Fractal dimension of invariants

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Summary

Introduction
- Objectives
- Semigroups
- Topological dimension
- Hausdorff dimension
- Fractal dimension

Projection of compact sets with finite fractal dimension

Dimension of invariant compacts

Exponential attractors

Estimative of the fractal dimension for gradient systems

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2. One motivation for the study and computation of fractal dimension;
3. Estimate of fractal dimension of negatively invariant sets;
4. Construction of exponential fractal attractors;
5. Estimatives on the fractal dimension of gradient-like attractors.

Semigroups

Definition: Let (X,d) be a metric space. A family $\{S(n) : n \in \mathbb{N}\} \subset \mathcal{C}(X)$ is called a (discrete) semigroup in X if it satisfies

(i) $S(0)x = x$ for all $x \in X$;

(ii) $S(n)S(m) = S(n+m)$ for all $n, m \in \mathbb{N}$;

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(ii) $S(n)S(m) = S(n+m)$ for all $n, m \in \mathbb{N}$;

Given two subsets A, B of X we define the Hausdorff semidistance between A and B by

$$\operatorname{dist}_H(A,B) = \sup_{x\in A} \inf_{y\in B} d(x,y).$$

Semigroups

A bounded subset A of X attracts bounded sets if for every bounded subset $B \subset X$ we have

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A subset $A \subset X$ is invariant by $\{S(t) : t \geqslant 0\}$ if $S(n)A = A$ for all $n \in \mathbb{N}$.

Semigroups

We say that $\xi : \mathbb{Z} \to X$ is a global solution for the semigroup $\{S(n) : n \in \mathbb{N}\}$ if $S(n)\xi(m) = \xi(n+m)$ for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

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A set $\mathcal{A}$ is called an global attractor for the semigroup $\{S(n) : n \in \mathbb{N}\}$ if it is compact, invariant and attracts bounded subsets of X under the action of $\{S(n) : n \in \mathbb{N}\}$.

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Given a subset B of X we define the unstable set of B by

$$\begin{aligned} W^u(B) = \{y \in X : &\text{ there is a global solution } \xi : \mathbb{Z} \to X \\ &\text{such that } \xi(0) = y \text{ and } \xi(m) \overset{m \to -\infty}{\longrightarrow} B\} \end{aligned}$$

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We can also define for each $\epsilon > 0$ the ϵ-unstable set of B by

$$\begin{aligned} W^u_\epsilon(B) = \{y \in X : &\text{ there is a global solution } \xi : \mathbb{Z} \to X \\ &\text{such that } \xi(0) = y,\ \xi(m) \overset{m\to-\infty}{\longrightarrow} B, \\ &\text{and } \operatorname{dist}_H(\xi(m), B) \leqslant \epsilon \text{ for all } m \leqslant 0\}. \end{aligned}$$

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Analogously, we can define the ϵ-stable set $W^s_\epsilon(B)$ of a subset $B \subset X$ by

$$W^s(B) = \{y \in X : S^m(y) \overset{m\to\infty}{\longrightarrow} B \text{ and } \mathrm{dist}_H(S(m)y, B) \leqslant \epsilon\}$$

Semigroups

Finally, for a subset $B \subset X$, we define the ω-limit set of B by

$$\omega(B) = \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geqslant m} S(n)B}.$$

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We say that a point $e \in X$ is an equilibrium point for the semigroup $\{S(n) : n \in \mathbb{N}\}$ if the set $\{e\}$ is invariant by $\{S(n) : n \in \mathbb{N}\}$; that is, $S(n)e = e$ for all $n \in \mathbb{N}$.

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Let $\{S(n) : n \in \mathbb{N}\}$ be a semigroup in a Banach space X with a finite set $\mathcal{E} = \{e_1, \ldots, e_n\}$ of equilibrium points and a global attractor $\mathcal{A}$. We say that $\mathcal{A}$ is a gradient-like attractor if

$$\mathcal{A} = \bigcup_{i=1}^{n} W^u(e_i).$$

Topological dimension

Let K be a topological space. We say that K has finite topological dimension if there exists a integer $n \geqslant 0$ such that every open cover $\mathcal{U}$ of K has a refinement $\mathcal{U}'$ such that every point of K belongs at most to $n+1$ elements of $\mathcal{U}'$.
The dimension $\dim(K)$ of K is defined as the least integer n with this property.

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Property 1: $\dim(\mathbb{R}^n) = n$.

Property 2: If K is a compact topological space then $\dim(K) < \infty$ and K is homeomorphic to a subset of $\mathbb{R}^{2\dim(\mathrm{K})+1}$.

Hausdorff dimension

Let (X,d) be a metric space, $\alpha > 0$ and $\epsilon > 0$. If $A \subset X$, we define

$$\mu_\epsilon^{(\alpha)}(A) = \inf\left\{ \sum_{i=1}^{\infty} (\mathrm{diam}(B_i))^\alpha,\ A \subset \bigcup_{i=1}^{\infty} B_i,\ \mathrm{diam}(B_i) < \epsilon \right\},$$

with the convention $\inf \varnothing = \infty$.

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with the convention $\inf \varnothing = \infty$.

Since $\mu_\epsilon^{(\alpha)}(A)$ increases as ϵ decreases, we can define

$$\mu^{(\alpha)}(A) = \lim_{\epsilon \to 0} \mu_\epsilon^{(\alpha)}(A).$$

Hausdorff dimension

Proposition 1: Let $0 < \alpha < \alpha'$. If $\mu^{(\alpha)}(A) < \infty$ then $\mu^{(\alpha')}(A) = 0$ and also, if $\mu^{(\alpha')}(A) > 0$ then $\mu^{(\alpha)}(A) = \infty$.

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In view of this proposition we can define the Hausdorff dimension of A by

$$\dim_H(A) = \inf\{\alpha > 0 : \ \mu^{(\alpha)}(A) = 0\} = \sup\{\alpha > 0 : \ \mu^{(\alpha)}(A) = \infty\}.$$

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Remark: We know that $\dim(K) \leqslant \dim_H(K)$.

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Corollary 1: Let $f : X \to Y$ be a Lipschitz continuous function, $A \subset X$ and $G(f,A) = \{(x, f(x)) : \; x \in A\}$ the graph of f restricted to A. Then $\dim_H(G(f,A)) = \dim_H(A)$.

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Proposition 3: Let $\{A_j\}_{j\in\mathbb{N}}$ be a sequence of sets in X and $A = \bigcup_{j=1}^\infty A_j$. Then

$$\dim_H(A) = \sup_{j\in\mathbb{N}} \dim_H(A_j).$$

Hausdorff dimension

Property: Let $\{S(n) : n \in \mathbb{N}\}$ be a discrete semigroup in a Banach space with a finite set $\mathcal{E} = \{e_1, \ldots, e_n\}$ of equilibrium points and a gradient-like global attractor $\mathcal{A}$. Assume that $S(1)$ is a Lipschitz continuous map and that each local unstable set $W^u_{loc}(e_i)$ is a graph of a Lipschitz function with domain Q_iX, where Q_i is a finite rank projection. Then

$$\dim_H(\mathcal{A}) = \max_{i=1,\ldots,n} \dim_H(Q_iX).$$

Fractal dimension

Let K be a compact metric space and, for $r > 0$ we define $N(r, K)$ as the minimum number of balls with radius r necessary to cover K. The fractal dimension of K is defined by

$$c(K) = \limsup_{r \to 0} \frac{\log N(r, K)}{\log(1/r)}.$$

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We can also see that the fractal dimension is the number $c(K)$ for which given $\epsilon > 0$ there exists a $\delta > 0$ such that for $0 < r < \delta$ we have

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Remark: It is easily seen that $\dim_H(K) \leqslant c(K)$ (but they can be different).

Fractal dimension

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Proposition 5: Let K, Y be two metric spaces with K compact and $f : K \to Y$ a Lipschitz continuous function. Then $c(f(K)) \leqslant c(K)$.

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Exercise: If K is a compact subset of a metric space X, $\alpha > 0$ and $\eta \in (0,1)$ show that

$$c(K) = \limsup_{n\to\infty} \frac{\log N(\alpha\eta^n, K)}{-\log(\alpha\eta^n)}.$$

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Projection of compact sets

If X is a Banach space and Y is a closed subspace of X, we define

$$\mathcal{P}(X,Y) = \{P \in \mathcal{L}(X) : \ P^2 = P \text{ and } P(X) = Y\},$$

with the uniform topology of operators.

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with the uniform topology of operators.

Theorem 2 (Mañé, Lemma 1.1): If $\dim_H(K-K) < \infty$ and Y is a subspace of X with $\dim_H(K-K) + 1 < \dim Y < \infty$ then the set $\{P \in \mathcal{P}(X,Y) : \ P|_K \text{ is injective}\}$ is residual in $\mathcal{P}(X,Y)$.

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Corollary 3: If $c(K) < \infty$ and Y is a subspace of X with $2c(K)+1 < \dim Y < \infty$ then the set $\{P \in \mathcal{P}(X,Y) : \ P|_K \text{ is injective}\}$ is residual in $\mathcal{P}(X,Y)$.

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The Banach-Mazur distance

Definition: Let X, Y be two isomorphic normed vector spaces. We define the Banach-Mazur distance between X and Y by

$$\mathrm{d}_{BM}(X,Y) = \log(\inf\{\|T\|_{\mathcal{L}(X,Y)}\|T^{-1}\|_{\mathcal{L}(Y,X)} : T \in \mathcal{L}(X,Y),\ T^{-1} \in \mathcal{L}(Y,X)\}).$$

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Remark: We can easily see that for two normed vector spaces X and Y with the same finite dimension, $\mathrm{d}_{BM}(X,Y) = 0$ if and only if X and Y are isometrically isomorphic.

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Remark: We can easily see that for two normed vector spaces X and Y with the same finite dimension, $\mathrm{d}_{BM}(X,Y) = 0$ if and only if X and Y are isometrically isomorphic.

We will denote by $\mathbb{K}^m_\infty$ the space $\mathbb{K}^m$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the $\|\cdot\|_\infty$ norm; that is, if $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{K}^m_\infty$ we have

$$\|\mathbf{z}\|_\infty = \max_{i=1,\ldots,m} |z_i|_{\mathbb{K}}.$$

Auerbach's base

We are interested to prove the estimate $\mathrm{d}_{BM}(X, \mathbb{K}^m_\infty) \leqslant \log m$, where X is a Banach space m-dimensional. For this purpose we will make use of an Auerbach's base for X.

Auerbach's base

We are interested to prove the estimate $\mathrm{d}_{BM}(X, \mathbb{K}^m_\infty) \leqslant \log m$, where X is a Banach space m-dimensional. For this purpose we will make use of an Auerbach's base for X.

Lemma 1: Let X be a n-dimensional normed vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then, there are bases $\{x_1, \ldots, x_n\}$ for X and $\{x_1^*, \ldots, x_n^*\}$ for X^* such that $\|x_i\| = \|x_i^*\| = 1$ for all $i = 1, \ldots, n$ and $x_i^*(x_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$. In this conditions $\{x_1, \ldots, x_n\}$ is called an Auerbach's base for X.

Estimate on the Banach-Mazur distance

Proposition 6: Let Y be a m-dimensional Banach space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then $\mathrm{d}_{BM}(Y, \mathbb{K}^m_\infty) \leqslant \log m$.

Estimate on the Banach-Mazur distance

Proposition 6: Let Y be a m-dimensional Banach space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then $\mathrm{d}_{BM}(Y, \mathbb{K}^m_\infty) \leqslant \log m$.

This result gives an improvement on the estimate done by Mañé, that under the same hypotheses stated that

$$\mathrm{d}_{BM}(Y, \mathbb{K}^m_\infty) \leqslant \log(m2^m).$$

Dimension of invariant compacts

Lemma 2: Let X be a Banach space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). If Y is a m-dimensional subspace of X we have that

(i) If $\mathbb{K} = \mathbb{R}$ then

$$N(\rho, B_r^Y(0)) \leqslant (m+1)^m \left(\frac{r}{\rho}\right)^m, \ 0 < \rho \leqslant r.$$

(ii) If $\mathbb{K} = \mathbb{C}$ then

$$N(\rho, B_r^Y(0)) \leqslant (m+1)^{2m} \left(\frac{\sqrt{2}r}{\rho}\right)^{2m}, \ 0 < \rho \leqslant r.$$

Moreover, the balls can be taken with centers in Y.

Dimension of invariant compacts

Before we continue, we define for X_1, X_2 two Banach spaces the following set

$$\begin{aligned}\mathcal{L}_\lambda(X_1, X_2) = \{T \in \mathcal{L}(X_1, X_2) : {} & T = L + C, \text{ with} \\ & C \text{ compact and } \|L\|_{\mathcal{L}(X_1, X_2)} < \lambda\}.\end{aligned}$$

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Lemma 3: Let X be a Banach space and $T \in \mathcal{L}_{\lambda/2}(X)$. Then there exists a finite dimensional subspace Z of X such that

$$\mathrm{dist}_H(T[B_1^X(0)], T[B_1^Z(0)]) < \lambda.$$

Dimension of invariant compacts

Lemma 4: Let X be a Banach space over $\mathbb{K}$, Y a m-dimensional subspace of X, $\lambda > 0$ and $T \in \mathcal{L}(X)$ such that $\mathrm{dist}_H(T[B_1^X(0)], T[B_1^Y(0)]) < \lambda$. Then, for all $r > 0$ and $\gamma > 0$:

(i) If $\mathbb{K} = \mathbb{R}$ then

$$N((1+\gamma)\lambda r, T(B_r^X(0))) \leqslant (m+1)^m \left(\frac{\|T\|_{\mathcal{L}(X)} + \lambda}{\gamma\lambda}\right)^m.$$

(ii) If $\mathbb{K} = \mathbb{C}$ then

$$N((1+\gamma)\lambda r, T(B_r^X(0))) \leqslant 2^m (m+1)^{2m} \left(\frac{\|T\|_{\mathcal{L}(X)} + \lambda}{\gamma\lambda}\right)^{2m}.$$

Dimension of invariant compacts

Lemma 5: Let K be a compact subset of a Banach space X over $\mathbb{K}$ and $f : X \to X$ a continuously differentiable function in a neighborhood of K. Assume that K is negatively invariant for f; that is, $f(K) \supset K$ and also assume that exist $0 < \alpha < 1$ and $M \geqslant 1$ such that for every $x \in K$

$$N(\alpha, D_x f(B_1^X(0))) \leqslant M.$$

Then

$$c(K) \leqslant \frac{\log M}{-\log \alpha}.$$

Dimension of invariant sets

Definition: For $T \in \mathcal{L}(X)$ we define

$$\nu_\lambda(T) = \min\{n \in \mathbb{N} : \text{there exists a } n\text{-dimensional subspace } Z \text{ of } X \text{ such that } \operatorname{dist}_H(T[B_1^X(0)], T[B_1^Z(0)]) < \lambda\},$$

with the convention $\min \varnothing = \infty$.

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$$\nu_\lambda(T) = \min\{n \in \mathbb{N} : \text{there exists a } n\text{-dimensional subspace } Z \text{ of } X \\ \text{such that } \operatorname{dist}_H(T[B_1^X(0)], T[B_1^Z(0)]) < \lambda\},$$

with the convention $\min \varnothing = \infty$.

Remark: Note that by Lemma 3 we have that if $T \in \mathcal{L}_{\lambda/2}(X)$ then $\nu_\lambda(T) < \infty$.

Dimension of invariant compacts

Theorem 3: Let X be a Banach space over $\mathbb{K}$, $U \subset X$ an open subset and $f : U \to X$ a continuously differentiable map. Assume that $K \subset U$ is a compact subset and that $D_x f \in \mathcal{L}_{\lambda/2}$, for some $0 < \lambda < \frac{1}{2}$, for all $x \in K$. Then $n = \sup\limits_{x \in K} \nu_\lambda(D_x f)$ and $D = \sup\limits_{x \in K} \|D_x f\|$ are finite and

$$N(2\lambda, D_x f[B_1^X(0)]) \leqslant \left[(n+1)\frac{\sqrt{\alpha}D}{\lambda}\right]^{\alpha n}.$$

Moreover, if $f(K) \supset K$, then

$$c(K) \leqslant \alpha n \left[\frac{\log((n+1)\sqrt{\alpha}D/\lambda)}{-\log(2\lambda)}\right],$$

where $\alpha = 1$ if $\mathbb{K} = \mathbb{R}$ or $\alpha = 2$ if $\mathbb{K} = \mathbb{C}$.

Dimension of invariant compacts

Theorem 4: Let X be a Banach space over $\mathbb{K}$, $U \subset X$ an open subset and $f : U \to U$ a continuously differentiable map. If $K \subset U$ is a compact subset such that $f(K) = K$ and there exists an $\epsilon > 0$ such that $D_x f \in \mathcal{L}_{1-\epsilon}(X)$ for all $x \in K$, then

$$c(K) < \infty.$$

Dimension of invariant compacts

Theorem 4: Let X be a Banach space over $\mathbb{K}$, $U \subset X$ an open subset and $f : U \to U$ a continuously differentiable map. If $K \subset U$ is a compact subset such that $f(K) = K$ and there exists an $\epsilon > 0$ such that $D_x f \in \mathcal{L}_{1-\epsilon}(X)$ for all $x \in K$, then

$$c(K) < \infty.$$

Corollary 4: Let X be a real Banach space and assume that $T \in \mathcal{C}^1(X)$ is such that $\{T^n : n \geqslant 0\}$ has a global attractor $\mathcal{A}$ and $D_x T$ has finite rank $\nu(x)$ with $\sup\limits_{x \in \mathcal{A}} \nu(x) \doteq \nu < \infty$. Then

$$c(\mathcal{A}) \leqslant \nu.$$

Dimension of invariant compacts

Corollary 5: Let X be a real Banach space, $L, K \in \mathcal{C}^1(X)$. If $T = L + K$, assume that the discrete semigroup $\{T^n :\ n \geqslant 0\}$ has a global attractor $\mathcal{A}$. Assume that K has finite rank in $\mathcal{A}$; that is, $R(D_x K) \subset Y(x)$ where $Y(x)$ is a finite dimensional subspace of X with $\sup\limits_{x\in\mathcal{A}} \dim(Y(x)) := \nu < \infty$, and that L satisfies

$$\sup_{x\in\mathcal{A}} \|D_{T^{n-1}(x)} L \circ \cdots \circ D_x L\| \leqslant c(n),\ n \in \mathbb{N},$$

where $c(n) \to 0$, when $n \to \infty$. Then

$$c(\mathcal{A}) \leqslant \nu.$$

Summary

Introduction
- Objectives
- Semigroups
- Topological dimension
- Hausdorff dimension
- Fractal dimension

Projection of compact sets with finite fractal dimension

Dimension of invariant compacts

Exponential attractors

Estimative of the fractal dimension for gradient systems

Exponential attractors

Definition: Let $\{S^n : n \geqslant 0\}$ be a semigroup in a metric space X. We say that $\mathcal{M}$ is an exponential attractor for $\{S^n : n \geqslant 0\}$ if it is compact, positively invariant (that is, $S^n(\mathcal{M}) \subset \mathcal{M}$), $c(\mathcal{M}) < \infty$ and there is a constant $\gamma > 0$ such that

$$\lim_{n\to\infty} e^{\gamma n}\mathrm{dist}_H(S^n(B), \mathcal{M}) = 0,$$

for every bounded subset $B \subset X$.

Exponential attractors

To continue, we will need a result concerning the existence of an attractor for a semigroup $\{S^n : n \in \mathbb{N}\}$.

Lemma: Let $\{S^n : n \in \mathbb{N}\}$ be a semigroup in a metric space X. Assume that it satisfies the following conditions:

(i) There exists a bounded set $B_0 \subset X$ which attracts points;

(ii) For every bounded set $B \subset X$, there is a $n_B \in \mathbb{N}$ such that $S^{n_B}(B)$ is precompact.

Then $\{S^n : n \in \mathbb{N}\}$ has a global attractor $\mathcal{A}$.

Exponential attractors

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two Banach spaces and assume that $(X, \|\cdot\|_X)$ is compactly immersed in $(Y, \|\cdot\|_Y)$; that is $X \subset Y$ and the bounded subsets of $(X, \|\cdot\|_X)$ are relatively compacts in $(Y, \|\cdot\|_Y)$.

Exponential attractors

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two Banach spaces and assume that $(X, \|\cdot\|_X)$ is compactly immersed in $(Y, \|\cdot\|_Y)$; that is $X \subset Y$ and the bounded subsets of $(X, \|\cdot\|_X)$ are relatively compacts in $(Y, \|\cdot\|_Y)$.

Let also $S : X \to X$ be a continuous function such that

(i) $\{S^n : n \geqslant 0\}$ is bounded dissipative; that is, there exists a bounded subset $B_0 \subset X$ such that for every bounded subset $B \subset X$ there exists $n_B \in \mathbb{N}$ such that $S^n(B) \subset B_0$ for all $n \geqslant n_B$.

(ii) There exists a constant $K > 0$ such that $\|Sx - Sy\|_X \leqslant K\|x - y\|_Y$ for all $x, y \in B_0$.

Exponential attractors

Theorem 5: In the previous conditions, for all $\nu \in (0,1)$, $\{S^n : n \geqslant 0\}$ has an exponential attractor $\mathcal{M}_\nu$ and if $N(\nu, A)$ denotes the minimum number of balls with radius ν in $(Y, \|\cdot\|_Y)$ necessaries to cover $A \subset Y$, then $\mathcal{M}_\nu$ can be chosen in such a way that

$$c(\mathcal{M}_\nu) \leqslant \frac{\log N\left(\frac{\nu}{2K}, B_1^X(0)\right)}{\log\left(\frac{1}{\nu}\right)}.$$

The semigroup $\{S^n : n \geqslant 0\}$ has a global attractor $\mathcal{A}$ with $c(\mathcal{A}) \leqslant c(\mathcal{M}_\nu)$.

Proof: We know that, since B_0 is bounded, there is a $n_{B_0} \in \mathbb{N}$ such that $S^{n_{B_0}}(B_0) \subset B_0$. Considering iterates of S if necessary we can assume that $n_{B_0} = 1$.
Let $\nu \in (0,1)$, $N_0 = N(\frac{\nu}{2K}, B_0)$ and $V_0 = \{x_1, \ldots, x_{N_0}\} \subset B_0$ such that

$$B_0 \subset \bigcup_{i=1}^{N_0} B^Y_{\frac{\nu}{K}}(x_i).$$

Since $S(B_0) \subset B_0$, we have that

$$S(B_0) = S\left(\bigcup_{i=1}^{N_0} B^Y_{\frac{\nu}{K}}(x_i) \cap B_0\right) = \bigcup_{i=1}^{N_0} B^X_{\nu}(Sx_i) \cap S(B_0). \qquad (*)$$

Let $V_1 = S(V_0)$ and $N_\nu = N(\frac{\nu}{2K}, B_1^X(0))$ the minimum number of balls with radius $\nu/2K$ in Y necessary to cover $B_1^X(0) \subset Y$. So there exists $V_2 = \{x_{ij} : i = 1, \ldots, N_0;\ j = 1, \ldots, N_\nu\} \subset S(B_0)$ such that

$$S(B_0) = \bigcup_{i=1}^{N_0} B_\nu^X(Sx_i) \cap S(B_0) = \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} B^Y_{\frac{\nu^2}{K}}(x_{ij}) \cap S(B_0),$$

and proceeding as before, there exists
$V_3 = \{x_{ijk} : i = 1, \ldots, N_0;\ j, k = 1, \ldots, N_\nu\} \subset S^2(B_0)$ such that

$$\begin{aligned} S^2(B_0) &= \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} S(B^Y_{\frac{\nu^2}{K}}(x_{ij}) \cap B_0) \cap S^2(B_0) = \\ &= \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} B^X_{\nu^2}(Sx_{ij}) \cap S^2(B_0) = \\ &= \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} \bigcup_{k=1}^{N_\nu} B^Y_{\frac{\nu^3}{K}}(x_{ijk}) \cap S^2(B_0), \end{aligned}$$

and

$$S^3(B_0) = \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} \bigcup_{k=1}^{N_\nu} S(B^Y_{\frac{\nu^3}{K}}(x_{ijk}) \cap B_0) \cap S^3(B_0) = \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} \bigcup_{k=1}^{N_\nu} B^X_{\nu^3}(Sx_{ijk}) \cap S^3(B_0).$$

Proceeding inductively we obtain $V_n \subset S^{n-1}(B_0)$, $\#V_n = N_0 N_\nu^{n-1}$ and

$$S^n(B_0) \subset \bigcup_{x \in V_n} B^X_{\nu^n}(Sx).$$

By the equation $(*)$ we know that $S(B_0)$ is precompact in X, so the semigroup $\{S^n : n \in \mathbb{N}\}$ is bounded dissipative and eventually compact, which means that $\{S^n : n \in \mathbb{N}\}$ has a global attractor $\mathcal{A}$.
Clearly $\mathcal{A} \subset S^n(B_0)$ for all $n \in \mathbb{N}$ and hence $N(\nu, \mathcal{A}) \leqslant N_0 N_\nu^{n-1}$ (in X).
Then

$$c(\mathcal{A}) \leqslant \frac{\log N_\nu}{-\log \nu} < \infty.$$

Now we note that

$$\operatorname{dist}_H(S^n(B_0), V_n) \leqslant \nu_n,$$

and also

$$\max\{\operatorname{dist}_H(S(V_n), V_{n+1}), \operatorname{dist}_H(V_{n+1}, S(V_n))\} \leqslant \nu^n.$$

Define $E_0 = V_0$, $E_{n+1} = V_{n+1} \cup S(E_n)$ and $\mathcal{M}_\nu = \overline{\cup_{n\in\mathbb{N}} E_n}^X$. Clearly we have that $S(\mathcal{M}_\nu) \subset \mathcal{M}_\nu$ and also

$$\operatorname{dist}_H(S^n(B_0), \mathcal{M}_\nu) \leqslant \nu^n = e^{-n\log(1/\nu)}.$$

Since B_0 absorbs bounded sets, given $B \subset X$ any bounded set, there is a constant $C(B) > 0$ such that

$$\operatorname{dist}_H(S^n B, \mathcal{M}_\nu) \leqslant C(B) e^{-n\log(1/\nu)}, \text{ for all } n \in \mathbb{N}.$$

It remains to show that $c(\mathcal{M}_\nu) < \infty$. Firstly we note that $E_{n+j} \subset S^n(B_0)$ for all $j \in \mathbb{N}$ and then (if we assume that B_0 is closed):

$$\mathcal{M}_\nu \subset E_1 \cup \cdots \cup E_n \cup S^n(B_0).$$

But $\#(E_1 \cup \cdots \cup E_n) \leqslant (n-1)^2 N_0 N_\nu^{n-1}$ and $N(\nu^n, S^n(B_0)) \leqslant N_0 N_\nu^{n-1}$ in X, hence

$$N(\nu^n, \mathcal{M}_\nu) \leqslant [(n-1)^2 + 1] N_0 N_\nu^{n-1},$$

and then

$$c(\mathcal{M}_\nu) \leqslant \limsup_{n\to\infty} \frac{\log\{[(n-1)^2+1]N_0 N_\nu^{n-1}\}}{-\log v^n} = \frac{\log N_\nu}{-\log \nu},$$

which completes the proof. ■

Exponential attractors

Corollary 6: Let X, Y be two Banach spaces with X compactly immersed in Y and $S : X \to X$ continuous such that $\{S^n : n \geqslant 0\}$ has a global attractor $\mathcal{A}$. If $\|Sx - Sy\|_X \leqslant K\|x - y\|_Y$ for all $x, y \in \mathcal{A}$ e for some $K > 0$, then $c(\mathcal{A}) < \infty$.

Exponential attractors

Corollary 6: Let X, Y be two Banach spaces with X compactly immersed in Y and $S : X \to X$ continuous such that $\{S^n : n \geqslant 0\}$ has a global attractor $\mathcal{A}$. If $\|Sx - Sy\|_X \leqslant K\|x - y\|_Y$ for all $x, y \in \mathcal{A}$ e for some $K > 0$, then $c(\mathcal{A}) < \infty$.

Remark:

1. In many cases the global attractor has finite fractal dimension and exponentially attracts bounded sets.
2. There are (although rarely) global attractors that are not exponential and, in these cases, might still be possible to construct an exponential attractor.

Entropy numbers

Remark: If X and Y are Banach spaces, de entropy numbers $e_k(T)$ of $T \in \mathcal{L}(X,Y)$ are defined by

$$e_k(T) = \inf \left\{ \epsilon > 0 : T(B_1^X(0)) \subset \bigcup_{i=1}^{2^{k-1}} B_\epsilon^Y(y_j),\ y_j \in Y,\ 1 \leqslant j \leqslant 2^{k-1} \right\}.$$

Informally, we can say $e_k(T)$ is the root of $\log_2 N(\epsilon, T(B_1^X(0))) = k-1$. In many situations $e_k(T)$ tends to zero when k tends to infinite. In such cases, we can find $\nu \in (0, \frac{1}{2})$ such that $e_k(T) \leqslant \frac{\nu}{K}$ $(K = \|T\|_{\mathcal{L}(X,Y)})$ and for this k we have that $N(\frac{\nu}{K}, T(B_1^X(0))) \leqslant 2^{k-1}$.

Entropy numbers

Assume that X has infinite dimension and X is compactly immersed in Y. Let $I : X \to Y$ be the inclusion map. It is easy to see that $0 < e_k(I) < \infty$ for every $k \in \mathbb{N}$. Also, in this case, he have that

$$\frac{(k-1)\log 2}{-\log e_k(I)} \xrightarrow{k\to\infty} \infty.$$

This is simple to see using the following result:

Proposition: If Z is a subspace of Y with $\dim Z = m$ then $c(B_1^Z(0)) = \alpha m$, where $\alpha = 1$ if $\mathbb{K} = \mathbb{R}$ or $\alpha = 2$ if $\mathbb{K} = \mathbb{C}$..

Exponential attractors

Theorem 6: Let X be a Banach space and $S \in \mathcal{C}(X)$. Assume that the semigroup $\{S^n : n \geqslant 0\}$ has a global attractor $\mathcal{A}$ in X. Let Y be a Banach space with X compactly immersed in Y and assume that $S = L + C : X \to X$ with $L, C \in \mathcal{C}(X)$ such that, for all $x, y \in \mathcal{A}$, for some $\lambda \in (0, \frac{1}{2})$ and some $K > 0$,

$$\|Lx - Ly\|_X \leqslant \lambda \|x - y\|_X, \quad \|Cx - Cy\|_X \leqslant K\|x - y\|_Y. \qquad (*)$$

Then $c(\mathcal{A}) \leqslant \frac{\log N\left(\frac{\nu}{K}, B_1^X(0)\right)}{\log\left(\frac{1}{2(\lambda+\nu)}\right)}$, for each $\nu \in (0, \frac{1}{2} - \lambda)$.

Moreover, if B_0 is an absorbing set with the property that $S(B_0) \subset B_0$ and $(*)$ is valid for all $x, y \in B_0$, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists an exponential attractor $\mathcal{M}_\nu$ for $\{S^n : n \geqslant 0\}$ and $c(\mathcal{M}_\nu) \leqslant \frac{\log N\left(\frac{\nu}{K}, B_1^X(0)\right)}{\log\left(\frac{1}{2(\lambda+\nu)}\right)}$.

Proof: Let $\nu \in (0, \frac{1}{2} - \lambda)$. From the compactness of $\mathcal{A}$, we can find $N_0 = N(\lambda + \nu, \mathcal{A})$ in X and points $\{x_1, \ldots, x_{N_0}\} \subset \mathcal{A}$ such that

$$\mathcal{A} = \bigcup_{i=1}^{N_0} B^X_{2(\lambda+\nu)}(x_i) \cap \mathcal{A}.$$

Since $S(x) = S(y) - (Lx - Ly) - (Cx - Cy)$ we have that

$$\begin{aligned}\mathcal{A} = S(\mathcal{A}) = \bigcup_{i=1}^{N_0} S(B^X_{2(\lambda+\nu)}(x_i) \cap \mathcal{A}) \cap \mathcal{A} = \\ = \bigcup_{i=1}^{N_0} \Big\{ L(B^X_{2(\lambda+\nu)}(x_i) \cap \mathcal{A}) + C(B^X_{2(\lambda+\nu)}(x_i) \cap \mathcal{A}) \Big\} \cap \mathcal{A} = \\ = \bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} B^X_{2(\lambda_\nu)^2}(Lx_i + y_{ij}) \cap \mathcal{A},\end{aligned}$$

where $N_\nu = N(\frac{\nu}{K}, B^X_1(0))$ is the minimum number of balls in Y of radius ν/K necessary to cover $B^X_1(0)$ and for some choice of $\{y_{ij} : i = 1, \ldots, N_0;\ j = 1, \ldots, N_\nu\}$.

Now there are $\{x_{ij} : i = 1, \dots, N_0;\ j = 1, \dots, N_\nu\}$ in $\mathcal{A}$ such that

$$\bigcup_{i=1}^{N_0} \bigcup_{j=1}^{N_\nu} B^X_{2^2(\lambda+\nu)^2}(x_{ij}) \cap \mathcal{A}.$$

Following this procedure we obtain a set V_n with $\#V_n = N_0 N_\nu^{n-1}$ in $\mathcal{A}$ such that

$$\mathcal{A} = \bigcup_{x \in V_n} B^X_{[2(\lambda+\nu)]^n}(x) \cap \mathcal{A}.$$

Thus

$$c(\mathcal{A}) \leqslant \limsup_{n\to\infty} \frac{\log N_0 N_\nu^{n-1}}{n \log(\frac{1}{2(\lambda+\nu)})} = \frac{\log N_\nu}{\log(\frac{1}{2(\lambda+\nu)})}.$$

Let B_0 be an absorbing set and suppose that $S(B_0) \subset B_0$. Let $R > 0$ be such that $B_0 \subset B^X_R(b_0)$ for some $b_0 \in B_0$. Then

$$\begin{aligned} S(B_0) = S(B^X_R(b_0) \cap B_0) &= (L(B^X_R(b_0) \cap B_0) + C(B^X_R(b_0) \cap B_0)) \cap S(B_0) = \\ &= (B^X_{\lambda R}(Lb_0) + \bigcup_{i=1}^{N_\nu} B^X_{\nu R}(Cx_i)) \cap S(B_0) = \\ &= \bigcup_{i=1}^{N_\nu} B^X_{(\lambda+\nu)R}(Lb_0 + Cy_i) \cap S(B_0), \end{aligned}$$

and so, we can choose $\{x_1, \ldots, x_{N_\nu}\}$ in $S(B_0)$ such that

$$S(B_0) = \bigcup_{i=1}^{N_\nu} B^X_{2(\lambda+\nu)R}(x_i) \cap S(B_0).$$

Since $S(x) = S(y) - (Lx - Ly) - (Cx - Cy)$ we obtain

$$\begin{aligned}
S^2(B_0) &= \bigcup_{i=1}^{N_\nu} S(B^X_{2(\lambda+\nu)R}(x_i) \cap B_0) \cap S^2(B_0) = \\
&= \bigcup_{i=1}^{N_\nu} \Big\{ L(B^X_{2(\lambda+\nu)R}(x_i) \cap B_0) + C(B^X_{2(\lambda+\nu)R}(x_i) \cap B_0) \Big\} \cap S^2(B_0) = \\
&= \bigcup_{i=1}^{N_\nu} \left\{ B^X_{2\lambda(\lambda+\nu)R}(Lx_i) + \bigcup_{j=1}^{N_\nu} B^X_{2\nu(\lambda+\nu)R}(Cy_{ij}) \right\} \cap S^2(B_0) = \\
&= \bigcup_{i=1}^{N_\nu} \bigcup_{j=1}^{N_\nu} B^X_{2(\lambda+\nu)^2 R}(Lx_i + Cy_{ij}) \cap S^2(B_0),
\end{aligned}$$

for some choice of $\{y_{ij} : 1 \leqslant i, j \leqslant N_\nu\}$ in X.

Thus there are $\{x_{ij} : 1 \leqslant i, j \leqslant N_\nu\}$ in $S^2(B_0)$ such that

$$S^2(B_0) = \bigcup_{i=1}^{N_\nu} \bigcup_{j=1}^{N_\nu} B^X_{2^2(\lambda+\nu)^2 R}(x_{ij}) \cap S^2(B_0).$$

Following this procedure we construct sets V_n with $\#V_n = N_\nu^n$, $V_n \subset S^n(B_0)$ such that

$$S^n(B_0) \bigcup_{x \in V_n} B^X_{[2(\lambda+\nu)]^n R}(x) \cap S^n(B_0).$$

Hence $\mathrm{dist}_H(S^n(B_0), V_n) \leqslant [2(\lambda+\nu)]^n R$, and also

$$\max\{\mathrm{dist}_H(V_{n+1}, S(V_n)), \mathrm{dist}_H(S(V_n), V_{n+1})\} \leqslant [2(\lambda+\nu)]^{n+1} R.$$

Define $E_0 = V_0 \doteq \{b_0\}$, $E_{n+1} = V_{n+1} \cup S(E_n)$ for $n \in \mathbb{N}$. Set $\mathcal{M}_\nu = \overline{\cup_{n\in\mathbb{N}} E_n}^X$. As before $S(\mathcal{M}_\nu) \subset \mathcal{M}_\nu$ and $\mathcal{M}_\nu$ exponentially attracts bounded sets. It remains to show that $c(\mathcal{M}_\nu) < \infty$. As before, $E_{n+j} \subset S^n(B_0)$ for all $j \in \mathbb{N}$ and hence (again, we can assume without loss of generality that B_0 is closed) we have that

$$\mathcal{M}_\nu \subset E_1 \cup \cdots \cup E_n \cup S^n(B_0).$$

But $\#(E_1 \cup \cdots \cup E_n) \leqslant (n-1)^2 N_\nu^n$ and $N([2(\lambda+\nu)]^n R, S^n(B_0)) \leqslant N_\nu^n$ in X, hence

$$N([2(\lambda+\nu)]^n R, \mathcal{M}_\nu) \leqslant [(n-1)^2 + 1] N_\nu^n,$$

and then

$$c(\mathcal{M}_\nu) \leqslant \limsup_{n\to\infty} \frac{\log\{[(n-1)^2+1] N_\nu^n\}}{-\log[2(\lambda+\nu)]^n R} = \frac{\log N_\nu}{\log\left(\frac{1}{2(\lambda+\nu)}\right)},$$

which completes the proof. ∎

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Projection of compact sets with finite fractal dimension

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Introduction

Let $\{T^n : n \in \mathbb{N}\}$ be a semigroup. We say that an invariant set $\Xi \subset X$ for the semigroup $\{T^n : n \in \mathbb{N}\}$ is an isolated invariant set if there is an $\epsilon > 0$ such that Ξ is the maximal invariant subset of $\mathcal{O}_\epsilon(\Xi)$.
A disjoint family of isolated invariant sets is a family $\{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant sets with the property that, for some $\epsilon > 0$,

$$\mathcal{O}_\epsilon(\Xi_i) \cap \mathcal{O}_\epsilon(\Xi_j) = \varnothing,\ 1 \leq i < j \leq n.$$

Introduction

Let $\{T^n : n \in \mathbb{N}\}$ be a semigroup which has a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. A homoclinic structure associated to $\mathbf{\Xi}$ is a subset $\{\Xi_{k_1}, \cdots, \Xi_{k_p}\}$ of $\mathbf{\Xi}$ $(p \leq n)$ together with a set of global solutions $\{\phi_1, \cdots, \phi_p\}$ such that

$$\Xi_{k_j} \overset{m \to -\infty}{\longleftarrow} \phi_j(m) \overset{m \to \infty}{\longrightarrow} \Xi_{k_{j+1}},\ 1 \leq j \leq p,$$

where $\Xi_{k_{p+1}} := \Xi_{k_1}$.

Introduction

Let $\{T^n : n \in \mathbb{N}\}$ be a semigroup with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. We say that $\{T^n : n \in \mathbb{N}\}$ is a (discrete)generalized gradient-like semigroup relative to $\mathbf{\Xi}$ if

(i) For any global solution $\xi : \mathbb{N} \to \mathcal{A}$ there are $1 \leq i, j \leq n$ such that

$$\Xi_i \overset{m \to -\infty}{\longleftarrow} \xi(m) \overset{m \to \infty}{\longrightarrow} \Xi_j.$$

(ii) There is no homoclinic structure associated to $\mathbf{\Xi}$.

Introduction

Let $\{T^n : n \in \mathbb{N}\}$ be a semigroup with a global attractor $\mathcal{A}$. We say that a non-empty subset Ξ of $\mathcal{A}$ is a local attractor if there is an $\epsilon > 0$ such that $\omega(\mathcal{O}_\epsilon(\Xi)) = \Xi$. The repeller Ξ^* associated to a local attractor Ξ is the set defined by

$$\Xi^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi = \varnothing\}.$$

The pair (Ξ, Ξ^*) is called attractor-repeller pair for $\{T(t) : t \geq 0\}$.

Introduction

Let $\{T^n : n \in \mathbb{N}\}$ be a semigroup with a global attractor $\mathcal{A}$. We say that a non-empty subset Ξ of $\mathcal{A}$ is a local attractor if there is an $\epsilon > 0$ such that $\omega(\mathcal{O}_\epsilon(\Xi)) = \Xi$. The repeller Ξ^* associated to a local attractor Ξ is the set defined by

$$\Xi^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi = \varnothing\}.$$

The pair (Ξ, Ξ^*) is called attractor-repeller pair for $\{T(t) : t \geq 0\}$.

Remark: Note that if Ξ is a local attractor, then Ξ^* is closed and invariant.

Abstract result

Proposition 7: Let $\{T^n : \ n \in \mathbb{N}\}$ be a discrete semigroup with global attractor $\mathcal{A}$. Let $S = T_{|_{\mathcal{A}}}$ and assume that S is Lipschitz continuous with Lipschitz constant $c > 1$. Let (A, A^*) be an attractor-repeller pair in $\mathcal{A}$, and assume that there exist constants $M \geqslant 1$ and $\omega > 0$ such that, for all K compact subset of $\mathcal{A}$ with $K \cap A^* = \varnothing$, we have $\mathrm{dist}_{\mathrm{H}}(S^n K, A) \leqslant M e^{-\omega n}$, for all $n \in \mathbb{N}$. Assume also that there is a neighbourhood B of A^* in $\mathcal{A}$ such that $\overline{B} \cap A = \varnothing$.
Then

$$c(B) \leqslant c(\mathcal{A}) \leqslant \max\left\{ \frac{\omega + \ln(c)}{\omega} c(B), c(A) \right\}.$$

Proof: Clearly, since $B \subset \mathcal{A}$, $c(B) \leqslant c(\mathcal{A})$. We only have to prove the right inequality. For this, we divide the proof in four steps:
Step 1: Define $\Omega_n = S^n(\mathcal{A} \setminus B) \setminus S^{n+1}(\mathcal{A} \setminus B)$, for all $n \in \mathbb{N}$. Note that $\Omega_0 = (\mathcal{A} \setminus B) \setminus S(\mathcal{A} \setminus B) \subset S(B) \setminus B \subset S(B)$ and therefore $c(\Omega_0) \leqslant c(S(B)) = c(B)$, because $B \subset S(B)$ and S is a Lipschitz continuous function.
Now we obtain an estimate on the minimum number of r-balls $N(r, \Omega_k)$ necessary to cover Ω_k in terms of the numbers of balls necessary to cover Ω_0. Let $n_0^{r,k} = N(r/c^k, \Omega_0)$ and $\{x_1, \ldots, x_{n_0^{r,k}}\}$ a finite sequence of points in Ω_0 such that

$$\Omega_0 \subset \bigcup_{i=1}^{n_0^{r,k}} B(x_i, r/c^k).$$

Set, for each $i = 1, \ldots, n_0^{r,k}$, $\xi_i = S^k(x_i) \in \Omega_k$. Then, for each $y \in \Omega_k$ there exists $z \in \Omega_0$ such that $y = S^k(z)$, $z \in B(x_i, r/c^k)$ for some $i = 1, \ldots, n_0^{r,k}$ and we have

$$\|y - \xi_i\| = \|S^k(z) - S^k(x_i)\| \leqslant c^k \|z - x_i\| < r, \text{ for all } y \in \Omega_k.$$

So, we just proved $\Omega_k \subset \cup_{i=1}^{n_0^{r,k}} B(\xi_i, r)$, which gives $N(r, \Omega_k) \leqslant n_0^{r,k}$.
Step 2: Given $r > 0$, since $\mathrm{dist}_{\mathrm{H}}(S^n(\mathcal{A} \setminus B), A) \leqslant Me^{-\omega n}$ for all $n \geqslant 0$, there exists $n_0(r) = \lceil \frac{1}{\omega} \ln(\frac{M}{r}) \rceil$ such that

$$G(r) := \left(\bigcup_{j \geqslant n_0(r)} \Omega_j \right) \cup A \subset \mathcal{O}_r(A),$$

where $\mathcal{O}_r(A)$ denotes the r-neighborhood of A.

So, if $A \subset \cup_{i=1}^{N(r,A)} B(x_i, r)$ with $x_i \in A$ for all $i = 1, \ldots, N(r, A)$, then $\mathcal{O}_r(A) \subset \cup_{i=1}^{N(r,A)} B(x_i, 2r)$ therefore $N(2r, \mathcal{O}_r(A)) \leqslant N(r, A)$. We conclude that $N\left(r, G(\frac{r}{2})\right) \leqslant N(\frac{r}{2}, A)$.

Step 3: From Step 1, if $H(r) := \bigcup_{j=0}^{n_0(r)} \Omega_j$ we have

$$N(r, H(r)) \leqslant n_0(r) \max_{k=0,\ldots,n_0(r)} N(r/c^k, \Omega_0) = n_0 N(r/c^{n_0(r)}, \Omega_0),$$

since $c > 1$.

Step 4: First, note that for each $r > 0$, we have that $\mathcal{A} = B \cup G(\frac{r}{2}) \cup H(\frac{r}{2})$ and therefore

$$\begin{aligned} N(r,\mathcal{A}) &\leqslant 3\max\{N(r,B);\ N(r,H(r/2));\ N(r,G(r/2))\} \\ &\leqslant 3\max\{N(r,B);\ N\left(r/2,H(r/2)\right);\ N(r/2,A)\} \\ &\leqslant 3\max\{N(r,B);\ n_0(r/2)N(r/c^{n_0(r/2)},\Omega_0);\ N(r/2,A)\}. \end{aligned}$$

As the logarithm function is increasing, we obtain

$$\begin{aligned} \ln N(r,\mathcal{A}) \leqslant \ln 3 + \max\{ &\ln N(r,B);\ \ln n_0(r/2) + \\ &\ln N(r/c^{n_0(r/2)},\Omega_0);\ \ln N(r/2,A)\}. \end{aligned}$$

Hence

$$\frac{\ln N(r,\mathcal{A})}{\ln(1/r)} \leqslant \frac{\ln 3}{\ln(1/r)} + \max\left\{ \frac{\ln N(r,B)}{\ln(1/r)};\ \frac{\ln n_0(r/2)}{\ln(1/r)} + \frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(1/r)}; \frac{\ln N(r/2,A)}{\ln(1/r)} \right\}.$$

Obviously, $\limsup_{r\to 0^+} \dfrac{\ln 3}{\ln(1/r)} = 0$. Now, we compute the other terms:

(a)

$$\limsup_{r\to 0^+} \frac{\ln n_0(r/2)}{\ln(1/r)} = \limsup_{r\to 0^+} \frac{\ln 1/\omega}{\ln(1/r)} + \limsup_{r\to 0^+} \frac{\ln(\ln(2M/r))}{\ln(1/r)} = 0;$$

(b)

$$\limsup_{r\to 0^+}\frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(1/r)} = \limsup_{r\to 0^+}\frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(c^{n_0(r/2)}/rc^{n_0})}$$
$$= \limsup_{r\to 0^+}\frac{1}{1-\frac{n_0(r/2)\ln c}{\ln(c^{n_0(r/2)}/r)}}\frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(c^{n_0(r/2)}/r)},$$

but

$$\limsup_{r\to 0^+}\frac{1}{1-\frac{n_0(r/2)\ln c}{\ln(c^{n_0(r/2)}/r)}} = \limsup_{r\to 0^+}\left(\frac{n_0(r/2)\ln(c)}{\ln(1/r)}+1\right),$$

and since $\frac{1}{\omega}\ln(\frac{2M}{r}) \leqslant n_0 \leqslant \frac{1}{\omega}\ln(\frac{2M}{r})+1$,

$$\limsup_{r\to 0^+}\left(\frac{n_0(r/2)\ln(c)}{\ln(1/r)}+1\right) = \frac{\omega+\ln(c)}{\omega},$$

which shows that

$$\limsup_{r\to 0^+}\frac{\ln N(r/c^{n_0(r/2)},\Omega_0)}{\ln(1/r)} \leqslant \frac{\omega+\ln(c)}{\omega}c(\Omega_0).$$

(c)

$$\limsup_{r\to 0^+} \frac{\ln N(r/2,A)}{\ln(1/r)} = \limsup_{r\to 0^+} \frac{\ln N(r/2,A)}{\ln(2/2r)}$$
$$\limsup_{r\to 0^+} \frac{1}{1+\frac{\ln(1/2)}{\ln(1/r)}} \frac{\ln N(r/2,A)}{\ln(2/r)} \leqslant c(A).$$

Joining (a), (b) and (c), we obtain

$$c(\mathcal{A}) \leqslant \max\left\{ c(B), \frac{\omega+\ln(c)}{\omega}c(\Omega_0), c(A)\right\} \leqslant \max\left\{ \frac{\omega+\ln(c)}{\omega}c(B), c(A)\right\},$$

using the fact $c(\Omega_0) \leqslant c(B)$. The proof is now complete. ■

Gradient-like systems

Let $\{T^n : n \in \mathbb{N}\}$ be a generalized gradient-like semigroup with global attractor $\mathcal{A}$, and $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ a family of associated isolated invariant sets. We say that an isolated invariant set Ξ_i is a source, if $W^s_{loc}(\Xi_i) \cap \mathcal{A} = \Xi_i$; and a sink if $W^u(\Xi_i) = \Xi_i$. Otherwise, we say that Ξ_i is a saddle.

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Proposition 8: Let $\{T^n : n \in \mathbb{N}\}$ be a generalized gradient-like semigroup with global attractor $\mathcal{A}$ and $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ the associated isolated invariant sets. Then, there is at least one source and at least one sink.

Gradient-like systems

Theorem 7: Let $\{T^n : n \in \mathbb{N}\}$ be a discrete generalized gradient-like semigroup with global attractor $\mathcal{A}$ and $\boldsymbol{\Xi} = \{\Xi_1, \ldots, \Xi_p\}$ the associated isolated invariant sets. Assume that the restriction $T_{|\mathcal{A}}$ to $\mathcal{A}$ of the operator T is a Lipschitz continuous function with Lipschitz constant $c > 1$ and assume also that there exist constants $M > 1$ and $\omega > 0$ such that for every attractor-repeller pair (A, A^*) in $\mathcal{A}$ and every compact subset $K \subset \mathcal{A}$ with $K \cap A^* = \varnothing$ we have

$$\mathrm{dist}_{\mathrm{H}}(T^n(K), A) \leqslant Me^{-\omega n}, \text{ for all } n \geqslant 0.$$

Finally, assume that the local unstable manifolds $\{W^u_{loc}(\Xi_i), i, \ldots, p\}$ are given as graphs of Lipschitz functions. Under these conditions

$$\max_{i=1,\ldots,p} c(W^u_{loc}(\Xi_i)) \leqslant c(\mathcal{A}) \leqslant \frac{\omega + \ln(c)}{\omega} \max_{i=1,\ldots,p} c(W^u_{loc}(\Xi_i)).$$

Proof: Since $\{T^n : \quad n \in \mathbb{N}\}$ is a discrete gradient-like semigroup, there exists at least one source. Let Ξ_i one of these sources and B_i a neighbourhood of Ξ_i in $\mathcal{A}$ such that $B_i \subset W^u_{loc}(\Xi_i)$ and $T(B_i) \subset W^u_{loc}(\Xi_i)$, so that $c(B_i) = c(T(B_i)) = c(W^u_{loc}(\Xi_i))$ Now, it is easy to see that $\Xi_i = A_i^*$, where $A_i = \cup_{j \neq i} W^u_{loc}(\Xi_j)$. By Proposition 7,

$$c(B_i) \leqslant c(\mathcal{A}) \leqslant \max\left\{\frac{\omega + \ln(c)}{\omega} c(B_i), c(A_i)\right\},$$

that is

$$c(W^u_{loc}(\Xi_i)) \leqslant c(\mathcal{A}) \leqslant \max\left\{\frac{\omega + \ln(c)}{\omega} c(W^u_{loc}(\Xi_i)), c(A_i)\right\}.$$

Now, restrict the operator T to the attractor A_i. Thus, we have a discrete generalized gradient-like semigroup with attractor A and $\tilde{\Xi}^1 = \tilde{\Xi} \setminus \{\Xi_i\}$, which has at least one source Ξ_k, with $k \neq i$. We can use the same argument above to prove that

$$c(W^u_{loc}(\Xi_k)) \leqslant c(A_i) \leqslant \max\left\{\frac{\omega + \ln(c)}{\omega} c(W^u_{loc}(\Xi_k)), c(A_k)\right\}.$$

And joining these two results, we obtain

$$\begin{aligned}\max_{j=i,k} c(W^u_{loc}(\Xi_j)) \leqslant \ & c(\mathcal{A}) \\ & \leqslant \max\left\{\frac{\omega + \ln(c)}{\omega} c(W^u_{loc}(\Xi_i)), \frac{\omega + \ln(c)}{\omega} c(W^u_{loc}(\Xi_k)), c(A_k)\right\}.\end{aligned}$$

This process must stop, since there are just a finite number of isolated invariant sets, and proceeding inductively we obtain the desired result. ∎

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