Results and their proofs

Short Course - August 16^{th}, 18^{th} and 19^{th}

Hausdorff dimension

Proposition 1: Let $0 < \alpha < \alpha'$. If $\mu^{(\alpha)}(A) < \infty$ then $\mu^{(\alpha')}(A) = 0$ and also, if $\mu^{(\alpha')}(A) > 0$ then $\mu^{(\alpha)}(A) = \infty$.

Proof: It suffices to prove the first statement, as the second one is its contrapositive. If $\mu^{(\alpha)}(A) < \infty$, for any $\delta > 0$ there exists $\{B_j\}_{j\in\mathbb{N}}$ with $A \subset \cup_{j=1}^{\infty} B_j$, $\text{diam}(B_j) \leqslant \delta$, and

$$\sum_{j=1}^{\infty} (\text{diam}(B_j)^{\alpha} \leqslant \mu^{(\alpha)}(A) + 1.$$

But for $\alpha' > \alpha$,

$$\sum_{j=1}^{\infty} (\text{diam}(B_j))^{\alpha'} \leqslant \delta^{\alpha'-\alpha} \sum_{j=1}^{\infty} (\text{diam}(B_j))^{\alpha} \leqslant \delta^{\alpha'-\alpha} [\mu^{(\alpha)}(A) + 1],$$

so $\mu_{\delta}^{(\alpha')} \leqslant \delta^{\alpha'-\alpha}[\mu^{(\alpha)}(A) + 1] \xrightarrow{\delta\to 0} 0$ and $\mu^{(\alpha')}(A) = 0$. ■

Theorem 1: Let (X,d) be a metric space. For each $\alpha > 0$ and $\delta > 0$, $\mu_{\delta}^{(\alpha)} : 2^X \to [0,\infty]$ is an outer measure.

Proof: Fix $\delta > 0$. Clearly $\mu_{\delta}^{(\alpha)}(\emptyset) = 0$ and $\mu_{\delta}^{(\alpha)}(A) \leqslant \mu_{\delta}^{(\alpha)}(B)$ whenever $A \subset B$. If $\varepsilon > 0$, $\{A_j\}_{j=1}^{\infty}$ is a sequence in 2^X and for each $j \in \mathbb{N}$ there exists a sequence $\{B_i^j\}_{i=1}^{\infty}$ with $A_j \subset \bigcup_{i=1}^{\infty} B_i^j$, $\text{diam}(B_i^j) < \delta$, for all $i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} (\text{diam}(B_i^j))^{\alpha} \leqslant \mu_{\delta}^{\alpha}(A_j) + \varepsilon 2^{-j}$, then $\bigcup_{j=1}^{\infty} A_j \subset \bigcup_{j,i=1}^{\infty} B_i^j$ and

$$\mu_{\delta}^{(\alpha)}\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \sum_{i,j=1}^{\infty} (\text{diam}(B_i^j))^{\alpha} \leqslant \sum_{j=1}^{\infty} \mu_{\delta}^{(\alpha)}(A_j) + \varepsilon.$$

It follows that $\mu_{\delta}^{(\alpha)}\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \sum_{j=1}^{\infty} \mu_{\delta}^{(\alpha)}(A_j)$. The result now follows. ■

Proposition 2: Let (X,d), (Y,ρ) be two metric spaces and $A \subset X$. If $f : X \to Y$ is a Lipschitz continuous function then $\dim_H(f(A)) \leqslant \dim_H(A)$.

Proof: If $\mu^{(\alpha)}(A) = \infty$ we have nothing to do. Assume that $\mu^{(\alpha)}(A) < \infty$, given $\varepsilon > 0$, there exists $\delta_{\varepsilon} > 0$ such that, for all $0 < \delta < \delta_{\varepsilon}$, there is a cover of A by sets B_j such that $\operatorname{diam}(B_j) \leqslant C^{-1}\delta$ and

$$\sum_{j=1}^{\infty} (\operatorname{diam}(B_j))^{\alpha} \leqslant \mu^{(\alpha)}(A) + \varepsilon.$$

The sets $B_{j'} = f(B_j)$ cover $f(A)$ and $\operatorname{diam}(B_{j'}) \leqslant C \operatorname{diam}(B_j) \leqslant \delta$, and

$$\mu_{\delta}^{(\alpha)}(f(A)) \leqslant C^{\alpha}\mu^{(\alpha)}(A) + C^{\alpha}\varepsilon.$$

Making $\delta \to 0$ and then letting $\varepsilon \to 0$ we have that

$$\mu_{\delta}^{(\alpha)}(f(A)) \leqslant C^{\alpha}\mu^{(\alpha)}(A),$$

and so, if $\mu^{(\alpha)}(A) = 0$ then $\mu_{\delta}^{(\alpha)}(f(A)) = 0$ which proves that $\dim_H(f(A)) \leqslant \dim_H(A)$. ■

Corollary 1: Let $f : X \to Y$ be a Lipschitz continuous function, $A \subset X$ and $G(f,A) = \{(x, f(x)) : x \in A\}$ the graph of f restricted to A. Then $\dim_H(G(f,A)) = \dim_H(A)$.

Proof: This is a simple consequence of the last proposition, since the maps

$$A \ni x \mapsto (x, f(x)) \in G(f,A),$$
$$G(f,A) \ni (x, f(x)) \mapsto x \in A$$

are both Lipschitz continuous functions. ■

Proposition 3: Let $\{A_j\}_{j\in\mathbb{N}}$ be a sequence of sets in X and $A = \bigcup_{j=1}^{\infty} A_j$. Then

$$\dim_H(A) = \sup_{j\in\mathbb{N}} \dim_H(A_j).$$

Proof: The inequality $\dim_H(A) \geqslant \sup_{j\in\mathbb{N}} \dim_H(A_j)$ follows from monotonicity. Now let $\alpha > \sup_{j\in\mathbb{N}} \dim_H(A_j)$. Then $\mu^{(\alpha)}(A_j) = 0$ for all $j \in \mathbb{N}$ and so $\mu^{(\alpha)}(A) = 0$, which implies that $\dim_H(A) \leqslant \sup_{j\in\mathbb{N}} \dim_H(A_j)$, and the result is proved. ■

Property: Let $\{S(n) : n \in \mathbb{N}\}$ be a discrete semigroup in a Banach space with a finite set $\mathscr{E} = \{e_1, \ldots, e_n\}$ of equilibrium points and a gradient-like global attractor $\mathscr{A}$. Assume that $S(1)$ is a Lipschitz continuous map and that each local unstable set $W^u_{loc}(e_i)$ is a graph of a Lipschitz function with domain $Q_i X$, where Q_i is a finite rank projection. Then

$$\dim_H(\mathscr{A}) = \max_{i=1,\ldots,n} \dim_H(Q_i X).$$

Proof: From the Corollary 3 and Proposition 2 we know that

$$\dim_H(W^u_{loc}(e_i)) = \dim_H(Q_iX) < \infty, \text{ for each } i = 1,\dots,n,$$
$$\dim_H(S(m)W^u_{loc}(e_i)) \leqslant \dim_H(W^u_{loc}(e_i)), \text{ for each } i = 1,\dots,n \text{ and all } m \in \mathbb{N}.$$

We easily see that $W^u(e_i) = \cup_{m\geqslant 0} S(m)W^u_{loc}(e_i)$ and using Proposition 3 we have that

$$\begin{aligned}\dim_H(Q_iX) &= \dim_H(W^u_{loc}(e_i)) \leqslant \dim_H(W^u(e_i)) = \\ &= \dim_H\left(\bigcup_{n\geqslant 0} S(m)W^u_{loc}(e_i)\right) \leqslant \sup_{n\in\mathbb{N}} \dim_H(S(m)W^u_{loc}(e_i)) \leqslant \\ &\leqslant \dim_H(W^u_{loc}(e_i)) = \dim_H(Q_iX),\end{aligned}$$

and hence $\dim(W^u(e_i)) = \dim(Q_iX)$ for all $i = 1,\dots,n$. So, since $\mathscr{A} = \cup_{i=1}^n W^u(e_i)$, we have that

$$\dim_H(\mathscr{A}) = \max_{i=1,\dots,n} \dim_H(Q_iX).$$

■

Fractal dimension

Remark: It is easily seen that $\dim_H(K) \leqslant c(K)$.

Proof: If $c(K) = \infty$ we have nothing to prove. Assume that $c(K) < \infty$. We know that given $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that

$$N(r,K) \leqslant \left(\frac{1}{r}\right)^{c(K)+\varepsilon}, \text{ for } 0 < r < \delta_\varepsilon.$$

But for any $\alpha > 0$ we have that

$$\mu^{(\alpha)}_{2r}(K) \leqslant (2r)^\alpha N(r,K) \leqslant 2^\alpha r^{\alpha - c(K) - \varepsilon}, \text{ for } 0 < r < \delta_\varepsilon.$$

So, for every $\eta > 0$:

$$\mu^{(c(K)+\eta)}_{2r}(K) \leqslant 2^\alpha r^{\eta-\varepsilon}.$$

Then, for $\eta = 2\varepsilon$:

$$\mu^{(c(K)+2\varepsilon)}_{2r}(K) \leqslant 2^\alpha r^{\varepsilon},$$

and taking $r \to 0$ we have

$$\mu^{(c(K)+2\varepsilon)}(K) = 0, \text{ for every } \varepsilon > 0,$$

which implies that $\dim_H(K) \leqslant c(K)$. ■

Proposition 4: Let X be a normed vector space and K_1, K_2 compact subsets of X. Then $c(K_1+K_2) \leqslant c(K_1)+c(K_2)$.

Proof: Note that, if $N_i = N(r,K_i)$ there are $x_1^i, \cdots, x_N^i$ in K such that $K_i \subset \cup_{j=1}^{N_i} B_r(x_j^i)$ and consequently $K_1+K_2 \subset \cup_{i=1}^{N_2} \cup_{j=1}^{N_1} \left(B_r(x_j^1)+B_r(x_i^2)\right)$. Since $\operatorname{diam}(B_r(x_j^1)+B_r(x_i^2)) \leqslant 4r$ we have that $N(4r,K_1+K_2) \leqslant N(r,K_1)N(r,K_2)$ and consequently

$$c(K_1+K_2) = \limsup_{r\to 0} \frac{\log N(4r,K_1+K_2)}{\log(1/4r)} \leqslant \limsup_{r\to 0} \frac{\log N(r,K_1)N(r,K_2)}{\log(1/4r)}$$
$$\leqslant \limsup_{r\to 0} \frac{\log N(r,K_1)}{\log(1/4r)} + \limsup_{r\to 0} \frac{\log N(r,K_2)}{\log(1/4r)} = c(K_1)+c(K_2).$$

■

Corollary 2: Let X be a normed vector space and K a compact subset of X. Then $c(K-K) \leqslant 2c(K)$.

Proof: Is an straightforward consequence of the previous proposition. ■

Proposition 5: Let K, Y be two metric spaces with K compact and $f: K \to Y$ a Lipschitz continuous function. Then $c(f(K)) \leqslant c(K)$.

Proof: The result is immediate if $c(K) = \infty$. If $c(K) < \infty$ we have that, for $C > 0$ the Lipschitz constant of f:

$$f(K) \subset \bigcup_{j=1}^{N(\varepsilon,K)} f(B_\varepsilon^K(x_j)) \subset \bigcup_{j=1}^{N(\varepsilon,K)} B_{C\varepsilon}^Y(Tx_j).$$

Hence, $N(C\varepsilon, T(K)) \leqslant N(\varepsilon, K)$ and

$$c(f(K)) = \lim_{\varepsilon\to 0} \frac{N(C\varepsilon, f(K))}{\log\left(\frac{1}{C\varepsilon}\right)} \leqslant \lim_{\varepsilon\to 0} \frac{N(\varepsilon,K)}{\log\left(\frac{1}{C\varepsilon}\right)} = c(K).$$

■

Projection of compact sets with finite fractal dimension

In this parts of our notes, we will develop some theory in order to prove the result about the projection of compact sets with finite fractal dimension.

Let X be a Banach space and Y be a finite dimensional subspace of X. Denote by $\mathscr{P}(X,Y)$ the subset of $\mathscr{L}(X)$ consisting of the projections with range Y.

Let X be a Banach space, $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of compact subsets of X and $K=\cup_{n\in\mathbb{N}}K_n$.

Lemma 0.0.1 *Given $r>0$, $n\in\mathbb{N}$, if*

$$A_{n,r}=\{v-w\in X: v,w\in K_n \text{ and } \|v-w\|_X\geqslant r\},$$

then $A_{n,r}$ is a compact subset of X.

Proof: We first show that $A_{n,r}$ is closed. In fact, $y_k=v_k-w_k\overset{k\to\infty}{\longrightarrow}y$, with $v_k,w_k\in K_n$, $\|v_k-w_k\|_X$. Using the compactness of K_n, we may assume, passing to subsequences, that $v_k\overset{k\to\infty}{\longrightarrow}v\in K_n$ and $w_k\overset{k\to\infty}{\longrightarrow}w\in K_n$. Hence $y=v-w$ and $\|v-w\|_X\geqslant r$ and $y\in A_{n,r}$. That proves that $A_{n,r}$ is closed.

Now note that, $A_{n,r}\subset K_n-K_n$. Since $X\times X\ni(x,y)\mapsto x-y\in X$ is continuous, it follows that K_n-K_n is compact and consequently $A_{n,r}$ is a compact in X. ∎

Lemma 0.0.2 *Define*

$$\mathscr{P}_{n,r}=\{P\in\mathscr{P}(X,Y):\ \operatorname{diam}(P^{-1}(y)\cap K_n)<r,\ \forall\, y\in Y\} \tag{1}$$

If $P\in\mathscr{P}(X,Y)$, then $P\in\mathscr{P}_{n,r}$ if and only if $P^{-1}(0)\cap A_{n,r}=\emptyset$. This is saying that the projections in $\mathscr{P}_{n,r}$ are exactly those that are injective in $A_{n,r}$.

Proof: Assume that there is a $y\in P^{-1}(0)\cap A_{n,r}$, then $Py=0$ and there are $v,w\in K_n$ such that $y=v-w$, $\|v-w\|_X\geqslant r$. If $z=Pv=Pw$ then $\operatorname{diam}\big(P^{-1}(z)\cap K_n\big)\geqslant r$ and $P\notin\mathscr{P}_{n,r}$.

On the other hand, if $P^{-1}(0)\cap A_{n,r}=\emptyset$, for all $y\in Y$ and $v,w\in P^{-1}(y)\cap K_n$, we have that $v-w\in P^{-1}(0)$ ($P(v-w)=y-y=0$) and therefore we must have that $\|v-w\|_X<r$. Since $P^{-1}(y)\cap K_n$ is compact, it follows that $\operatorname{diam}(P^{-1}(y)\cap K_n)<r$. This completes the proof. ∎

Lemma 0.0.3 *If $\mathscr{P}_{n,r}$ defined by (1), then $\mathscr{P}_{n,r}$ is open in $\mathscr{P}(X,Y)$ with the uniform operator topology.*

Proof: The proof of this result is based on the characterization of $\mathscr{P}_{n,r}$ given in Lemma 0.0.2. Given a projection $P\in\mathscr{P}_{n,r}$, note that

$$\varepsilon=\inf_{x\in P^{-1}(0)}\operatorname{dist}(x,A_{n,r})=\inf_{x\in X}((I-P)x,A_{n,r})>0.$$

Choose $s > 0$ such that $B_{s-\varepsilon}(0) \supset A_{n,r}$ and $\bar{P} \in \mathscr{P}(X,Y)$ such that $\|P - \bar{P}\|_{\mathscr{L}(X)} < \frac{\varepsilon}{s}$. Then

$$\begin{aligned}\inf_{x\in\bar{P}^{-1}(0)} \operatorname{dist}(x, A_{n,r}) &= \inf_{x\in B_s(0)} \operatorname{dist}((I-\bar{P})x, A_{n,r}) \\ &\geqslant \inf_{x\in B_s(0)} \operatorname{dist}((I-P)x, A_{n,r}) - \sup_{x\in B_s(0)} \|\bar{P}x - Px\|_X \\ &= \varepsilon - s\|\bar{P} - P\|_{\mathscr{L}(X)} > 0.\end{aligned}$$

It follows that $\bar{P}^{-1}(0) \cap A_{n,r} = \emptyset$ and $\bar{P} \in \mathscr{P}_{n,r}$, proving that $\mathscr{P}_{n,r}$ is open. ∎

Lemma 0.0.4 *There is a sequence* $\{\phi_i\}_{i\in\mathbb{N}}$ *in* X' *such that, if* $x \in K$ *and* $\phi_i(x) = 0$ *for all* $i \in \mathbb{N}$*, then* $x = 0$.

Proof: Now consider the closure W of the subspace of X generated by K. Since K is countable union of compact sets, it is separable and consequently W is a separable Banach space. From the fact that $B_1^{W'}(0)$ is compact and metrizable in the weak star topology $\sigma(W', W)$, it follows that there is a dense sequence $\{\phi_i\}_{i\in\mathbb{N}}$ in $(B_1^{W'}(0), \sigma(W', W))$. Now if $x \in W$ and $\phi_i(x) = 0$ for all $n \in N$, it follows that $\phi(x) = 0$ for all $\phi \in B_1^{W'}(0)$ and we have that $x = 0$. The desired sequence is now obtained extending ϕ_i to X through Hahn-Banach theorem, for each $n \in \mathbb{N}$. ∎

Lemma 0.0.5 $\displaystyle\bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \mathscr{P}_{n,\frac{1}{m}} = \{P \in \mathscr{P}(X,Y) : P \text{ is injective in } K\}$

Proof: Let $P \in \mathscr{P}(X,Y)$. Since $K = \bigcup_{n\in\mathbb{N}} K_n$ we have that, P is injective in K if and only if P is injective in K_n for all $n \in \mathbb{N}$ if and only if $P^{-1}(y) \cap K_n$ is either the empty set or a unitary set for all $y \in Y$ and for all $n \in \mathbb{N}$ if and only if $\operatorname{diam}(P^{-1}(y) \cap K_n) < \frac{1}{m}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ if and only if $P \in \bigcap_{n\in N} \bigcap_{m\in\mathbb{N}} \mathscr{P}_{n,\frac{1}{m}}$. ∎

Recall the definition of quotient space.

Definition 0.0.6 *Let* X *be a Banach space over a field* $\mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*) and* Z *be a closed subspace of* X*. For each* $x \in X$*, let* $[x] := \{z \in X : z - x \in Z\}$,

$$X/Z := \{[x] : x \in X\}$$

and define

- $\lambda[x] := [\lambda x]$, $\lambda \in \mathbb{K}$, $x \in X$,
- $[x] + [y] := [x + y]$, $x, y \in X$,
- $\|[x]\|_{X/Z} := \inf_{y\in Z} \|x + y\|_X$, $x \in X$.

The set X/Z with the above operations (addition and scalar multiplication) is a vector space and $X/Z \ni [x] \mapsto \|[x]\|_{X/Z} \in [0,\infty)$ is a norm which makes X/Z a Banach space. The space X/Z is called the quotient space *of X and Z.*

Recall that a subset of a metric space W is said *residual* in W if it is countable intersection of open dense subsets W.

Corollary 3 [Mañé, Lemma 1.1]: If $c(K) < \infty$ and Y is a finite dimensional subspace of X with $2c(K)+1 < \dim Y < \infty$, then $\{P \in \mathscr{P}(X,Y) : P|_K$ is injective $\}$ is residual in $\mathscr{P}(X,Y)$.

Proof: From Lemma 0.0.4, it is sufficient to show that $\mathscr{P}_{n,r}$ is dense in $\mathscr{P}(X,Y)$ for each $n \in \mathbb{N}$ and for each $r > 0$.

Let Q be the quotient map from X onto $Z = X/Y$. Then,

$$Q(A_{n,r}) \setminus \{0\} = \bigcup_{m\in\mathbb{N}} \left\{ Q(v) : v \in A_{n,r},\ \|Q(v)\|_Z \geqslant \frac{1}{m} \right\}$$

where, each $\left\{Q(v) : v \in A_{n,r},\ \|Q(v)\|_Z \geqslant \frac{1}{m}\right\}$ is compact. It follows from Lemma 0.0.4 that there is a sequence $\{\phi_i\}_{i\in\mathbb{N}}$ in Z', $\phi_i : Z \to \mathbb{R}$, such that, if $\phi_i(z) = 0$ for all $i \in \mathbb{N}$, then $z = 0$.

Let

$$A_{n,r,i,j} = \{v \in A_{n,r} : |\phi_i(Q(v))| \geqslant 1/j\}.$$

Then

$$P^{-1}(0) \cap A_{n,r} = \cup_{i\in\mathbb{N}} \cup_{j\in\mathbb{N}} \left(P^{-1}(0) \cap A_{n,r,i,j}\right),$$

$$\mathscr{P}_{n,r,i,j} = \left\{P \in \mathscr{P}(X,Y) : P^{-1}(0) \cap A_{n,r,i,j} = \emptyset\right\}$$

and

$$\mathscr{P}_{n,r} = \cap_{i\in\mathbb{N}} \cap_{j\in\mathbb{N}} \mathscr{P}_{n,r,i,j}.$$

Since $A_{n,r}$ is compact (see Lema 0.0.1) it follows easily that $A_{n,r,i,j}$ is compact. Proceeding exactly as in the proof of Lemma 0.0.3 we obtain that $\mathscr{P}_{n,r,i,j}$ is open. Hence, the proof is reduced to the proof that each for each $n,i,j \in \mathbb{N}$ and $r > 0$, $\mathscr{P}_{n,r,i,j}$ is dense in $\mathscr{P}(X,Y)$.

Let $P_0 \in \mathscr{P}(X,Y)$ and define $\phi : Y \setminus \{0\} \to S = \{y \in Y : \|y\|_X = 1\}$ by $\phi(y) = y/\|y\|_X$. Then

$$\phi(P_0(A_{n,r}) \setminus \{0\}) = \cup_{\varepsilon>0} \phi(P_0(A_{n,r}) \cap [Y \setminus B^Y_\varepsilon(0)])$$

and, from Proposition 3,

$$\dim_H \phi(P_0(A_{n,r})) \leqslant \sup_{\varepsilon>0} \dim_H(\phi(P_0(A_{n,r}) \cap [Y \setminus B^Y_\varepsilon(0)])).$$

Note that, ϕ restricted to $P_0(A_{n,r}) \cap [Y \backslash B_\varepsilon^Y(0)]$ is Lipschitz continuous. Consequently, from Proposition 2, Corollary 2, Remark ($\dim_H(K) \leqslant c(K)$) and monotonicity of the Hausdorff dimension,

$$\dim_H(\phi(P_0(A_{n,r}) \cap [Y \backslash B_\varepsilon^Y(0)])) \leqslant \dim_H(A_{n,r}) \leqslant 2c(K_n) \leqslant 2c(K).$$

From this we obtain that there exists $u \in S \backslash \phi(P_0(A_{n,r}))$. In fact, if that is not the case, then $S \subset \phi(P_0(A_{n,r}))$ and

$$\dim(Y) - 1 = \dim_H(S) \leqslant \dim_H(\phi(P_0(A_{n,r}))) \leqslant 2c(K)$$

which contradicts our assumption.

Given $\varepsilon > 0$, $i, j \in \mathbb{N}$ we define

$$P_\varepsilon(x) = P_0(x) + \varepsilon\phi_i(Q(x))u.$$

Since $P_\varepsilon \in \mathscr{L}(X)$ with range in Y and recalling that if $y \in Y$, then $Qy = 0$ it is easy to see that $P_\varepsilon \in \mathscr{P}(X,Y)$.

If $P_\varepsilon(x) = 0$ we have that

$$P_0(x) = -\varepsilon\phi_i(Q(x))u.$$

If in addition $x \in A_{n,r,i,j}$ we have that $\phi_i(Q(x)) \neq 0$ and $P_0(x) \neq 0$. Hence

$$u = -(\varepsilon\phi_i(Q(x)))^{-1}P_0 x.$$

Since $u \in S$, $u = \phi(u)$ we have that $\pm u = \phi(P_0(x)) \in \phi(P_0(A_{n,r}))$ and consequently $u \in \phi(P_0(A_{n,r}))$ contradicting the choice of u and showing that $P_\varepsilon \in \mathscr{P}_{n,r,i,j}$.

Since $\|P_\varepsilon - P_0\|_{\mathscr{L}(X)} \xrightarrow{\varepsilon \to 0} 0$ we have that $\mathscr{P}_{n,r,i,j}$ is dense in $\mathscr{P}(X,Y)$. ∎

Dimension of invariant compacts

Lemma 1: Let X be a n-dimensional normed vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then, there are bases $\{x_1, \ldots, x_n\}$ for X and $\{x_1^*, \ldots, x_n^*\}$ for X^* such that $\|x_i\| = \|x_i^*\| = 1$ for all $i = 1, \ldots, n$ and $x_i^*(x_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$. In this conditions $\{x_1, \ldots, x_n\}$ is called an Auerbach's base for X.

Proof: Let $\mathscr{B} = \{v_1, \ldots, v_n\}$ be a base for X. Given a n-set of vectors $\{y_1, \ldots, y_n\}$ in X, let $\hat{y}_j$ be the column matrix of the coordinates y_j in the base $\mathscr{B}$ and consider the function $X^n \ni (y_i, \ldots, y_n) \mapsto \det[\hat{y}_1, \ldots, \hat{y}_n] \in \mathbb{K}$, where X^n is the product of n copies of X.

Let B be the closed unit ball in X and B^n the product of n copies of B. If $\{x_1,\ldots,x_n\}$ is a point where the map $\det[\cdot,\ldots,\cdot]$ attains a maximum in B^n, $\{x_1,\ldots,x_n\}$ is a base for X, since $\det[\hat{x}_1,\ldots,\hat{x}_n]\neq 0$, and each x_j has norm 1 (since otherwise, we could multiply it by a number larger than one and still remain in B, which would contradict the choice of $\{x_1,\ldots,x_n\}$). Define

$$x_j^*(x)=\frac{\det[\hat{\mathrm{x}}_1,\ldots,\hat{\mathrm{x}}_{\mathrm{j}-1},\hat{\mathrm{x}},\hat{\mathrm{x}}_{\mathrm{j}+1},\ldots,\hat{\mathrm{x}}_{\mathrm{n}}]}{\det[\hat{x}_1,\ldots,\hat{x}_n]}.$$

It is clear that $x_j^*(x_i)=\delta_{ij}$ and that $\|x_j^*\|_{X^*}=1$, $1\leqslant i,j\leqslant n$. ∎

Proposition 6: Let Y be a m-dimensional Banach space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$). Then $\mathrm{d}_{BM}(Y,\mathbb{K}^m_\infty)\leqslant \log m$.

Proof: Let $\{x_1,\ldots,x_m\}$ be an Auerbach's base for Y and $\{x_1^*,\ldots,x_m^*\}$ the corresponding base for Y^*. Define a linear transform $T:\mathbb{K}^m_\infty\to Y$ by

$$T(\mathbf{z})=\sum_{j=1}^m z_jx_j.$$

Then

$$\|T(\mathbf{z})\|_Y=\left\|\sum_{j=1}^m z_jx_j\right\|_Y\leqslant \sum_{j=1}^m |z_j|\leqslant m\|\mathbf{z}\|_\infty,$$

and thus

$$\|T\|_{\mathscr{L}(\mathbb{K}^m_\infty,Y)}\leqslant m.$$

By the other hand, if $x=\sum_{j=1}^m z_jx_j\in Y$, since $z_j=x_j^*(x)$, we have

$$\|T^{-1}(x)\|_\infty=\|\mathbf{z}\|_\infty=\max_{j=1,\ldots,m}|z_j|=\max_{j=1,\ldots,m}|x_j^*(x)|\leqslant\|x\|_Y,$$

which implies that

$$\|T^{-1}\|_{\mathscr{L}(Y,\mathbb{K}^m_\infty)}\leqslant 1,$$

and proves our result. ∎

Lemma 2: Let X be a Banach space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$). If Y is a m-dimensional subspace of X we have that

(i) If $\mathbb{K}=\mathbb{R}$ then

$$N(\rho,B_r^Y(0))\leqslant(m+1)^m\left(\frac{r}{\rho}\right)^m,\ 0<\rho\leqslant r.$$

(ii) If $\mathbb{K}=\mathbb{C}$ then

$$N(\rho, B_r^Y(0)) \leqslant (m+1)^{2m}\left(\frac{\sqrt{2}r}{\rho}\right)^{2m},\ 0<\rho\leqslant r.$$

Moreover, the balls can be taken with centers in Y.

Proof: We will prove the case when $\mathbb{K}=\mathbb{R}$.
Since Y is m-dimensional, $\mathrm{d}_{BM}(Y,\mathbb{R}^m_\infty)\leqslant \log m$; in particular, there is a linear isomorphism $T:\mathbb{R}^m_\infty\to Y$ with $\|T\|\|T^{-1}\|\leqslant m$.

Since $B_r^Y(0)=TT^{-1}(B_r^Y(0))\subset T(B^{\mathbb{R}^m_\infty}_{\|T^{-1}\|r}(0))$ and $B^{\mathbb{R}^m_\infty}_{\|T^{-1}\|r}(0)$ can be covered by

$$\left(1+\frac{\|T^{-1}\|r}{\rho/\|T\|}\right)^m=\left(1+\frac{\|T^{-1}\|\|T\|r}{\rho}\right)^m\leqslant\left(1+m\frac{r}{\rho}\right)^m\leqslant(m+1)^m\left(\frac{r}{\rho}\right)^m,$$

balls with radius $\frac{\rho}{\|T\|}$, then $B_r^Y(0)$ can be covered by the same number of balls with radius ρ. ■

Lemma 3: Let X be a Banach space and $T\in\mathscr{L}_{\lambda/2}(X)$. Then there exists a finite dimensional subspace Z of X such that

$$\mathrm{dist}_H(T[B_1^X(0)],T[B_1^Z(0)])<\lambda.$$

Proof: Write $T=L+C$, where $C\in\mathscr{K}(X)$ and $L\in\mathscr{L}(X)$ with $\|L\|_{\mathscr{L}(X)}<\lambda/2$. We first will show that for every $\varepsilon>0$ there is a finite dimensional subspace Z such that

$$\mathrm{dist}_H(C[B_1^X(0)],C[B_1^Z(0)])<\varepsilon.$$

Assume that this is not true; that is, there exists an $\varepsilon>0$ such that

$$\mathrm{dist}_H(C[B_1^X(0)],C[B_1^Z(0)])\geqslant\varepsilon_0,$$

for every finite dimensional subspace Z of X. Choose some $x_1\in X$ with $\|x_1\|_X=1$, and let $Z_1=\mathrm{span}\{x_1\}$. Then

$$\mathrm{dist}_H(C[B_1^X(0)],C[B_1^{Z_1}(0)])\geqslant\varepsilon_0,$$

and so there is a $x_2\in X$ with $\|x_2\|_X\leqslant 1$ such that

$$\|Cx_2-Cx_1\|\geqslant\varepsilon_0.$$

With $Z_2 = \operatorname{span}\{x_1, x_2\}$, we can find a $x_3 \in X$ with $\|x_3\|_X \leqslant 1$ such that

$$\|Cx_3 - Cx_i\| \geqslant \varepsilon_0, \text{ for } i = 1, 2.$$

Inductively we can construct a sequence $\{x_j\}$ in X with $\|x_j\|_X \leqslant 1$ such that

$$\|Cx_i - Cx_j\| \geqslant \varepsilon_0, \text{ for } i \neq j,$$

which contradicts the fact that C is a compact operator.

Now let $\tilde{\lambda} < \lambda$ such that $2\|L\|_{\mathscr{L}(X)} < \lambda < \lambda$. From what we made above, we can choose a finite dimensional subspace Z of X such that

$$\operatorname{dist}_H(C[B_1^X(0)], C[B_1^Z(0)]) < \lambda - \tilde{\lambda}.$$

If $x \in B_1^X(0)$ and $z \in B_1^Z(0)$, then

$$\|Tx - Tz\|_X \leqslant \|L(x - z)\|_X + \|Cx - Cz\|_X \leqslant \tilde{\lambda} + \|Cx - Cz\|_X.$$

Thus

$$\operatorname{dist}_H(T[B_1^X(0)], T[B_1^Z(0)]) \leqslant \tilde{\lambda} + \operatorname{dist}_H(C[B_1^X(0)], C[B_1^Z(0)]) < \lambda,$$

which proves the result. ∎

Lemma 4: Let X be a Banach space over $\mathbb{K}$, Y a m-dimensional subspace of X, $\lambda > 0$ and $T \in \mathscr{L}(X)$ such that $\operatorname{dist}_H(T[B_1^X(0)], T[B_1^Y(0)]) < \lambda$. Then, for all $r > 0$ and $\gamma > 0$:

(i) If $\mathbb{K} = \mathbb{R}$ then

$$N((1+\gamma)\lambda r, T(B_r^X(0))) \leqslant (m+1)^m \left(\frac{\|T\|_{\mathscr{L}(X)} + \lambda}{\gamma\lambda} \right)^m.$$

(ii) If $\mathbb{K} = \mathbb{C}$ then

$$N((1+\gamma)\lambda r, T(B_r^X(0))) \leqslant 2^m (m+1)^{2m} \left(\frac{\|T\|_{\mathscr{L}(X)} + \lambda}{\gamma\lambda} \right)^{2m}.$$

Proof: We will do the proof when $\mathbb{K} = \mathbb{R}$. By linearity of T it is sufficient to do the case $r = 1$.

Let $\tilde{r} = \|T\|_{\mathscr{L}(X)} + \lambda$. Cover the ball $B_{\tilde{r}}^{T(Y)}(0)$ by balls $B_{\gamma\lambda}$, $1 \leqslant i \leqslant k$. By Lemma 3, we can take

$$k \leqslant (m+1)^m \left(\frac{\tilde{r}}{\gamma\lambda} \right)^m.$$

The proof will be complete if we can show that

$$T(B_1^X(0)) \subset \bigcup_{i=1}^{n} B^X_{(1+\gamma)\lambda}(x_i).$$

If $v \in B_1^X(0)$ ($\|v\|_X \leqslant 1$), since $\mathrm{dist}_H(T[B_1^X(0)], T[B_1^Y(0)]) < \lambda$, there is $y \in B_1^Y(0)$ such that

$$\|Tv - Ty\|_X < \lambda.$$

Hence $\|Ty\|_X \leqslant \|Tv - Ty\|_X + \|Tv\|_X \leqslant \lambda + \|T\|_{\mathscr{L}(X)} = \tilde{r}$.

Thus $Ty \in B_{\tilde{r}}^{T(Y)}(0)$ and we can choose $1 \leqslant i \leqslant k$ such that $\|Ty - x_i\|_X \leqslant \gamma\lambda$. Then

$$\|Tv - x_i\|_X \leqslant \|Tv - Ty\|_X + \|Ty - x_i\|_X < (1+\gamma)\lambda,$$

which proves the result; ■

Lemma 5: Let K be a compact subset of a Banach space X over $\mathbb{K}$ and $f : X \to X$ a continuously differentiable function in a neighborhood of K. Assume that K is negatively invariant for f; that is, $f(K) \supset K$ and also assume that exist $0 < \alpha < 1$ and $M \geqslant 1$ such that for every $x \in K$

$$N(\alpha, D_x f(B_1^X(0))) \leqslant M.$$

Then

$$c(K) \leqslant \frac{\log M}{-\log \alpha}.$$

Proof: We will first show that we can ensure bounds for the minimum number of balls necessary to cover $f(B_r^X(x))$ when r is small.

Since f is differentiable and K is compact, for each $\eta > 0$ there is a $r_0(\eta) > 0$ such that

$$f(B_r^X(x)) \subset f(x) + D_x f[B_r^X(0)] + B_{\eta r}^X(0),$$

for $0 < r \leqslant r_0(\eta)$ and any $x \in K$.

It follows that

$$N((\alpha+\eta)r, f(B_r^X(x))) \leqslant M,$$

for all $0 < r \leqslant r_0(\eta)$.

Now we fix $0 < \eta < 1 - \alpha$ and let $r_0 = r_0(\eta) > 0$. We cover K with $N(r_0, K)$ balls of radius r_0 and then we apply f to every element of this cover. Since $f(K) \supset K$ we still get a cover of K with sets given by $f(B_{r_0}^X)$ for some $x \in K$.

By the previous argument, each element of this cover can be covered by M balls of radius $(\alpha+\eta)r_0$ and follows that

$$N((\alpha+\eta)r_0, K) \leqslant M N(r_0, K).$$

Applying this argument k times, we have that

$$N((\alpha+\eta)^k r_0, K) \leqslant M^k N(r_0, K).$$

Thus

$$c(K) \leqslant \limsup_{k\to\infty} \frac{\log[M^k N(r_0,K)]}{-\log[(\alpha+\eta)^k r_0]} = \frac{\log M}{-\log(\alpha+\eta)},$$

and since $0 < \eta < 1-\alpha$ is arbitrary

$$c(K) \leqslant \frac{\log M}{-\log \alpha}.$$

■

Theorem 3: Let X be a Banach space over $\mathbb{K}$, $U \subset X$ an open subset and $f : U \to X$ a continuously differentiable map. Assume that $K \subset U$ is a compact subset and that $D_x f \in \mathscr{L}_{\lambda/2}$, for some $0 < \lambda < \frac{1}{2}$, for all $x \in K$. Then $n = \sup_{x\in K} \nu_\lambda(D_x f)$ and $D = \sup_{x\in K} \|D_x f\|$ are finite and

$$N(2\lambda, D_x f[B_1^X(0)]) \leqslant \left[(n+1)\frac{\sqrt{\alpha}D}{\lambda}\right]^{\alpha n}.$$

Moreover, if $f(K) \supset K$, then

$$c(K) \leqslant \alpha n \left[\frac{\log((n+1)\sqrt{\alpha}D/\lambda)}{-\log(2\lambda)}\right],$$

where $\alpha = 1$ if $\mathbb{K} = \mathbb{R}$ or $\alpha = 2$ if $\mathbb{K} = \mathbb{C}$.

Proof: Again, we will prove the case $\mathbb{K} = \mathbb{R}$. First we show that $n = \sup_{x\in K} \nu_\lambda(D_x f)$ is finite. By Lemma 3, for each $xinK$ there is a finite dimensional subspace Z_x if X such that

$$\mathrm{dist}_H(D_x f[B_1^X(0)], D_x f[B_1^{Z_x}(0)]) < \lambda.$$

Since $K \ni x \mapsto D_x f \in \mathscr{L}(X)$ is continuous, there is a $\delta_x > 0$ such that

$$\mathrm{dist}_H(D_y f[B_1^X(0)], D_y f[B_1^{Z_x}(0)]) < \lambda,$$

for all $y \in B_{\delta_x}^X(x)$; that is $\nu_\lambda(D_y f) \leqslant \nu_\lambda(D_x f)$, for all $y \in B_{\delta_x}^X(x)$. Since K is compact and $\{B_{\delta_x}^X(x)\}_{x\in K}$ covers K, we have that $n < \infty$.

Now since $n = \sup_{x\in K} \nu_\lambda(D_x f) < \infty$, for each $x \in K$, there exists a finite dimensional subspace Z_x of X with $\dim Z_x \leqslant n$ such that

$$\mathrm{dist}_H(D_x f[B_1^X(0)], D_x f[B_1^{Z_x}(0)]) < \lambda.$$

In order to make the notation cleaner, we will omit the subscribe x in Z_x and set $T \doteq D_x f$.

Noting that $\dim T(Z) \leqslant n$, we can cover the ball $B^{T(Z)}_{\|T\|}(0)$ by balls $B^X_\lambda(y_i)$, $1 \leqslant i \leqslant k$, such that $y_i \in B^X_{\|T\|}(0)$ and

$$k \leqslant (n+1)^n \left(\frac{\|T\|}{\lambda}\right)^n.$$

Hence

$$T(B^Z_1(0)) \subset B^{T(Z)}_{\|T\|}(0) \subset \bigcup_{i=1}^{k} B^X_\lambda(y_i).$$

The proof will be complete if we can show that

$$T(B^X_1(0)) \subset \bigcup_{i=1}^{k} B^X_{2\lambda}(y_i).$$

But if $x \in B^X_1(0)$ then there is $y \in T(B^Z_1(0))$ such that $\|Tx - y\|_X < \lambda$. Since $y \in T(B^Z_1(0))$, there is $1 \leqslant i \leqslant k$ such that $\|y - y_i\|_X < \lambda$, thus $\|Tx - y_i\|_X < 2\lambda$. The result now follows since n is uniform on K and the estimate is a straightforward applications of Lemma 5. ■

Theorem 4: Let X be a Banach space over $\mathbb{K}$, $U \subset X$ an open subset and $f : U \to U$ a continuously differentiable map. If $K \subset U$ is a compact subset such that $f(K) = K$ and there exists an $\varepsilon > 0$ such that $D_x f \in \mathscr{L}_{1-\varepsilon}(X)$ for all $x \in K$, then

$$c(K) < \infty.$$

Proof: For each $y \in K$, $D_y f \in \mathscr{L}_{1-\varepsilon}(X)$; that is $D_y f = L_y + C_y$ with $\|L_y\| < 1 - \varepsilon$ and $C \in \mathscr{K}(X)$.

Since

$$D_x f^n = D_{f^{n-1}(x)} f \circ \cdots \circ D_x f = L + C,$$

where $D_{f^{n-j}(x)} f = L_j + C_j$ and $L = L_1 \circ \cdots \circ L_n$ we have $\|L\| < (1-\varepsilon)^n$ and $C \in \mathscr{K}(X)$. It follows that for some $n_0 \in \mathbb{N}$ sufficiently large, $g = f^{n_0}$ is such that $D_x g \in \mathscr{L}_{\lambda/2}(X)$ for all $x \in K$ and some $0 < \lambda < \frac{1}{2}$. The result now follows from Theorem 3 applied to $g = f^{n_0}$. ■

Corollary 4: Let X be a real Banach space and assume that $T \in \mathscr{C}^1(X)$ is such that $\{T^n : n \geqslant 0\}$ has a global attractor $\mathscr{A}$ and $D_x T$ has finite rank $\nu(x)$ with $\sup\limits_{x \in \mathscr{A}} \nu(x) \doteq \nu < \infty$. Then

$$c(\mathscr{A}) \leqslant \nu.$$

Proof: Clearly, for each $\lambda > 0$ and $x \in \mathscr{A}$, $D_xT \in \mathscr{K}(X) \subset \mathscr{L}_{\lambda/2}(X)$. consequently, by Theorem 3, for $0 < \lambda < \frac{1}{2}$

$$c(\mathscr{A}) \leqslant \nu \left[\frac{\log[(\nu+1)D/\lambda]}{-\log(2\lambda)}\right],$$

and taking the limit when $\lambda \to 0$ we have that

$$c(\mathscr{A}) \leqslant \nu.$$

■

Corollary 5: Let X be a real Banach space, $L, K \in \mathscr{C}^1(X)$. If $T = L + K$, assume that the discrete semigroup $\{T^n : \ n \geqslant 0\}$ has a global attractor $\mathscr{A}$. Assume that K has finite rank in $\mathscr{A}$; that is, $R(D_xK) \subset Y(x)$ where $Y(x)$ is a finite dimensional subspace of X with $\sup\limits_{x\in\mathscr{A}} \dim(Y(x)) := \nu < \infty$, and that L satisfies

$$\sup_{x\in\mathscr{A}} \|D_{T^{n-1}(x)}L \circ \cdots \circ D_xL\| \leqslant c(n), \ n \in \mathbb{N},$$

where $c(n) \to 0$, when $n \to \infty$. Then

$$c(\mathscr{A}) \leqslant \nu.$$

Proof: We will prove that we can make λ as small as we want, without changing ν. For this we note that

$$\begin{aligned} D_xT^n =& D_{T^{n-1}(x)}T \circ \cdots \circ D_xT = (D_{T^{n-1}}L + D_{T^{n-1}}K) \circ \cdots \circ (D_xL + D_xK) = \\ =& D_{T^{n-1}}L \circ \cdots \circ D_xL + K_n \doteq L_n + K_n, \end{aligned}$$

where K_n is a compact operator with rank less than or equal ν.

Clearly there exists a subspace Z_n of X with $\dim Z_n \leqslant \nu$ such that

$$\mathrm{dist}_H(D_xT^n(B_1^X(0)), D_xT^n(B_1^{Z_n}(0))) \leqslant 2\|L_n\| \leqslant 2c(n).$$

Indeed, it suffices to choose Z_n such that $K_nB_1^{Z_n}(0) = K_nB_1^X(0)$, since if $x \in B_1^X(0)$ and $z \in B_1^{Z_n}(0)$ then

$$\|L_nx + K_nx - L_nz - K_nz\| \leqslant 2\|L_n\| + \|K_nx - K_nz\|,$$

but $\inf_{z\in B_1^{Z_n}(0)} \|K_nx - K_nz\| = 0$ and then

$$\mathrm{dist}_H(D_xT^n(B_1^X(0)), D_xT^n(B_1^{Z_n}(0))) \leqslant 2\|L_n\| \leqslant 2c(n).$$

Hence, given $\lambda > 0$, keeping ν fixed and taking n large, we can ensure that

$$D_xT^n \in \mathscr{L}_\lambda(X) \text{ and } \nu_\lambda(D_xT^n) \leqslant \nu.$$

Since $T^n\mathscr{A} = \mathscr{A}$ we have that

$$c(\mathscr{A}) \leqslant \nu\frac{\log[(\nu+1)D/\lambda]}{-\log(2\lambda)},$$

for each $0 < \lambda < \frac{1}{2}$ and taking $\lambda \to 0$ we have that

$$c(\mathscr{A}) \leqslant \nu$$

■

Exponential attractors

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two Banach spaces and assume that $(X, \|\cdot\|_X)$ is compactly immersed in $(Y, \|\cdot\|_Y)$; that is $X \subset Y$ and the bounded subsets of $(X, \|\cdot\|_X)$ are relatively compacts in $(Y, \|\cdot\|_Y)$.

Let also $S : X \to X$ be a continuous function such that

(i) $\{S^n : n \geqslant 0\}$ is bounded dissipative; that is, there exists a bounded subset $B_0 \subset X$ such that for every bounded subset $B \subset X$ there exists $n_B \in \mathbb{N}$ such that $S^n(B) \subset B_0$ for all $n \geqslant n_B$.

(ii) There exists a constant $K > 0$ such that $\|Sx - Sy\|_X \leqslant K\|x - y\|_Y$ for all $x, y \in B_0$.

Theorem 5: In the previous conditions, for all $\nu \in (0,1)$, $\{S^n : n \geqslant 0\}$ has an exponential attractor $\mathscr{M}_\nu$ and if $N(\nu, A)$ denotes the minimum number of balls with radius ν in $(Y, \|\cdot\|_Y)$ necessaries to cover $A \subset Y$, then $\mathscr{M}_\nu$ can be chosen in such a way that

$$c(\mathscr{M}_\nu) \leqslant \frac{\log N\left(\frac{\nu}{2K}, B_1^X(0)\right)}{\log\left(\frac{1}{\nu}\right)}.$$

The semigroup $\{S^n : n \geqslant 0\}$ has a global attractor $\mathscr{A}$ with $c(\mathscr{A}) \leqslant c(\mathscr{M}_\nu)$.

Proof: We know that, since B_0 is bounded, there is a $n_{B_0} \in \mathbb{N}$ such that $S^{n_{B_0}}(B_0) \subset B_0$. Considering iterates of S if necessary we can assume that $n_{B_0} = 1$.

Let $\nu \in (0,1)$, $N_0 = N(\frac{\nu}{2K}, B_0)$ and $V_0 = \{x_1, \ldots, x_{N_0}\} \subset B_0$ such that

$$B_0 \subset \bigcup_{i=1}^{N_0} B^Y_{\frac{\nu}{K}}(x_i).$$

Since $S(B_0) \subset B_0$, we have that

$$S(B_0) = S\left(\bigcup_{i=1}^{N_0} B^Y_{\frac{\nu}{K}}(x_i) \cap B_0\right) = \bigcup_{i=1}^{N_0} B^X_\nu(Sx_i) \cap S(B_0). \qquad (*)$$

Let $V_1 = S(V_0)$ and $N_\nu = N(\frac{\nu}{2K}, B^X_1(0))$ the minimum number of balls with radius $\nu/2K$ in Y necessary to cover $B^X_1(0) \subset Y$. So there exists $V_2 = \{x_{ij} : i = 1, \dots, N_0;\ j = 1, \dots, N_\nu\} \subset S(B_0)$ such that

$$S(B_0) = \bigcup_{i=1}^{N_0} B^X_\nu(Sx_i) \cap S(B_0) = \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} B^Y_{\frac{\nu^2}{K}}(x_{ij}) \cap S(B_0),$$

and proceeding as before, there exists $V_3 = \{x_{ijk} : i = 1, \dots, N_0;\ j,k = 1, \dots, N_\nu\} \subset S^2(B_0)$ such that

$$\begin{aligned} S^2(B_0) &= \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} S(B^Y_{\frac{\nu^2}{K}}(x_{ij}) \cap B_0) \cap S^2(B_0) = \\ &= \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} B^X_{\nu^2}(Sx_{ij}) \cap S^2(B_0) = \\ &= \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} \bigcup_{k=1}^{N_\nu} B^Y_{\frac{\nu^3}{K}}(x_{ijk}) \cap S^2(B_0), \end{aligned}$$

and

$$S^3(B_0) = \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} \bigcup_{k=1}^{N_\nu} S(B^Y_{\frac{\nu^3}{K}}(x_{ijk}) \cap B_0) \cap S^3(B_0) = \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} \bigcup_{k=1}^{N_\nu} B^X_{\nu^3}(Sx_{ijk}) \cap S^3(B_0).$$

Proceeding inductively we obtain $V_n \subset S^{n-1}(B_0)$, $\#V_n = N_0 N_\nu^{n-1}$ and

$$S^n(B_0) \subset \bigcup_{x \in V_n} B^X_{\nu^n}(Sx).$$

By the equation $(*)$ we know that $S(B_0)$ is precompact in X, so the semigroup $\{S^n : n \in \mathbb{N}\}$ is bounded dissipative and eventually compact, which means that $\{S^n : n \in \mathbb{N}\}$ has a global attractor $\mathscr{A}$.

Clearly $\mathscr{A} \subset S^n(B_0)$ for all $n \in \mathbb{N}$ and hence $N(\nu, \mathscr{A}) \leqslant N_0 N_\nu^{n-1}$ (in X). Then

$$c(\mathscr{A}) \leqslant \frac{\log N_\nu}{-\log \nu} < \infty.$$

Now we note that

$$\mathrm{dist}_H(S^n(B_0),V_n) \leqslant \nu_n,$$

and also

$$\max\{\mathrm{dist}_H(S(V_n),V_{n+1}),\mathrm{dist}_H(V_{n+1},S(V_n))\} \leqslant \nu^n.$$

Define $E_0 = V_0$, $E_{n+1} = V_{n+1} \cup S(E_n)$ and $\mathscr{M}_\nu = \overline{\cup_{n\in\mathbb{N}} E_n}^X$. Clearly we have that $S(\mathscr{M}_\nu) \subset \mathscr{M}_\nu$ and also

$$\mathrm{dist}_H(S^n(B_0),\mathscr{M}_\nu) \leqslant \nu^n = e^{-n\log(1/\nu)}.$$

Since B_0 absorbs bounded sets, given $B \subset X$ any bounded set, there is a constant $C(B) > 0$ such that

$$\mathrm{dist}_H(S^n B,\mathscr{M}_\nu) \leqslant C(B)e^{-n\log(1/\nu)}, \text{ for all } n \in \mathbb{N}.$$

It remains to show that $c(\mathscr{M}_\nu) < \infty$. Firstly we note that $E_{n+j} \subset S^n(B_0)$ for all $j \in \mathbb{N}$ and then (if we assume that B_0 is closed):

$$\mathscr{M}_\nu \subset E_1 \cup \cdots \cup E_n \cup S^n(B_0).$$

But $\#(E_1 \cup \cdots \cup E_n) \leqslant (n-1)^2 N_0 N_\nu^{n-1}$ and $N(\nu^n, S^n(B_0)) \leqslant N_0 N_\nu^{n-1}$ in X, hence

$$N(\nu^n,\mathscr{M}_\nu) \leqslant [(n-1)^2+1]N_0 N_\nu^{n-1},$$

and then

$$c(\mathscr{M}_\nu) \leqslant \limsup_{n\to\infty} \frac{\log\{[(n-1)^2+1]N_0 N_\nu^{n-1}\}}{-\log \nu^n} = \frac{\log N_\nu}{-\log \nu},$$

which completes the proof. ∎

Corollary 6: Let X,Y be two Banach spaces with X compactly immersed in Y and $S : X \to X$ continuous such that $\{S^n : n \geqslant 0\}$ has a global attractor $\mathscr{A}$. If $\|Sx - Sy\|_X \leqslant K\|x - y\|_Y$ for all $x, y \in \mathscr{A}$ e for some $K > 0$, then $c(\mathscr{A}) < \infty$.

Proof: This result follows from the proof of the previous theorem. ∎

Proposition: If Z is a subspace of Y with $\dim Z = m$ then $c(B_1^Z(0)) = \alpha m$, where $\alpha = 1$ if $\mathbb{K} = \mathbb{R}$ or $\alpha = 2$ if $\mathbb{K} = \mathbb{C}$.

Proof: We already know that

$$\left(\frac{\sqrt{\alpha}}{\varepsilon} - 1\right)^{\alpha m} \leqslant N(\varepsilon, B_1^{\mathbb{K}^m_\infty}(0)) \leqslant \left(\frac{\sqrt{\alpha}}{\varepsilon} + 1\right)^{\alpha m}$$

and

$$\alpha m = \lim_{\varepsilon\to 0} \frac{\log\left(\frac{\sqrt{\alpha}}{\varepsilon}-1\right)^{\alpha m}}{\log\frac{1}{\varepsilon}} \leqslant c(B_1^{\mathbb{K}_\infty^m}(0)) \leqslant \lim_{\varepsilon\to 0} \frac{\log\left(\frac{\sqrt{\alpha}}{\varepsilon}+1\right)^{\alpha m}}{\log\frac{1}{\varepsilon}} = \alpha m.$$

The result now follows since Z and $\mathbb{K}_\infty^m$ are isomorphic. ■

Theorem 6: Let X be a Banach space and $S \in \mathscr{C}(X)$. Assume that the semigroup $\{S^n : n \geqslant 0\}$ has a global attractor $\mathscr{A}$ in X. Let Y be a Banach space with X compactly immersed in Y and assume that $S = L + C : X \to X$ with $L, C \in \mathscr{C}(X)$ such that, for all $x, y \in \mathscr{A}$, for some $\lambda \in (0, \frac{1}{2})$ and some $K > 0$,

$$\|Lx - Ly\|_X \leqslant \lambda \|x-y\|_X, \quad \|Cx - Cy\|_X \leqslant K\|x-y\|_Y. \qquad (*)$$

Then $c(\mathscr{A}) \leqslant \frac{\log N\left(\frac{\nu}{K}, B_1^X(0)\right)}{\log\left(\frac{1}{2(\lambda+\nu)}\right)}$, for each $\nu \in (0, \frac{1}{2} - \lambda)$.

Moreover, if B_0 is an absorbing set with the property that $S(B_0) \subset B_0$ and $(*)$ is valid for all $x, y \in B_0$, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists an exponential attractor $\mathscr{M}_\nu$ for $\{S^n : n \geqslant 0\}$ and $c(\mathscr{M}_\nu) \leqslant \frac{\log N\left(\frac{\nu}{K}, B_1^X(0)\right)}{\log\left(\frac{1}{2(\lambda+\nu)}\right)}$.

Proof: Let $\nu \in (0, \frac{1}{2} - \lambda)$. From the compactness of $\mathscr{A}$, we can find $N_0 = N(\lambda + \nu, \mathscr{A})$ in X and points $\{x_1, \ldots, x_{N_0}\} \subset \mathscr{A}$ such that

$$\mathscr{A} = \bigcup_{i=1}^{N_0} B^X_{2(\lambda+\nu)}(x_i) \cap \mathscr{A}.$$

Since $S(x) = S(y) - (Lx - Ly) - (Cx - Cy)$ we have that

$$\begin{aligned}\mathscr{A} = S(\mathscr{A}) &= \bigcup_{i=1}^{N_0} S(B^X_{2(\lambda+\nu)}(x_i) \cap \mathscr{A}) \cap \mathscr{A} = \\ &= \bigcup_{i=1}^{N_0} \left\{ L(B^X_{2(\lambda+\nu)}(x_i) \cap \mathscr{A}) + C(B^X_{2(\lambda+\nu)}(x_i) \cap \mathscr{A}) \right\} \cap \mathscr{A} = \\ &= \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} B^X_{2(\lambda_\nu)^2}(Lx_i + y_{ij}) \cap \mathscr{A},\end{aligned}$$

where $N_\nu = N(\frac{\nu}{K}, B_1^X(0))$ is the minimum number of balls in Y of radius ν/K necessary to cover $B_1^X(0)$ and for some choice of $\{y_{ij} : i = 1, \ldots, N_0;\ j = 1, \ldots, N_\nu\}$.

Now there are $\{x_{ij} : i = 1, \ldots, N_0;\ j = 1, \ldots, N_\nu\}$ in $\mathscr{A}$ such that

$$\bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} B^X_{2^2(\lambda+\nu)^2}(x_{ij}) \cap \mathscr{A}.$$

Following this procedure we obtain a set V_n with $\#V_n = N_0 N_\nu^{n-1}$ in $\mathscr{A}$ such that

$$\mathscr{A} = \bigcup_{x\in V_n} B^X_{[2(\lambda+\nu)]^n}(x) \cap \mathscr{A}.$$

Thus

$$c(\mathscr{A}) \leqslant \limsup_{n\to\infty} \frac{\log N_0 N_\nu^{n-1}}{n\log(\frac{1}{2(\lambda+\nu)})} = \frac{\log N_\nu}{\log(\frac{1}{2(\lambda+\nu)})}.$$

Let B_0 be an absorbing set and suppose that $S(B_0) \subset B_0$. Let $R > 0$ be such that $B_0 \subset B^X_R(b_0)$ for some $b_0 \in B_0$. Then

$$\begin{aligned} S(B_0) = S(B^X_R(b_0)\cap B_0) &= (L(B^X_R(b_0)\cap B_0) + C(B^X_R(b_0)\cap B_0))\cap S(B_0) = \\ &= (B^X_{\lambda R}(Lb_0) + \bigcup_{i=1}^{N_\nu} B^X_{\nu R}(Cx_i)) \cap S(B_0) = \\ &= \bigcup_{i=1}^{N_\nu} B^X_{(\lambda+\nu)R}(Lb_0 + Cy_i) \cap S(B_0), \end{aligned}$$

and so, we can choose $\{x_1,\dots,x_{N_\nu}\}$ in $S(B_0)$ such that

$$S(B_0) = \bigcup_{i=1}^{N_\nu} B^X_{2(\lambda+\nu)R}(x_i) \cap S(B_0).$$

Since $S(x) = S(y) - (Lx - Ly) - (Cx - Cy)$ we obtain

$$\begin{aligned} S^2(B_0) &= \bigcup_{i=1}^{N_\nu} S(B^X_{2(\lambda+\nu)R}(x_i)\cap B_0) \cap S^2(B_0) = \\ &= \bigcup_{i=1}^{N_\nu} \Big\{ L(B^X_{2(\lambda+\nu)R}(x_i)\cap B_0) + C(B^X_{2(\lambda+\nu)R}(x_i)\cap B_0) \Big\} \cap S^2(B_0) = \\ &= \bigcup_{i=1}^{N_\nu} \left\{ B^X_{2\lambda(\lambda+\nu)R}(Lx_i) + \bigcup_{j=1}^{N_\nu} B^X_{2\nu(\lambda+\nu)R}(Cy_{ij}) \right\} \cap S^2(B_0) = \\ &= \bigcup_{i=1}^{N_\nu}\bigcup_{j=1}^{N_\nu} B^X_{2(\lambda+\nu)^2 R}(Lx_i + Cy_{ij}) \cap S^2(B_0), \end{aligned}$$

for some choice of $\{y_{ij} : 1 \leqslant i,j \leqslant N_\nu\}$ in X. Thus there are $\{x_{ij} : 1 \leqslant i,j \leqslant N_\nu\}$ in $S^2(B_0)$ such that

$$S^2(B_0) = \bigcup_{i=1}^{N_\nu}\bigcup_{j=1}^{N_\nu} B^X_{2^2(\lambda+\nu)^2 R}(x_{ij}) \cap S^2(B_0).$$

Following this procedure we construct sets V_n with $\#V_n = N_\nu^n$, $V_n \subset S^n(B_0)$ such that

$$S^n(B_0) \bigcup_{x\in V_n} B^X_{[2(\lambda+\nu)]^n R}(x) \cap S^n(B_0).$$

Hence $\mathrm{dist}_H(S^n(B_0), V_n) \leqslant [2(\lambda+\nu)]^n R$, and also

$$\max\{\mathrm{dist}_H(V_{n+1}, S(V_n)), \mathrm{dist}_H(S(V_n), V_{n+1})\} \leqslant [2(\lambda+\nu)]^{n+1} R.$$

Define $E_0 = V_0 \doteq \{b_0\}$, $E_{n+1} = V_{n+1} \cup S(E_n)$ for $n \in \mathbb{N}$. Set $\mathscr{M}_\nu = \overline{\cup_{n\in\mathbb{N}} E_n}^X$. As before $S(\mathscr{M}_\nu) \subset \mathscr{M}_\nu$ and $\mathscr{M}_\nu$ exponentially attracts bounded sets. It remains to show that $c(\mathscr{M}_\nu) < \infty$. As before, $E_{n+j} \subset S^n(B_0)$ for all $j \in \mathbb{N}$ and hence (again, we can assume without loos of generality that B_0 is closed) we have that

$$\mathscr{M}_\nu \subset E_1 \cup \cdots \cup E_n \cup S^n(B_0).$$

But $\#(E_1 \cup \cdots \cup E_n) \leqslant (n-1)^2 N_\nu^n$ and $N([2(\lambda+\nu)]^n R, S^n(B_0)) \leqslant N_\nu^n$ in X, hence

$$N([2(\lambda+\nu)]^n R, \mathscr{M}_\nu) \leqslant [(n-1)^2+1] N_\nu^n,$$

and then

$$c(\mathscr{M}_\nu) \leqslant \limsup_{n\to\infty} \frac{\log\{[(n-1)^2+1]N_\nu^n\}}{-\log[2(\lambda+\nu)]^n R} = \frac{\log N_\nu}{\log\left(\frac{1}{2(\lambda+\nu)}\right)},$$

which completes the proof. ∎

Estimative of the fractal dimension for gradient systems

Proposition 7: Let $\{T^n : n \in \mathbb{N}\}$ be a discrete semigroup with global attractor $\mathscr{A}$. Let $S = T_{|\mathscr{A}}$ and assume that S is Lipschitz continuous with Lipschitz constant $c > 1$. Let (A, A^*) be an attractor-repeller pair in $\mathscr{A}$, and assume that there exist constants $M \geqslant 1$ and $\omega > 0$ such that, for all K compact subset of $\mathscr{A}$ with $K \cap A^* = \varnothing$, we have $\mathrm{dist_H}(S^n K, A) \leqslant M e^{-\omega n}$, for all $n \in \mathbb{N}$. Assume also that there is a neighbourhood B of A^* in $\mathscr{A}$ such that $\overline{B} \cap A = \varnothing$.

Then

$$c(B) \leqslant c(\mathscr{A}) \leqslant \max\left\{\frac{\omega+\ln(c)}{\omega} c(B), c(A)\right\}.$$

Proof: Clearly, since $B \subset \mathscr{A}$, $c(B) \leqslant c(\mathscr{A})$. We only have to prove the right inequality. For this, we divide the proof in four steps:

Step 1: Define $\Omega_n = S^n(\mathscr{A} \setminus B) \setminus S^{n+1}(\mathscr{A} \setminus B)$, for all $n \in \mathbb{N}$. Note that $\Omega_0 = (\mathscr{A} \setminus B) \setminus S(\mathscr{A} \setminus B) \subset S(B) \setminus B \subset S(B)$ and therefore $c(\Omega_0) \leqslant c(S(B)) = c(B)$, because $B \subset S(B)$ and S is a Lipschitz continuous function.

Now we obtain an estimate on the minimum number of r-balls $N(r,\Omega_k)$ necessary to cover Ω_k in terms of the numbers of balls necessary to cover Ω_0. Let $n_0^{r,k} = N(r/c^k,\Omega_0)$ and $\{x_1,\dots,x_{n_0^{r,k}}\}$ a finite sequence of points in Ω_0 such that

$$\Omega_0 \subset \bigcup_{i=1}^{n_0^{r,k}} B(x_i, r/c^k).$$

Set, for each $i = 1,\dots,n_0^{r,k}$, $\xi_i = S^k(x_i) \in \Omega_k$. Then, for each $y \in \Omega_k$ there exists $z \in \Omega_0$ such that $y = S^k(z)$, $z \in B(x_i,r/c^k)$ for some $i = 1,\dots,n_0^{r,k}$ and we have

$$\|y-\xi_i\| = \|S^k(z)-S^k(x_i)\| \leqslant c^k\|z-x_i\| < r, \text{ for all } y \in \Omega_k.$$

So, we just proved $\Omega_k \subset \cup_{i=1}^{n_0^{r,k}} B(\xi_i,r)$, which gives $N(r,\Omega_k) \leqslant n_0^{r,k}$.

Step 2: Given $r>0$, since $\operatorname{dist}_{\mathrm{H}}(S^n(\mathscr{A}\setminus B),A) \leqslant Me^{-\omega n}$ for all $n \geqslant 0$, there exists $n_0(r) = \lceil \frac{1}{\omega}\ln(\frac{M}{r})\rceil$ such that

$$G(r) := \left(\bigcup_{j\geqslant n_0(r)} \Omega_j\right) \cup A \subset \mathscr{O}_r(A),$$

where $\mathscr{O}_r(A)$ denotes the r-neighborhood of A. So, if $A \subset \cup_{i=1}^{N(r,A)} B(x_i,r)$ with $x_i \in A$ for all $i = 1,\dots,N(r,A)$, then $\mathscr{O}_r(A) \subset \cup_{i=1}^{N(r,A)} B(x_i,2r)$ therefore $N(2r,\mathscr{O}_r(A)) \leqslant N(r,A)$. We conclude that $N\left(r,G(\frac{r}{2})\right) \leqslant N(\frac{r}{2},A)$.

Step 3: From Step 1, if $H(r) := \bigcup_{j=0}^{n_0(r)} \Omega_j$ we have

$$N(r,H(r)) \leqslant n_0(r) \max_{k=0,\dots,n_0(r)} N(r/c^k,\Omega_0) = n_0 N(r/c^{n_0(r)},\Omega_0),$$

since $c>1$.

Step 4: First, note that for each $r>0$, we have that $\mathscr{A} = B \cup G(\frac{r}{2}) \cup H(\frac{r}{2})$ and therefore

$$\begin{aligned} N(r,\mathscr{A}) &\leqslant 3\max\{N(r,B);\ N(r,H(r/2));\ N(r,G(r/2))\} \\ &\leqslant 3\max\{N(r,B);\ N(r/2,H(r/2));\ N(r/2,A)\} \\ &\leqslant 3\max\{N(r,B);\ n_0(r/2)N(r/c^{n_0(r/2)},\Omega_0);\ N(r/2,A)\}. \end{aligned}$$

As the logarithm function is increasing, we obtain

$$\ln N(r,\mathscr{A}) \leqslant \ln 3 + \max\{\ln N(r,B);\ \ln n_0(r/2) + \ln N(r/c^{n_0(r/2)},\Omega_0);\ \ln N(r/2,A)\}.$$

Hence

$$\frac{\ln N(r,\mathscr{A})}{\ln(1/r)} \leqslant \frac{\ln 3}{\ln(1/r)} + \max\left\{\frac{\ln N(r,B)}{\ln(1/r)};\ \frac{\ln n_0(r/2)}{\ln(1/r)} + \frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(1/r)};\frac{\ln N(r/2,A)}{\ln(1/r)}\right\}.$$

Obviously, $\limsup_{r\to 0^+}\frac{\ln 3}{\ln(1/r)} = 0$. Now, we compute the other terms:

(a)

$$\limsup_{r\to 0^+}\frac{\ln n_0(r/2)}{\ln(1/r)} = \limsup_{r\to 0^+}\frac{\ln 1/\omega}{\ln(1/r)} + \limsup_{r\to 0^+}\frac{\ln(\ln(2M/r))}{\ln(1/r)} = 0;$$

(b)

$$\begin{aligned}\limsup_{r\to 0^+}\frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(1/r)} &= \limsup_{r\to 0^+}\frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(c^{n_0(r/2)}/rc^{n_0})}\\ &= \limsup_{r\to 0^+}\frac{1}{1-\frac{n_0(r/2)\ln c}{\ln(c^{n_0(r/2)}/r)}}\frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(c^{n_0(r/2)}/r)},\end{aligned}$$

but

$$\limsup_{r\to 0^+}\frac{1}{1-\frac{n_0(r/2)\ln c}{\ln(c^{n_0(r/2)}/r)}} = \limsup_{r\to 0^+}\left(\frac{n_0(r/2)\ln(c)}{\ln(1/r)}+1\right),$$

and since $\frac{1}{\omega}\ln(\frac{2M}{r}) \leqslant n_0 \leqslant \frac{1}{\omega}\ln(\frac{2M}{r})+1$,

$$\limsup_{r\to 0^+}\left(\frac{n_0(r/2)\ln(c)}{\ln(1/r)}+1\right) = \frac{\omega+\ln(c)}{\omega},$$

which shows that

$$\limsup_{r\to 0^+}\frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(1/r)} \leqslant \frac{\omega+\ln(c)}{\omega}c(\Omega_0).$$

(c)

$$\begin{aligned}\limsup_{r\to 0^+}\frac{\ln N(r/2,A)}{\ln(1/r)} &= \limsup_{r\to 0^+}\frac{\ln N(r/2,A)}{\ln(2/2r)}\\ &\limsup_{r\to 0^+}\frac{1}{1+\frac{\ln(1/2)}{\ln(1/r)}}\frac{\ln N(r/2,A)}{\ln(2/r)} \leqslant c(A).\end{aligned}$$

Joining (a), (b) and (c), we obtain

$$c(\mathscr{A}) \leqslant \max\left\{c(B), \frac{\omega+\ln(c)}{\omega}c(\Omega_0), c(A)\right\} \leqslant \max\left\{\frac{\omega+\ln(c)}{\omega}c(B), c(A)\right\},$$

using the fact $c(\Omega_0) \leqslant c(B)$. The proof is now complete. ■

Proposition 8: Let $\{T(t) : t \geqslant 0\}$ be a generalized gradient-like semigroup with global attractor $\mathscr{A}$ and $\tilde{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ the associated isolated invariant sets. Then, there is at least one source and at least one sink.

Proof: Assume there are no sources. Then given Ξ_i, there exists a Ξ_j $(j \neq i)$ and a global solution ξ such that

$$\Xi_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \Xi_j.$$

Inductively, we can construct a homoclinic structure since there is a finite number of isolated invariant sets, which leads us to a contradiction. A similar argument proves the existence of a sink. ■

Theorem 7: Let $\{T^n : n \in \mathbb{N}\}$ be a discrete generalized gradient-like semigroup with global attractor $\mathscr{A}$ and $\tilde{\Xi} = \{\Xi_1, \ldots, \Xi_p\}$ the associated isolated invariant sets. Assume that the restriction $T_{|_{\mathscr{A}}}$ to $\mathscr{A}$ of the operator T is a Lipschitz continuous function with Lipschitz constant $c > 1$ and assume also that there exist constants $M > 1$ and $\omega > 0$ such that for every attractor-repeller pair (A, A^*) in $\mathscr{A}$ and every compact subset $K \subset \mathscr{A}$ with $K \cap A^* = \varnothing$ we have

$$\mathrm{dist_H}(T^n(K), A) \leqslant Me^{-\omega n}, \text{ for all } n \geqslant 0.$$

Finally, assume that the local unstable manifolds $\{W^u_{loc}(\Xi_i), i, \ldots, p\}$ are given as graphs of Lipschitz functions. Under these conditions

$$\max_{i=1,\ldots,p} c(W^u_{loc}(\Xi_i)) \leqslant c(\mathscr{A}) \leqslant \frac{\omega+\ln(c)}{\omega} \max_{i=1,\ldots,p} c(W^u_{loc}(\Xi_i)).$$

Proof: Since $\{T^n : \ n \in \mathbb{N}\}$ is a discrete gradient-like semigroup, there exists at least one source. Let Ξ_i one of these sources and B_i a neighbourhood of Ξ_i in $\mathscr{A}$ such that $B_i \subset W^u_{loc}(\Xi_i)$ and $T(B_i) \subset W^u_{loc}(\Xi_i)$, so that $c(B_i) = c(T(B_i)) = c(W^u_{loc}(\Xi_i))$ Now, it is easy to see that $\Xi_i = A_i^*$, where $A_i = \cup_{j\neq i} W^u_{loc}(\Xi_j)$. By Proposition **??**,

$$c(B_i) \leqslant c(\mathscr{A}) \leqslant \max\left\{\frac{\omega+\ln(c)}{\omega}c(B_i), c(A_i)\right\},$$

that is

$$c(W^u_{loc}(\Xi_i)) \leqslant c(\mathscr{A}) \leqslant \max\left\{\frac{\omega+\ln(c)}{\omega}c(W^u_{loc}(\Xi_i)), c(A_i)\right\}.$$

Now, restrict the operator T to the attractor A_i. Thus, we have a discrete generalized gradient-like semigroup with attractor A and $\tilde{\Xi}^1 = \tilde{\Xi} \setminus \{\Xi_i\}$, which has at least one source Ξ_k, with $k \neq i$. We can use the same argument above to prove that

$$c(W^u_{loc}(\Xi_k)) \leqslant c(A_i) \leqslant \max\left\{\frac{\omega+\ln(c)}{\omega}c(W^u_{loc}(\Xi_k)), c(A_k)\right\}.$$

And joining these two results, we obtain

$$\max_{j=i,k} c(W^u_{loc}(\Xi_j)) \leqslant c(\mathscr{A}) \leqslant \max\left\{\frac{\omega+\ln(c)}{\omega}c(W^u_{loc}(\Xi_i)), \frac{\omega+\ln(c)}{\omega}c(W^u_{loc}(\Xi_k)), c(A_k)\right\}.$$

This process must stop, since there are just a finite number of isolated invariant sets, and proceeding inductively we obtain the desired result. ∎