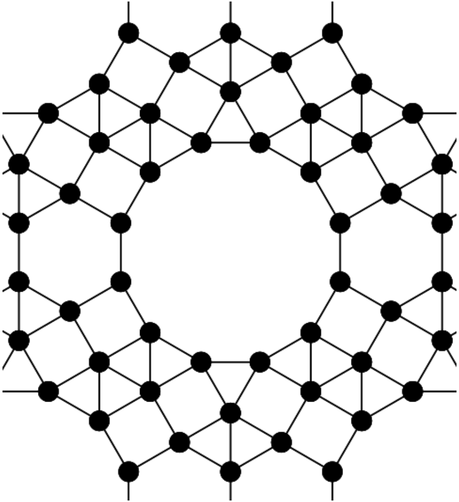


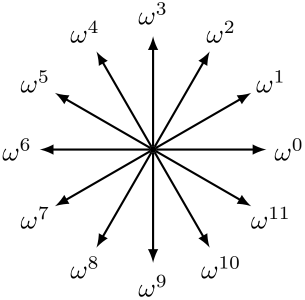
# Retícula (rede): aristas alineadas con algunas direcciones básicas



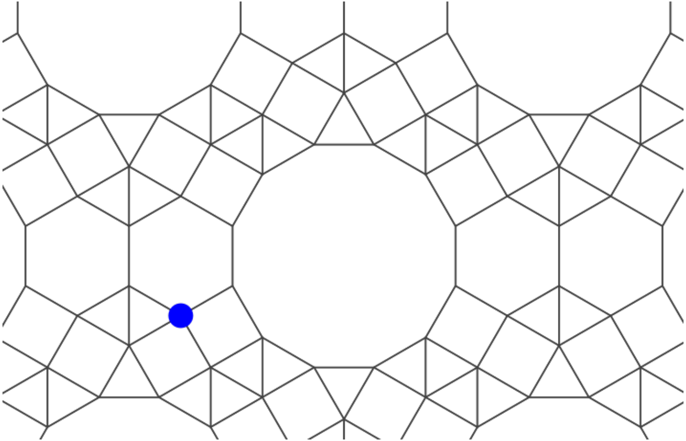
raíces de le unidad

$$\omega^{12} = 1, \quad \omega = e^{\frac{2\pi i}{12}}$$

$$\omega^n = e^{\frac{2\pi i}{12} n}, \quad n \in \{0, 1, \ldots, 11\}$$



# Coordenadas inteiras: vértices en $\mathbb{Z}[\omega]$



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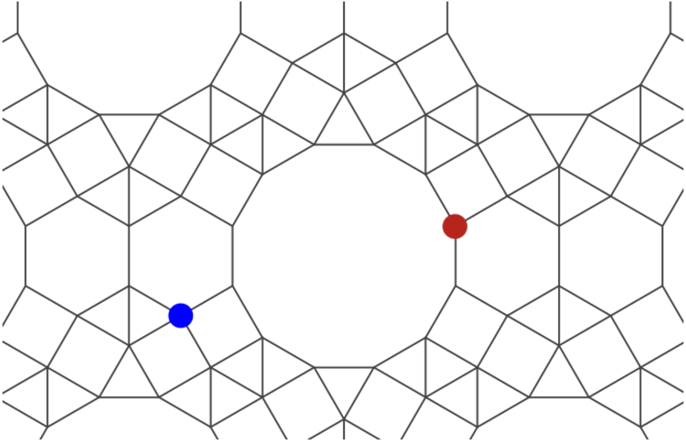

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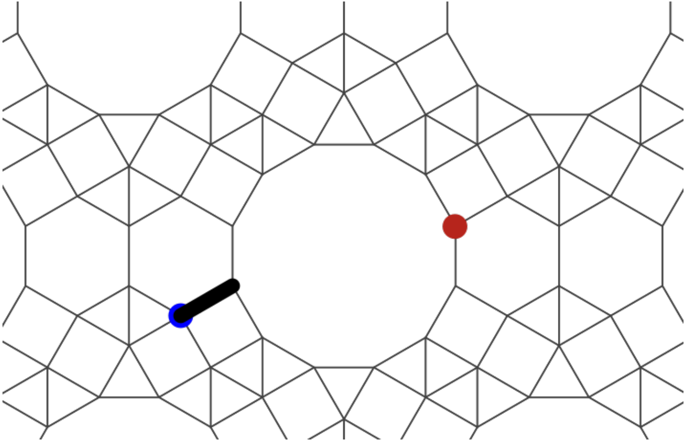

$\omega$

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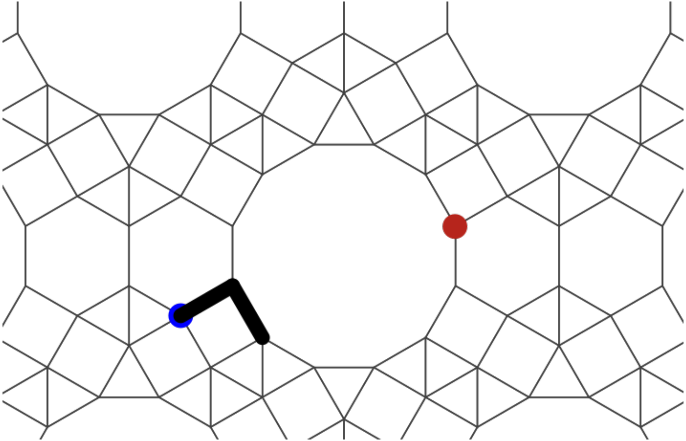

$\omega + \omega^{10}$

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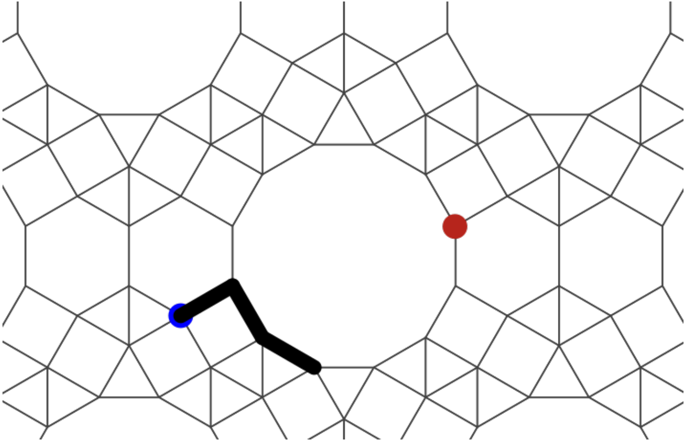
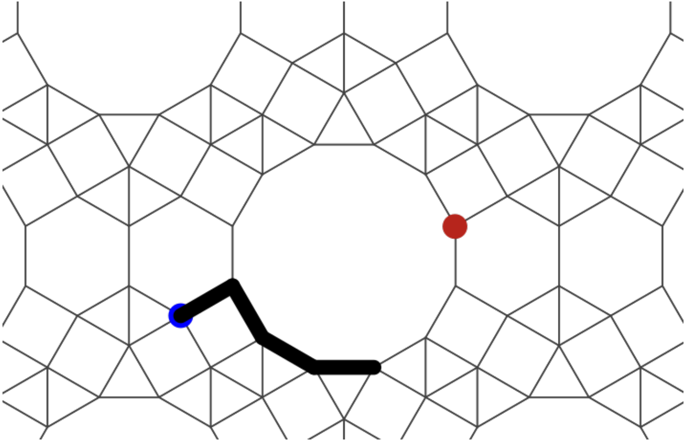


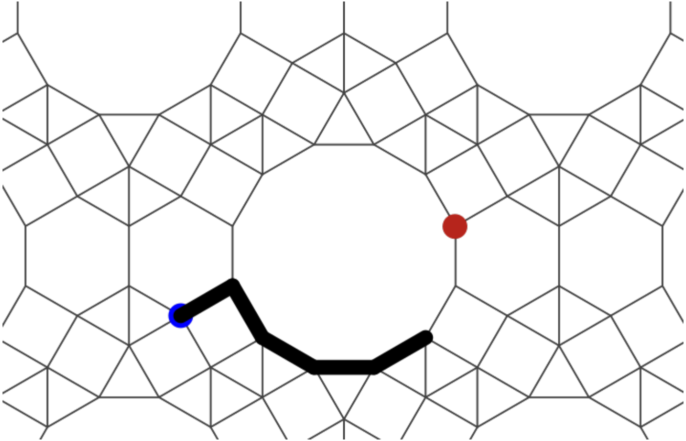
$\omega + \omega^{10} + \omega^{11}$

# Coordenadas inteiras: vértices en $\mathbb{Z}[\omega]$



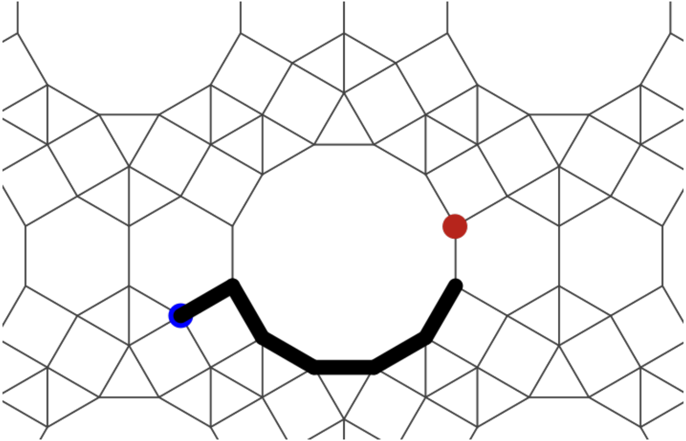
$\omega + \omega^{10} + \omega^{11} + \omega^{0}$

# Coordenadas inteiras: vértices en $\mathbb{Z}[\omega]$



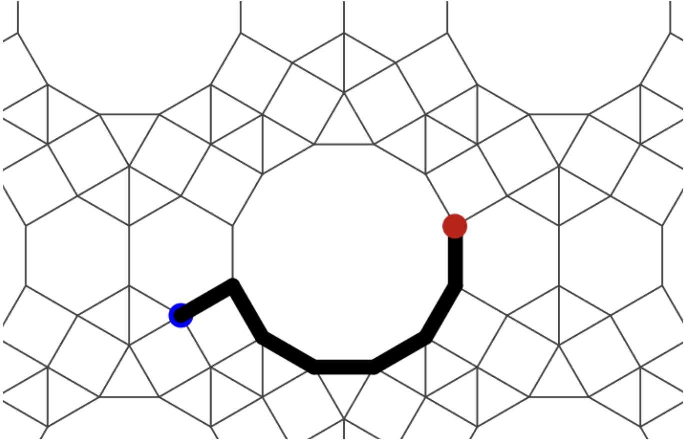
$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega$

# Coordenadas inteiras: vértices en $\mathbb{Z}[\omega]$



$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2}$$

# Coordenadas inteiras: vértices en $\mathbb{Z}[\omega]$



$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2} + \omega^{3}$$

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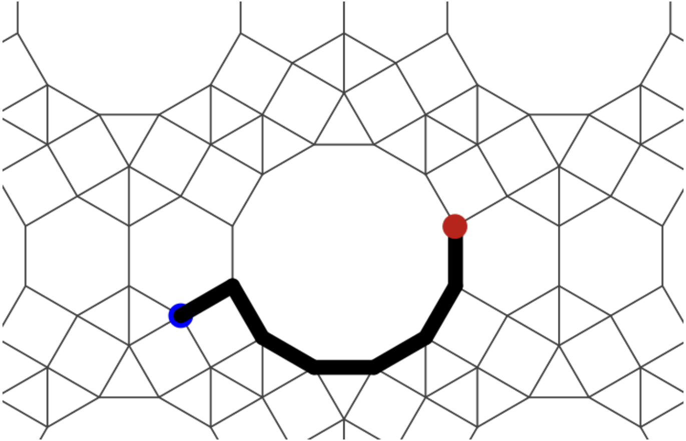
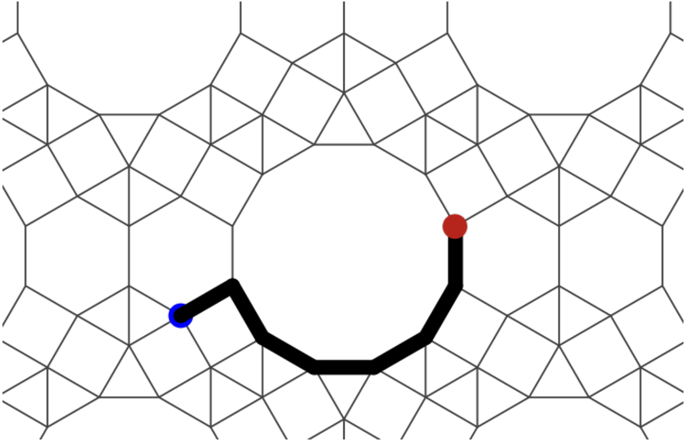


$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2} + \omega^{3} = \omega^{11} + \omega^{10} + \omega^{3} + \omega^{2} + 2\omega + 1$$

# Coordenadas inteiras: vértices en $\mathbb{Z}[\omega]$



$$\omega + \omega^{10} + \omega^{11} + \omega^{0} + \omega + \omega^{2} + \omega^{3} = \omega^{11} + \omega^{10} + \omega^{3} + \omega^{2} + 2\omega + 1 = V - O$$

# Coordenadas inteiras

Vértices e vetores de translação podem ser expressados como polinômios em $\omega$.

Porém, podemos alcançar um vértice por caminhos distintos...

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Polinômios en $\mathbb{Z}[\omega]$ podem se reduzir mod $\omega^4 - \omega^2 + 1$ ($12$º polinômio ciclotómico) para obter uma representação única!

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Logo,

$$\mathbb{Z}[\omega] = \mathbb{Z}1 + \mathbb{Z}\omega + \mathbb{Z}\omega^2 + \mathbb{Z}\omega^3$$

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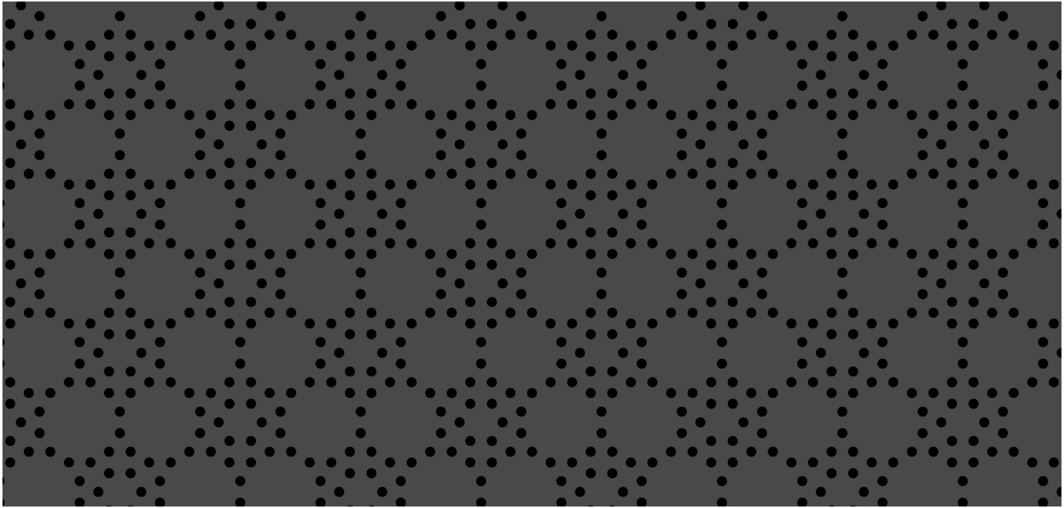
Logo,

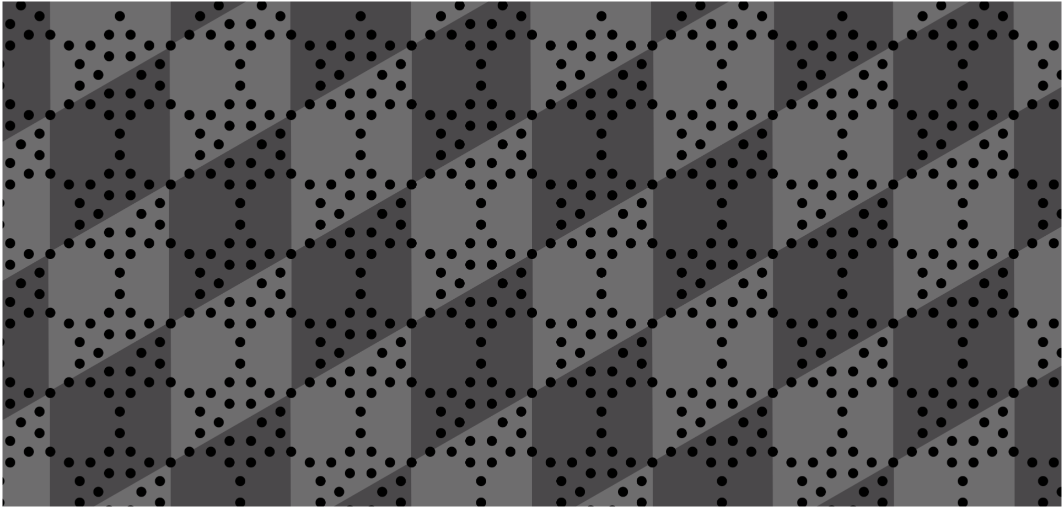
$$\mathbb{Z}[\omega] = \mathbb{Z}1 + \mathbb{Z}\omega + \mathbb{Z}\omega^2 + \mathbb{Z}\omega^3$$

$$
\begin{aligned}
\omega^0 & & &= [\ \ 1,\ \ 0,\ \ 0,\ \ 0] \\
\omega^1 & & &= [\ \ 0,\ \ 1,\ \ 0,\ \ 0] \\
\omega^2 & & &= [\ \ 0,\ \ 0,\ \ 1,\ \ 0] \\
\omega^3 & & &= [\ \ 0,\ \ 0,\ \ 0,\ \ 1] \\
\omega^4 &= -1 + \omega^2 & &= [-1,\ \ 0,\ \ 1,\ \ 0] \\
\omega^5 &= -\omega + \omega^3 & &= [\ \ 0, -1,\ \ 0,\ \ 1] \\
\omega^6 &= -1 & &= [-1,\ \ 0,\ \ 0,\ \ 0] \\
\omega^7 &= -\omega & &= [\ \ 0, -1,\ \ 0,\ \ 0] \\
\omega^8 &= -\omega^2 & &= [\ \ 0,\ \ 0, -1,\ \ 0] \\
\omega^9 &= -\omega^3 & &= [\ \ 0,\ \ 0,\ \ 0, -1] \\
\omega^{10} &= 1 - \omega^2 & &= [\ \ 1,\ \ 0, -1,\ \ 0] \\
\omega^{11} &= \omega - \omega^3 & &= [\ \ 0,\ \ 1,\ \ 0, -1]
\end{aligned}
$$

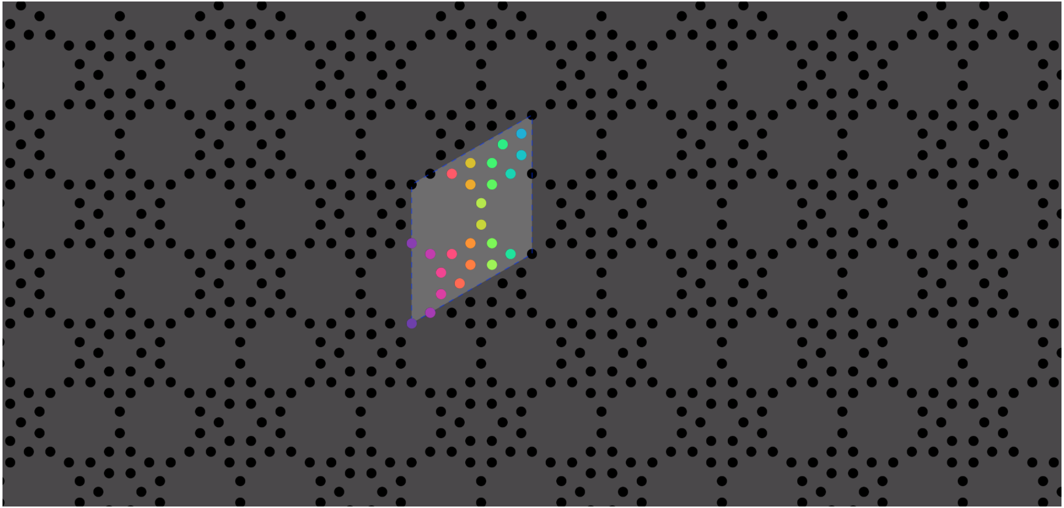
# Representação: periodicidade



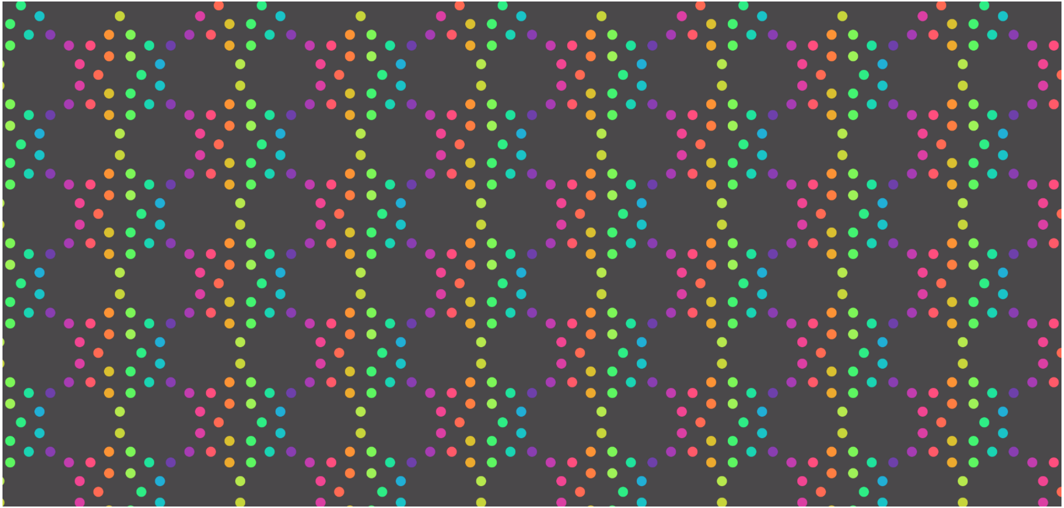
# Representação: equivalência por translação



# Representação: região fundamental

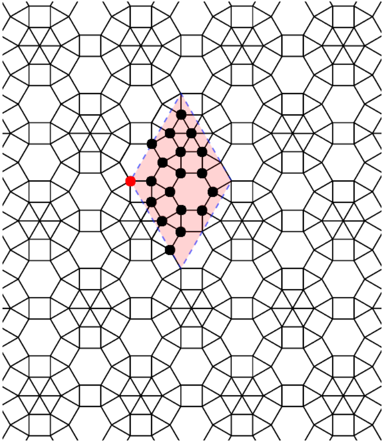


# Representação: sistema interligado de potos (Hilbert & Cohn-Vossen)



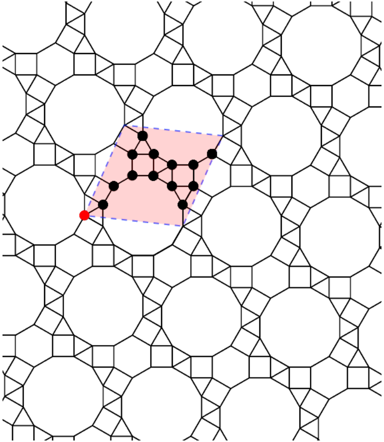
# Representação: exemplos

$$\left[\begin{array}{rrrr} 3 & 1 & -3 & -2 \\ 0 & 1 & 3 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & -2 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 2 & 1 & -1 & -2 \\ 2 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & -1 & -1 \\ 2 & 1 & 1 & -1 \\ 3 & 1 & 0 & -1 \end{array}\right] =$$



Galebach: t4105

# Representação: exemplos



Sa&Sa: HLMP, Galebach: t4030

$$= \left[\begin{array}{cccc} 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & -2 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & -1 \\ 2 & 3 & 0 & -1 \\ 2 & 3 & 1 & -1 \\ 2 & 3 & 1 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right]$$

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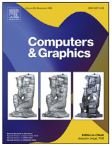



Technical Section

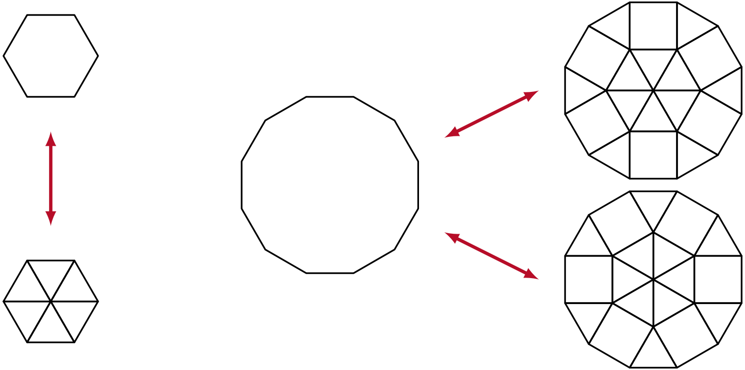
# An integer representation for periodic tilings of the plane by regular polygons

José Ezequiel Soto Sánchez[a], Tim Weyrich[b], Asla Medeiros e Sá[c], Luiz Henrique de Figueiredo[a,*]

[a] *IMPA, Rio de Janeiro, Brazil*
[b] *University College London, United Kingdom*
[c] *FGV EMAp, Rio de Janeiro, Brazil*

# Quadrados e triângulos

# *Densidade*: contém todas as de polígonos regulares



# Além disso...

- Quasicristais com simetria dodecagonal.

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- Ladrilhamentos aperiódicos por substituição.

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- Quasicristais com simetria dodecagonal.
- Ladrilhamentos aperiódicos por substituição.
- Empacotamentos *justos* de círculos.
- Modelagem de materia *mole*.

# Por exemplo (Oxborrow & Henley, 1993)

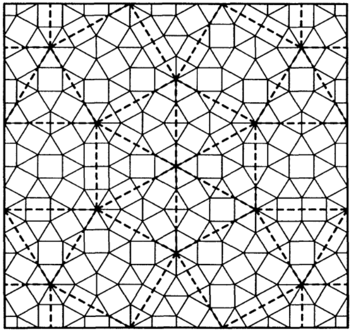
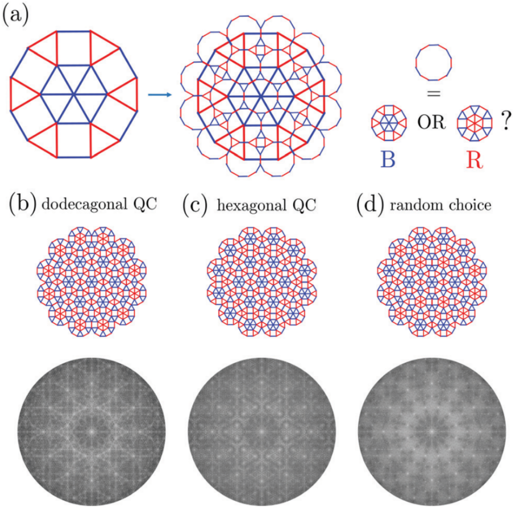
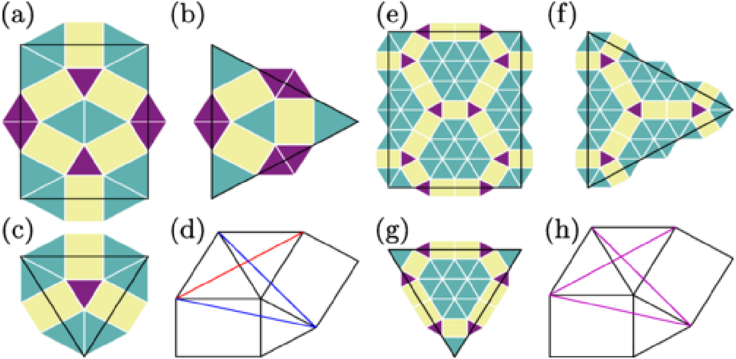


FIG. 5. Random-Stampfli inflation: a big parent square-triangle tiling (thick, dashed lines) and an offspring square-triangle tiling (thin, solid lines); both are periodic over the same, square unit cell (thick, solid lines).

# Por exemplo (Impéror-Clerc et al., 2021)

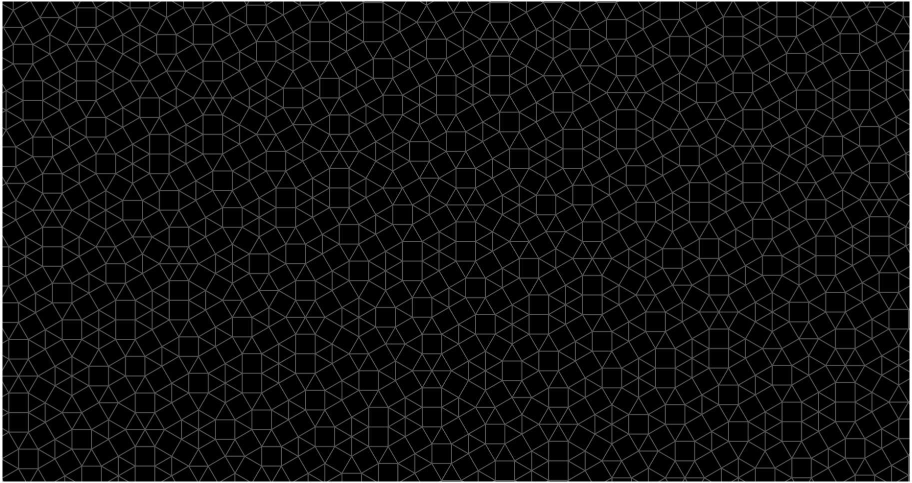


# Por exemplo (Archer et al., 2022)

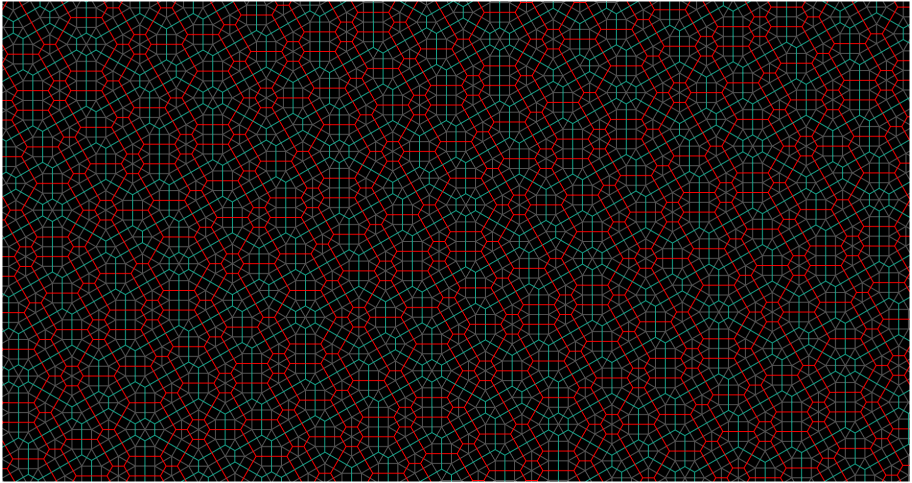


# *Origami e duais restritos*

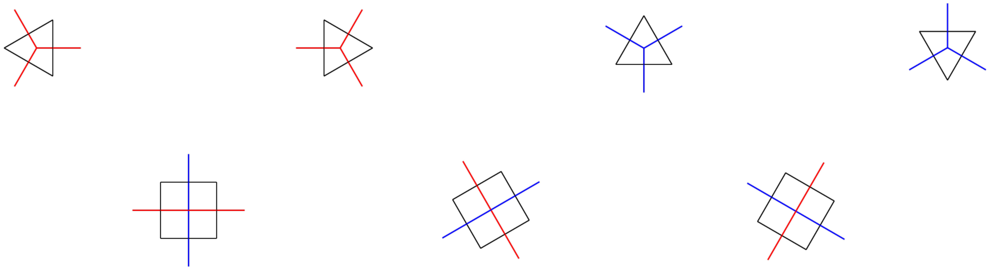
# Duais coloridos



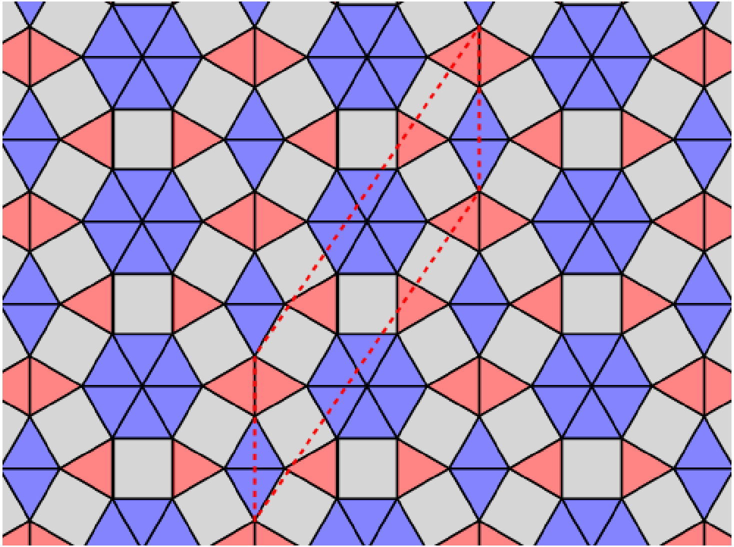
# Duais coloridos



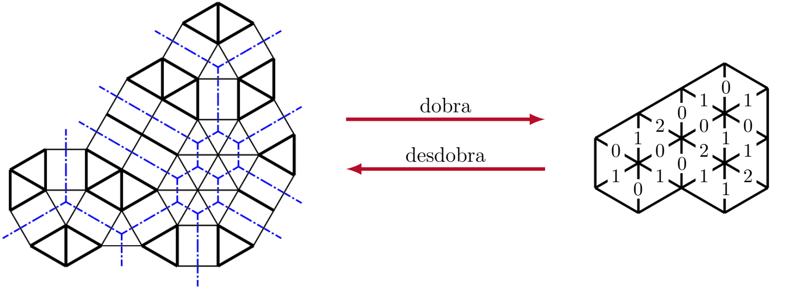
# Duais coloridos



# Duais e triângulos coloridos



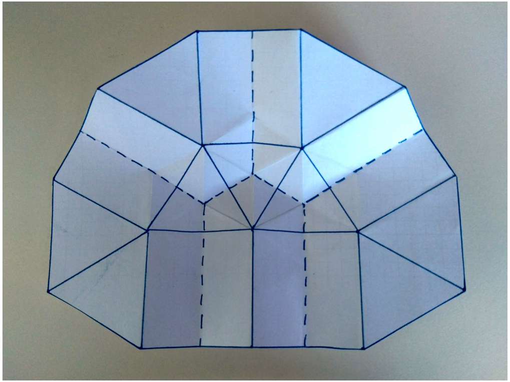
# Operações de origami



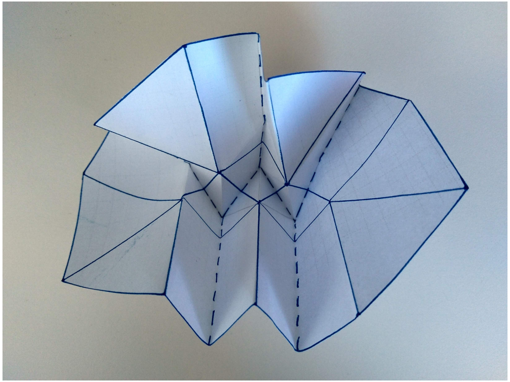

**Dobradura** no dual

Triângulos em $\Lambda(\omega, \omega^3)$
+ etiquetas nas arestas

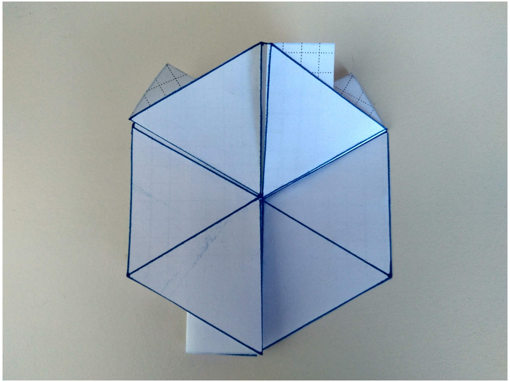
# Operações de origami



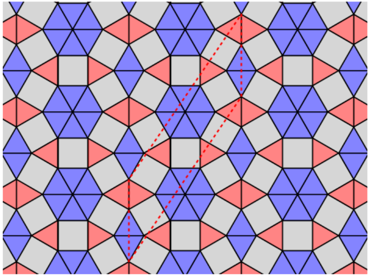
# Operações de origami

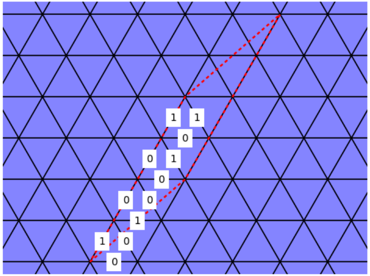


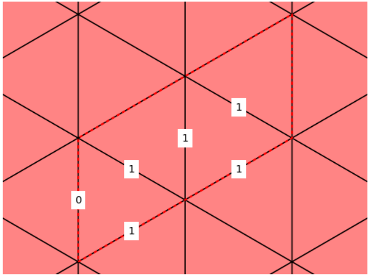
# Operações de origami



# Equivalência dupla

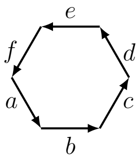




*Estrutura algebrica e estratégias generativas*

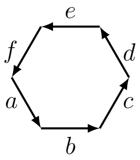
# Etiquetas: restrições geométricas




$$a\omega^{10} + b\omega^{0} + c\omega^{2} + d\omega^{4} + e\omega^{6} + f\omega^{8} = 0, \text{ isto é:}$$

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = 0.$$

# Etiquetas: restrições geomêtricas

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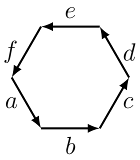
$\xi \in \mathbb{Z}^{3h}$ é o vetor de etiquetas do grafo hexagonal (triangular) con $h$ caras (vértices).

# Etiquetas: restrições geométricas

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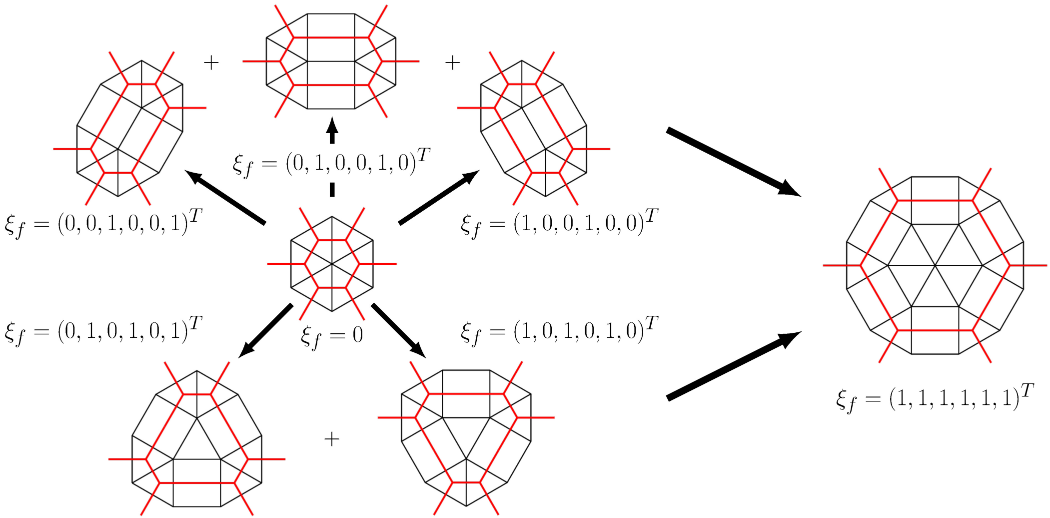
$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = 0.$$

$\xi \in \mathbb{Z}^{3h}$ é o vetor de etiquetas do grafo hexagonal (triangular) con $h$ caras (vértices).

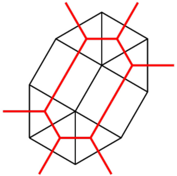
$\xi$ é um etiquetamento válido se e só se satisfaz as restrições geométricas em todos os vértices, um sistema linear esparso de $2h \times 3h$:

$$G\xi = 0, \quad \xi \geq 0$$

# Estructura algebrica e origami

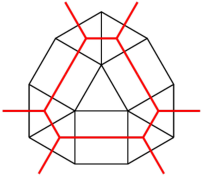


# Estructura algebrica e origami



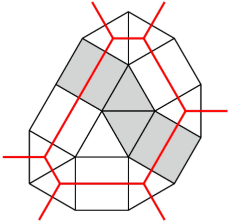
$\xi_f = (0,0,1,0,0,1)^T$

$+$



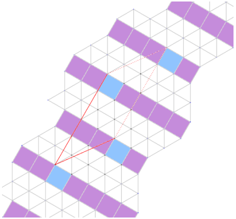
$\xi_f = (0,1,0,1,0,1)^T$

$=$

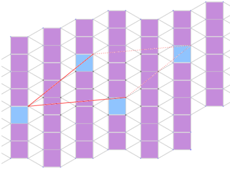


$\xi_f = (0,1,1,1,0,2)^T$

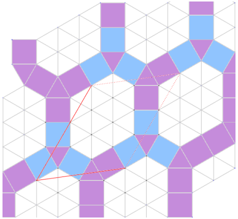
# Estructura algebrica e origami



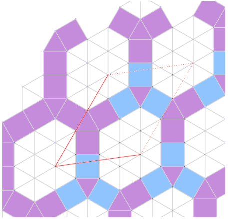
$X^4$



$X^{10}$



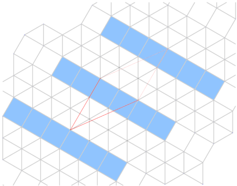
$X^1$



$X^5$

# Estructura algebrica e origami

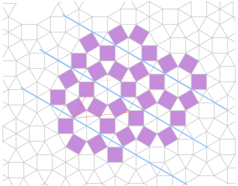
$2\Bigg($



$\Bigg)$

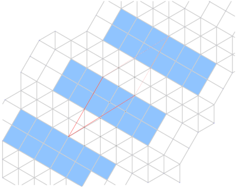
$(0,0,0,0,0,1,0,0,1,0,0,0)^T$

$+$



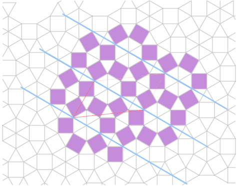
$(1,0,1,1,1,0,0,0,0,0,1,0)^T$

$=$



$(0,0,0,0,0,2,0,0,2,0,0,0)^T$

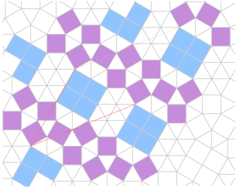
$+$



$(1,0,1,1,1,0,0,0,0,0,1,0)^T$

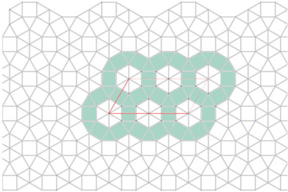
$=$

$\mathcal{T}$



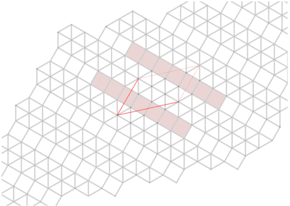
$(1,0,1,1,1,2,0,0,2,0,1,0)^T$

# Estructura algebrica e origami



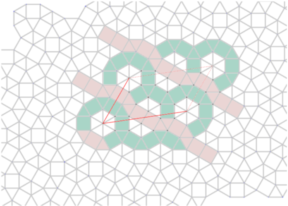
$X^1$

$+$



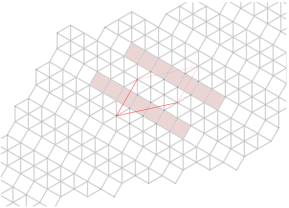
$X^3$

$=$



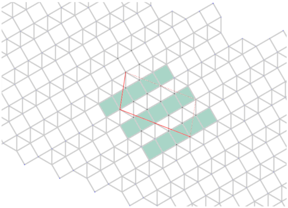
$X^1 + X^3$

# Estructura algebrica e origami



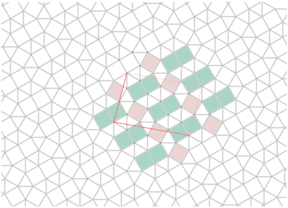
$X^3$

$+$

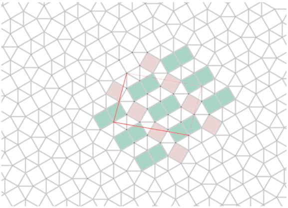


$X^6$

$=$

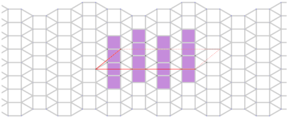


$X^3 + X^6$



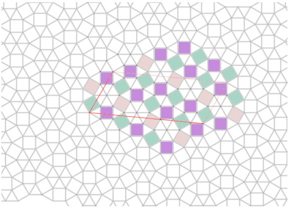
$X^3 + X^6$

$+$



$X^7$

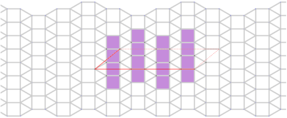
$=$



$X^3 + X^6 + X^7$

# Estructura algebrica e origami

$2 \Big($



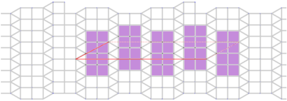
$\Big)$

$X^7$

$+$

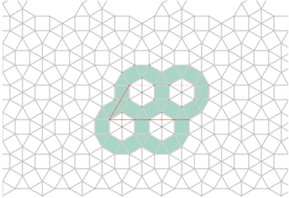
$X^9$

$=$



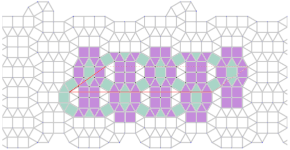
$2X^7$

$+$



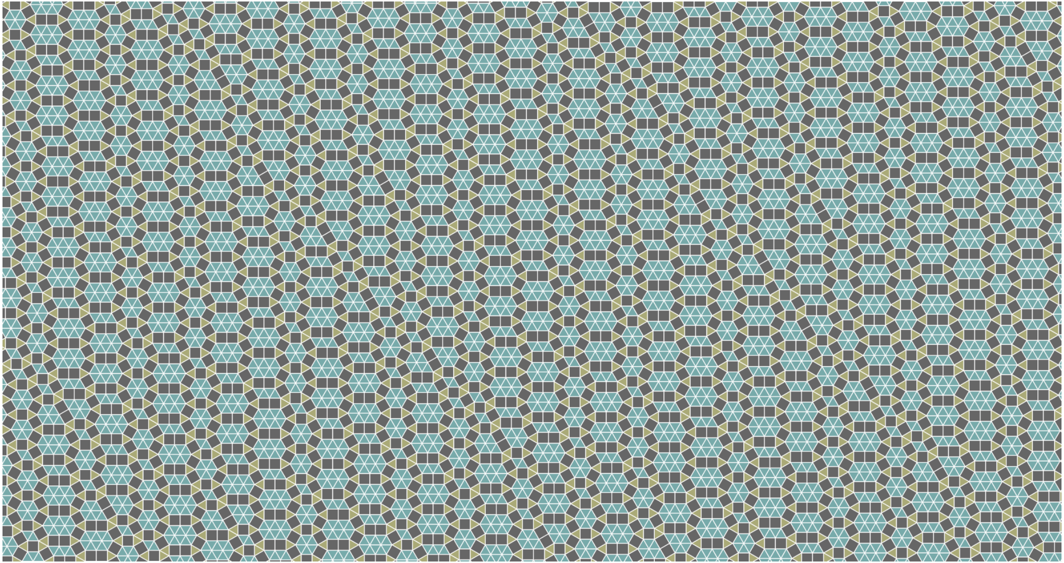
$X^9$

$=$

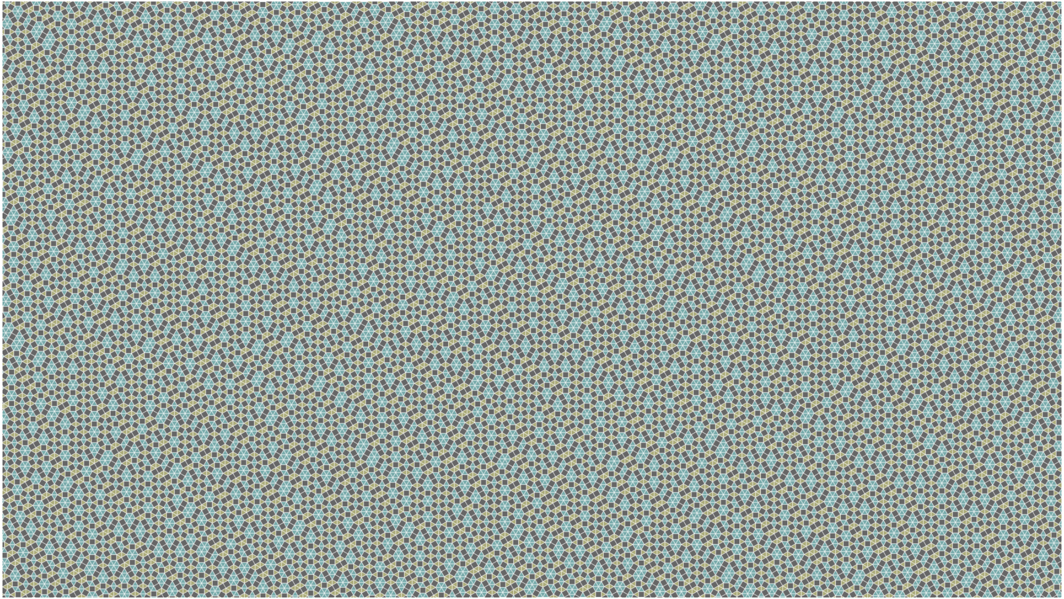


$2X^7 + X^9$

# Estratégias generativas: inflação alternada



# Estratégias generativas: inflação alternada



Obrigado LHF

Feliz aniversário!