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Duality and Implicit Differential Equations

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We prove in this paper some duality results concerning various types of implicit differential equations

$$F(x, y, \frac{dy}{dx}) = 0,$$

where F is a smooth function. These are used to deduce some geometric properties of surfaces in 3-space. April, 2000 ICMC-USP

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1. INTRODUCTION

In this paper we consider some duality results concerning various types of implicit differential equations (IDE's). Let

$$F(x, y, \frac{dy}{dx}) = 0 \tag{1}$$

be an implicit differential equation, where F is a smooth function in $(x, y, p) \in \mathbb{R}^3$. At points where the partial derivative $F_p \neq 0$, the above equation can be written locally in the form $\frac{dy}{dx} = g(x, y)$ and studied using the methods from the theory of ordinary differential equations.

When $F_p = 0$ the equation may define locally more than one direction in the plane. The qualitative study of such equations have received considerable attention in recent years. The cases that have been most intensively studied, with extensive applications in

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differential geometry and control theory, are those implicit differential equations that define at most two directions in the plane. This is the case locally for example when $F = F_p = 0$ but $F_{pp} \neq 0$. Note that in this situation, using the Division Theorem, one can reduce F to a quadratic equation in p. One way these equations are studied is by lifting the bi-valued direction field defined in the plane to a single field ξ on the surface $M = F^{-1}(0)$ in $\mathbb{R}^3$. (This field is determined by the restriction of the contact planes associated to the standard contact form $dy - pdx$ in $\mathbb{R}^3$.) If 0 is a regular value of F then M is smooth and the projection to the plane is a fold. The critical set of this projection is called the criminant and its image is the discriminant of the equation. The configuration of the solution curves of F at a point on the discriminant are determined by the pair (ξ, σ), where σ is the involution on M that interchanges points with the same image under the projection to $\mathbb{R}^2$. If ξ is regular then a smooth model is given by $dy^2 - xdx^2 = 0$ ([13]). The integral curves in this case are a family of cusps. If ξ has an elementary zero, with separatrices transverse to the criminant, and not killed under projection, then a smooth model is given by $dy^2 - (y - \lambda x^2)dx^2 = 0$, with $\lambda \neq 0, \frac{1}{16}$ (see [14], and [15] for applications to control theory).

There are three topological models, a well-folded saddle if $\lambda < 0$, a well-folded node if $0 < \lambda < \frac{1}{16}$ and a well-folded focus if $\frac{1}{16} < \lambda$. The family of cusps and the well-folded singularities are the only locally structurally stable configurations. Generic bifurcations of such equations in 1-parameter families are considered in [9] and [16].

A particular class of implicit differential equations are the so called binary differential equations (BDE's), that is differential equations of the form

$$a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2 = 0$$

where a, b, c are smooth real functions in (x, y). If $\mathbb{R}P$ denotes the real projective line consider in $\mathbb{R}^2 \times \mathbb{R}P$ the set M of points $(x, y, [\alpha : \beta])$ where $\delta(x, y) = (b^2 - ac)(x, y) \geq 0$ and the direction $[\alpha : \beta]$ is a solution of the BDE at (x, y). Note that here we allow $F = F_p = F_{pp} = 0$, i.e. $a = b = c = 0$ at the origin. The discriminant function $\delta = b^2 - ac$ of the BDE plays a key role. When δ has a Morse singularity the surface M is smooth and the approach highlighted above can be used to obtain topological models for the configurations of the solution curves of such equations ([3], [6]). Bifurcations of BDE's of Morse type with zero coefficients are studied in [7]. Other information concerning such equations, such as defining and computing their multiplicities and the deformations of the discriminant curve can be found in [8] and [10].

In [2] the first author studied the duals of the solution curves of equation (1). These are the solution curves of the differential equation obtained from equation (1) using the Legendre transformation. The duals of the dual curves (the original solutions of equation (1)) can be studied using the family of height functions on the family of dual curves. It is established that these form a family of cusps at fold points of the projection where ξ is regular (recovering the result mentioned above), and furthermore that there are no simple smooth models in the case when $F = F_p = F_{pp} = 0$ and $F_{ppp} \neq 0$. Note however that the dual only makes sense if we restrict to affine changes of co-ordinates in the (x, y)-plane. Consequently we can have diffeomorphic singular points of implicit differential equations yielding quite distinct Legendre transforms.

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It is consequently rather surprising that we can show, in this paper, that the well folded singularities are self-dual, that is, that generically the equation resulting from applying the Legendre transformation has the same type as the original (Section 2). This is also true in the case when the discriminant has a Morse singularity at a point where $F_{pp} \neq 0$ (Section 3). We also study the case of BDE's where $a = b = c = 0$ at the origin and the discriminant has a Morse singularity (Section 4). Although the Legendre transform refers only to the plane and the results associated with it are only affine invariant, rather surprisingly the theorems in Sections 2, 3, 4 have some interesting geometric consequences for surfaces which we discuss. Note that although much of our work is motivated by the case of BDE's, the Legendre transform of a BDE generally is not another BDE. For this reason we deal with general IDE's.

2. WELL FOLDED SINGULARITIES

The Legendre transformation in $\mathbb{R}^3$ is given by

$$X = p,\ Y = xp - y,\ P = x.$$

As in the introduction let M be the surface defined by $F(x, y, p) = 0$, where F is a smooth function. We shall only be considering the local behaviour of the integral curves of equation (1), so we assume that the points (x, y, p) under consideration are close to the origin. (We may clearly take $x = y = 0$ and suppose $p = 0$ by a rotation of the (x, y)-plane.)

The Legendre transformation of the surface M yields a surface N given as the zero-set of the function

$$G(X, Y, P) = F(P, XP - Y, X).$$

An important property of the Legendre transformation is that the solution curves of the implicit equation

$$G(X, Y, \frac{dY}{dX}) = 0 \tag{2}$$

are naturally dual to those of equation (1) (see for example [1]). So if $\mathbb{R}P^2$ denotes the affine projective plane, the curve representing all the tangent lines to a solution of equation (1) is a solution curve of equation (2), and conversely. The key is that the transform takes the canonical 1-form $dy - pdx$ to (minus) the canonical 1-form $dY - PdX$ in the other space. The objective here is to relate the configuration of the integral curves of F to those of G.

Recall that typically solution curves to IDE's have cusps, and that elementary geometry tells us that the dual of a cusp is (generically) an inflection. So when studying an IDE, its Legendre transform and their solution curves we should consider, in both cases, the cusp set and the inflexional set. We will be particularly interested in the configuration of these two sets at singular points of the IDE. Our aim is to analyse the configurations that can occur, and determine those that are dual. Recall, however, that the property that a curve have an inflexion at some point is not a diffeomorphism invariant, simply an affine one.

Throughout our investigation we will only allow ourselves affine changes of co-ordinates when simplifying normal forms.

We wish to study the singular points. Recall that the lifted field ξ on M can be written explicitly in the form

$$\xi = F_p\frac{\partial}{\partial x} + pF_p\frac{\partial}{\partial y} - (F_x + pF_y)\frac{\partial}{\partial p},$$

(see for example [2]). The corresponding line field is also determined by the kernel of the canonical 1-form on the tangent spaces to $F = 0$. So the condition for ξ to be singular is that $F = 0$, $F_p = 0$ and $F_x + pF_y = 0$. The first condition just says that we are on the surface, and the second that we are on the criminant. The third simply states, as we now show, that we are also on the locus of inflexions (under the Legendre transform these points get mapped to the cusp set of the dual). To see this differentiate F with respect to x to obtain $F_x + pF_y + F_p(dp/dx)$. This vanishes identically and we have an inflexion if and only if $dp/dx = 0$ that is $F_x + pF_y = 0$.

Suppose now that the projection $\pi : M \to \mathbb{R}^2$ is still a fold and ξ has an elementary singularity with separatrices transverse to the criminant, and their tangents not projecting to zero. Of course we cannot assume Davydov's normal form described in the introduction since the Legendre transform is only preserved (up to affine equivalence) by an *affine* change of co-ordinates.

We start by analysing the conditions for a well folded singularity.

LEMMA 2.2.1. *Assume that the point under consideration is* $(x, y, p) = (0, 0, 0)$. *Then the conditions for a well-folded singularity is determined by the 2-jet of* F *at* $(0, 0, 0)$. *Writing this 2-jet as*

$$j^2F = a_0p^2 + (b_0 + b_1x + b_2y)p + (c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2)$$

then the following is true.

(i) We have a fold point if $a_0 \neq 0,\ b_0 = 0$.

(ii) We have a zero of the lifted field if $c_1 = 0$, *and then* $F = 0$ *is locally smooth if and only if* $c_2 \neq 0$.

(iii) We have a well folded singularity if and only if $\lambda \neq 0, \frac{1}{16}$ *where* $\lambda = (4a_0c_3 - b_1^2 - b_1c_2)/4c_2^2$ *(these distinguish saddle/node/focus, that is degeneracies of the lifted field) and* $c_3 \neq 0$ *(the separatrix does not have tangent projecting to zero, i.e. is not vertical). A well-folded saddle-node bifurcation occurs when* $\lambda = 0$.

Proof: These are all fairly straightforward. Since we are at a fold point $F = F_p = 0,\ F_{pp} \neq 0$. We have a zero of the lifted field if, in addition, $F_x + pF_y = 0$. The linear part of the lifted field is $(2a_0p + b_1x)\partial/\partial x - ((b_1 + c_2)p + 2c_3x)\partial/\partial p$. The other assertions can be deduced from this.

Using this computation we can establish the following facts.

THEOREM 2.2.2. *(i) An IDE* $F = 0$ *has a fold (or worse) and a zero of the lifted field if and only if the same is true for its Legendre transform.*

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(ii) The set $F = 0$ is smooth if and only if $G = 0$ is smooth.

(iii) The IDE $F = 0$ (resp. $G = 0$) has a genuine fold if and only if $G = 0$ (resp. $F = 0$) has a non-vertical separatrix.

(iv) The lifted fields for F and G are equivalent.

(v) Consequently the well-folded singularities of the implicit differential equation $F(x, y, \frac{dy}{dx}) = 0$ are self-dual. This is also the case for the well-folded saddle-node bifurcation.

Proof: We have $G(X, Y, P) = F(P, XP - Y, X)$, and we can assume that the points under consideration are $(x, y, p) = (0, 0, 0)$ and $(X, Y, P) = (0, 0, 0)$. We denote the set of zeros of G by N. Note that the 2-jet of G at $(0, 0, 0)$ given by

$$j^2G = c_3P^2 + ((c_2 + b_1)X - c_4Y)P - c_2Y + a_0X^2 - b_2XY + c_5Y^2,$$

is determined by the 2-jet of F at $(0, 0, 0)$. The result now follows from the previous lemma by a straightforward calculation. For example we have

$$G_X = PFy + F_p = xF_y + F_p,\ G_Y = -F_y,\ G_P = F_x + XF_y = F_x + pF_y.$$

So at the origin, the projection $\pi : N, 0 \to \mathbb{R}^2, 0$ is a fold or worse if and only if $G_P = 0$, i.e. $F_x + pF_y = 0$, and we have a zero of the lifted field if and only if $G_X + PG_Y = 0$, that is $F_p = 0$. Indeed, as remarked above, the Legendre transform takes the canonical form in one space to that in the other. Consequently it takes M diffeomorphically to N and the integral curves of one lifted field to that of the other. We have a genuine fold if $G_{PP} \neq 0$ at the origin, that is if and only if $F_{xx} \neq 0$ at the origin, the condition that a separatrix is not vertical. The rest follows in the same vein.

Corollary 2.2.3. *As cusps dualise to inflexions, we deduce that there is a smooth curve of inflexion points tangential to the discriminant at well-folded singularities of equation (1), see Figure 1.*

At this stage it is worthwhile establishing various basic results about implicit differential equations and their Legendre transforms (or duals as we shall sometimes refer to them). We recall from [8] that we can associate to any germ of an IDE a multiplicity, which measures two numerical invariants. These are the number of cusps of the projection of $F = 0$ to the (x, y)-plane and the number of (non-degenerate, well-folded) zeros of the natural lifted field that emerge in a generic deformation of F.

In our case we are also interested in another phenomena.

Definition 2.2.4. *An undulation of the IDE $F = 0$ is a non-singular point on an integral curve where the curve has ≥ 4-point contact with its tangent line. It is said to be non-degenerate if we have exactly 4-point contact.*

Generally curves do not have undulations, but they do appear generically in 1-parameter families. Any IDE yields a natural 1-parameter family of curves, namely its solution curves, so we might expect (isolated) points of undulation for any given IDE.

Our aim now is to describe how an IDE, $F=0$ and its Legendre transform $G=0$ are related.

PROPOSITION 2.2.5. *(i) The set of germs IDE's, $F=0$, with the property that F and its Legendre transform G have finite multiplicity has a complement of infinite codimension.*

(ii) When both have finite multiplicity we can deform the IDE $F=0$ so that the deformed equation has finitely many cusps for the projection say $C(F)$, finitely many well-folded zeros of the lifted field $Z(F)$, and finitely many undulations $U(F)$. The same will be true for the Legendre transform G and $Z(F)=Z(G),\ C(F)=U(G),\ C(G)=U(F)$.

(iii) The zero sets $F=0$ and $G=0$ are diffeomorphic, the diffeomorphism preserving the integral curves of the lifted fields.

(iv) The initial parts of F and G are linearly equivalent.

Proof: (i) We showed in [8], that the set of germs of IDE's of non-finite multiplicity is of infinite codimension. The proof here is very similar and omitted.

(ii) These just follow from straightforward calculations, which again we omit. However we can explain the results geometrically. First note that the condition for a zero of the lifted field is that we have a cusp point on an integral curve, *and* an inflexion. Since these conditions are dual to each other, it follows that zeros of the field on $F=0$ correspond to zeros on $G=0$.

On the other hand an undulation can be thought of as two inflexions coming together, and the dual of this situation is that we have two cusp points coming together on the corresponding dual solution curves. This only occurs at a cusp of the projection. (More precisely we expect the undulation to be 'versally unfolded ', and the set of tangent lines to correspond to the discriminant of an A_2-singularity, that is yield a cusp. See for example [4], page 203.)

(iii) This is obvious, since the Legendre map is an involution, hence a diffeomorphism. Similarly for (iv).

We now wish to investigate the relative positions of the inflexion curve, the cusps and the separatrices (if any). We can do this by considering the corresponding curves on the surface $F=0$ and the way these project to the (x,y)-plane or via a more direct approach. We start with the former and an elementary lemma.

LEMMA 2.2.6. *Let $f:\mathbb{R}^2,0\to\mathbb{R}^2,0$ be a fold mapping, and let $C_j,\ j=1,\ 2$ be smooth curves through 0 in the source with tangents there transverse to the kernel of $df(0)$. Suppose further that $\sigma:\mathbb{R}^2,0\to\mathbb{R}^2,0$ is the involution in the source with $f(\sigma(x,y))=f(x,y)$ for all $(x,y)\in\mathbb{R}^2$. Then their images $f(C_j)$ are smooth and have 2-point contact if and only if C_1 and $\sigma(C_1)$ are transverse to C_2.*

Proof: We may suppose that $f(x,y)=(x,y^2)$, so $\ker df(0)$ is spanned by $(0,1)$ and $\sigma(x,y)=(x,-y)$. We can suppose that C_j is parametrised as $(t,g_j(t))$, since its tangent is transverse to $\ker df(0)$, and its image is parametrised by $(t,g_j(t)^2)$, so is clearly smooth. The contact between the two images is given by the first non-vanishing term in $(g_1(t)-$

$g_2(t))(g_1(t)+g_2(t))$. So we get two point contact if and only if $g_1'(0) \neq \pm g_2'(0)$. The result now follows.

In our case we now need to locate the kernel of the projection (at our base point $(0,0,0)$) and the linear part of the involution σ.

LEMMA 2.2.7. *Parametrising the surface $F = 0$ as above using the (x,p) co-ordinates, and denoting the projection $M, 0 \to \mathbb{R}^2, 0$ as above by f, we find that $\ker df(0)$ is spanned by $(0,1)$, the linear part of σ is $(x,p) \mapsto (x, -p - b_1x/a_0)$. Moreover the tangent to the critical set at $(0,0)$ is given by $(2a_0, -b_1)$, and that to the inflexion curve $(b_1 + c_2, -2c_3)$. Finally the separatrices have tangents which are the eigenvectors of*

$$\begin{pmatrix} b_1 & -2c_3 \\ 2a_0 & -b_1 - c_2 \end{pmatrix}.$$

Proof: We have seen that to order 2 we have $y = -(c_3x^2 + b_1xp + a_0p^2)/c_2$, and the first results follow easily. The singular points of the projection are given by $F = F_p = 0$, to first order by $2a_0p + b_1x = 0$. The condition for the locus of inflexions, we noted above was $F_x + pF_y = 0$. So to first order $(b_1 + c_2)p + 2c_3x = 0$. The final part follows from the linearisation of the lifted field.

Alternatively one can compute directly in the plane.

LEMMA 2.2.8. *In the case of the node and saddle the separatrices are given by curves parametrised to order 2 as $(t, \alpha t^2)$ where $4a_0\alpha^2 + (2b_1 + c_2)\alpha + c_3 = 0$.*

Proof: We know that to second order the solution curves will be parabolas tangent to the cusp set, so can be written in the form $y = \alpha x^2 + hot$. We simply substitute in the equation and set second order terms equal to 0.

Now we need to do some detailed calculations. It is not hard to see that by a linear change of co-ordinates we may suppose that in the above 2-jet we may take $a_0 = c_2 = 1$, and to simplify matters we will do so. This leaves us with a 2-parameter family parametrised by the coefficients b_1 and c_3. For each point in this space we have 2 or 4 curves in the (x,y) plane: the fold curve, the inflexion curve and any separatrices. Each is generically a parabola, all passing through the same point with the same tangent. We are interested in the way these parabolas nest. Clearly some change takes place when one of the following happens:

(a) we change the type of zero of the lifted field, i.e., focus/node/saddle;
(b) fold, inflexion curve or separatrices have an inflexion or a singularity at origin;
(c) some pair of fold/inflexion curve/separatrix sets have $> 2-$point contact.

We write u for b_1 and v for $4c_3$. Then we have the following sets of interest.

(i) $v = 0$: separatrices inflexional or singular;
(ii) $v = u^2$: cusp set inflexional (never singular);

(iii) $v = u^2 - 1$: inflexion curve inflexional;

(iv) $u + 1 = 0$: inflexion curve singular;

(v) $v = u(u+1)$: saddle/node change and also inflexion curve/fold set have $>$ 2-point contact;

(vi) $v = u(u+1) + 1/4$: focus/saddle node change;

(vii) $v = -2(u+1)$: the union of this set and (i) and (v) corresponds to inflexion curve/separatrices having $>$ 2-point contact.

When we take the dual of this BDE and normalise again we get an IDE of the same form with (u, v) replaced by $(-u-1, v)$. Note that this does give an involution on the (u, v) plane, which preserves the above sets.

Corollary 2.2.9. *There are 18 types of configuration of cusp set/ inflexion set/ separatrices for well-folded singular points of IDE's. They are as illustrated in Figure 1.*

Proof: This simply involves selecting one point from each of the regions bounded by the above curves and calculating.

Remark 2.2.10. *Note that the relative positions of the separatrices, cusp and inflexional sets are determined by their tangents on the surface $F = 0$.*

2.1. Application: Asymptotic curves on surfaces in $\mathbb{R}^3$

Suppose we are given a surface S in $\mathbb{R}^3$, and an asymptotic curve on that surface. So this curve has as its tangent at each point one of the asymptotic directions to S there. Alternatively the osculating plane to the curve at each point coincides with the tangent plane to the surface. Let us suppose that the surface is written as a graph $(x, y, h(x, y))$ for some smooth function h, and the asymptotic curve is parameterised $(t, \alpha(t), \beta(t))$. The condition for the curve to be asymptotic is that its tangent line at each point has 3-point contact (or higher) with the surface at each point. For a fixed parameter value of t the tangent line is $(t+s, \alpha(t)+s\alpha'(t), \beta(t)+s\beta'(t))$. The conditions for at least 3-point contact are (omitting the t's and $(t, \alpha(t))$ in any expression involving h)

$$\beta - h = h_x + h_y\alpha' - \beta' = h_{xx} + 2h_{xy}\alpha' + h_{yy}(\alpha')^2 = 0.$$

If we differentiate twice the first of these identities (which just tells us that the curve lies on the surface) we obtain

$$h_{xx} + 2h_{xy}\alpha' + h_{yy}(\alpha')^2 + h_y\alpha'' - \beta'' = 0 \qquad (3)$$

and combining this with the above we find that $h_y(t, \alpha(t))\alpha''(t) - \beta''(t) = 0$.

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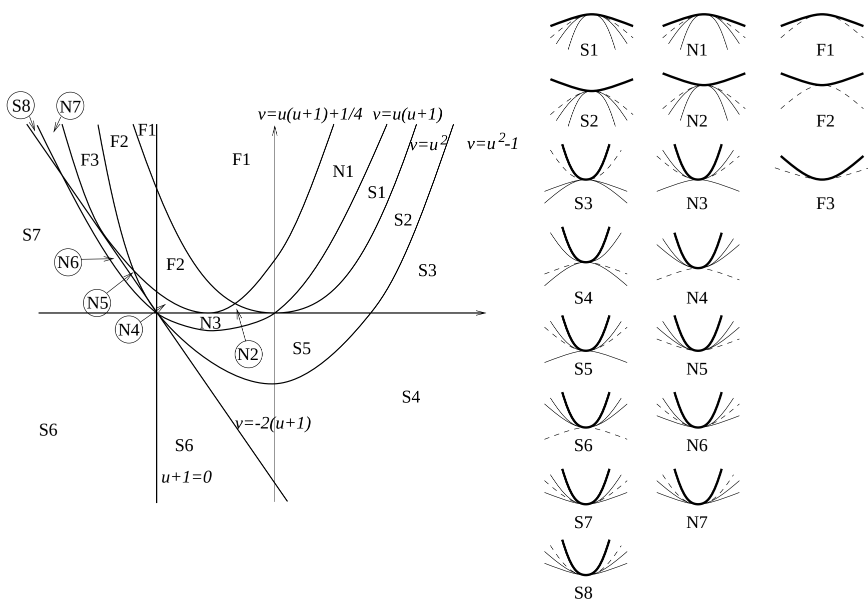

FIG. 1. Partition of the (u, v)-plane and the configurations of the cusp set (thick), inflexion set (dashed) and separatrices (thin).

Proposition 2.2.11. *Some plane projection of the asymptotic curve has an inflexion at a point if and only if the corresponding space curve can, after a change of co-ordinates in the source and an affine change in the 3-space be reduced to the form* (t, at^3+hot, bt^4+hot). *Following Scherbak [23] we call such points* $(1, 3, 4)$ *points.*

Proof: Recall that a plane curve has an inflexion if and only if its tangent line has at least 3-point contact with the curve. Now since asymptotic curves and plane projections are affine invariant we may assume that the form is of the indicated type and then clearly deduce that *any* projection yielding a nonsingular curve gives an inflexion. Conversely suppose (without loss of generality) that the projection results in $(t, \alpha(t))$. The condition for an inflexion at say $t = 0$ is $\alpha''(0) = 0$, but then $\beta''(0) = 0$ and the result follows.

So we are interested in $(1, 3, 4)$ points on the asymptotic curves. We now see that we may use the standard BDE $h_{yy}dy^2 + 2h_{xy}dxdy + h_{xx}dx^2 = 0$ for the asymptotic directions. The inflexions on the integral curves have an intrisic meaning, and hence so does the Legendre transform. (Note that a priori there is no reason why the Legendre transform of this equation is of any interest at all.) These points form the so-called *flecnodal curve.*

Now the well folded singularities of the asymptotic BDE correspond to cusps of the Gauss map. So after a linear change of co-ordinates we may write $h = y^2 + (ax^2y + bxy^2 + cy^3) + (dx^4 + \ldots)$ where $a^2 - 4d \neq 0$. A short calculation shows that the 2-jet of the BDE becomes

$$j^2F = p^2 + (2ax + 2by)p + (ay + 6dx^2 + \ldots).$$

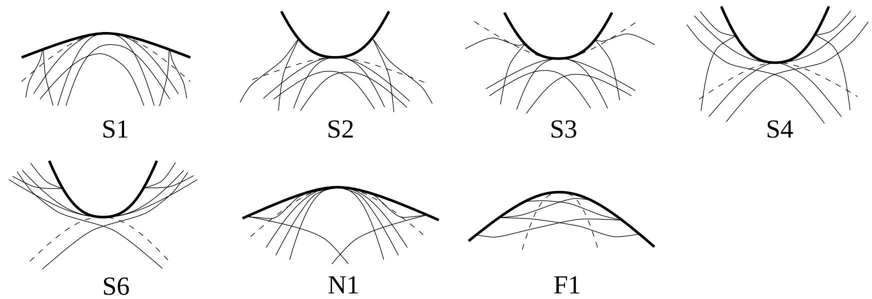

FIG. 2. Configurations of asymptotic lines at a cusp of Gauss.

To get a well folded singular point we need $a \neq 0$, $d \neq 0$, $25a^2 - 24d \neq 0$ and $a^2 - 4d \neq 0$. Now suppose we scale $y \mapsto ty$, then $p \mapsto tp$ and multiplying the equation by t^{-2} we obtain $p^2 + 2((a/t)x + by)p + ((a/t)y + (6d/t^2)x^2 + \ldots)$. If $a/t = 1$ then we have $p^2 + 2(x + 2by)p + (y + (6d/a^2)x^2 + \ldots)$. So in the notation preceding Corollary 2.2.9 we have $u = 2$ and $v = 6d/a^2$. Clearly as a and d vary we can obtain any point on the line $u = 2$. The exceptional points on this line are $v = -6,\ 0,\ 3\ 4,\ 6,\ 25/4$ and the type of folded singular point obtained are (moving from v large negative to v large positive) S6, S4, S3, S2, S1, N1, F1, as described in Figure 2. In other words there are 5 types of saddles but only one type of node and focus.

There are a number of other examples that we would like to consider, but which need to be viewed in a slightly different way. The following discussion explains the problems.

As we have seen many IDE's occur naturally in differential geometry. For example we can consider the lines of curvature on a surface in 3-space. We can of course parameterise that surface by a piece of the plane, and study the BDE in that plane, but there is no reason why the Legendre transform of this BDE or the curve of inflexions should have any geometric significance. On the other hand on the surface there are points where the integral curves have natural analogues of inflexions. These are the geodesic inflexions, where the curve crosses the plane spanned by its tangent line and the normal. It turns out that these points are often of considerable geometrical interest. We saw that for the asymptotic curves these geodesic inflexions are picked up as inflexions in the parameterising plane, but this is for very special reasons. To obtain information in other cases we need to proceed in a slightly different way.

Definition 2.2.12. *A 2-contact element of a curve in the plane is the $\mathcal{R}$-equiva-lence class of the 2-jet of a germ of an immersion $\mathbb{R}, 0 \to \mathbb{R}^2$. (That is we do not distinguish 2-jets of germs differing by a change of co-ordinates in the source.)*

Given a point u in the plane there are a 2-parameter family of such elements through u, determined by the slope of the tangent line through u and the curvature of the curve germ there. For those elements with non-zero curvature we can think of this space as $\mathbb{R}^2$ where a point v determines the centre of a circle through u, and the germ of this circle

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at u determines the contact element. Using this model the space of all contact elements (with non-zero curvature) can be thought of as $\mathbb{R}^2 \times \mathbb{R}^2$. Given a curve $\alpha : I \to \mathbb{R}^2$, the corresponding family of contact elements is parameterised (α, β) where $\beta(t)$ is the centre of curvature of the curve at $\alpha(t)$. We have seen that the following is true.

Lemma 2.2.13. *The 2-contact element of an integral curve of $F(x, y, p) = 0$ at (x, y) is determined by p and $((F_x + pF_y)/F_p)(x, y, p)$*

Now suppose given an IDE on a surface, and on that surface an integral curve through a point. Then clearly this yields a geodesic inflexion if it, and the curve obtained by slicing the surface by the plane containing its tangent and the normal, have ≥ 3-point contact on the surface. Looking at their inverse images in the parameterising space (contact is preserved by diffeomorphisms) this condition is determined by the corresponding 2-contact elements. So for each integral curve in the parameterising plane there is the family of 2-contact elements corresponding to the curve and a family determined by the normal slices of the surface. These two families have the same linear parts, that is determine the same 1-contact elements. We are interested in those points where the 2-contact elements coincide. Thinking of the space of 2-contact elements as $\mathbb{R}^2 \times \mathbb{R}^2$ again, we see that the fact that the elements are tangent means that the centre of the relevant circle for the normal slice family occur on the normals to the original curve, and the points sought are the intersection of this locus of centres with the evolute of the integral curve. The BDE will yield a family of integral curves and hence a family of such points. We will now show that we can generally use the previous models to describe the configuration of cusp locus, geodesic inflexions and separatrices, by showing how to carry out the required calculations.

We shall write our surface in Monge form $z = h(x, y)$.

Lemma 2.2.14. *Suppose given a point (x, y) in the parameterising plane and a direction with slope p there. Let $v(p)$ be the corresponding tangent direction at $(x, y, h(x, y))$ and $C(p)$ the curve given by the intersection of the plane spanned by the normal and $v(p)$ with the surface. Then its inverse image in the (x, y)-plane determines a 2-contact element represented by $(t, pt + qt^2)$ where*

$$q = (-h_y + ph_x)(h_{xx} + 2ph_{xy} + p^2 h_{yy})/2(1 + h_x^2 + h_y^2).$$

If the IDE for the curves on the surface are given by $F(x, y, p) = 0$ (F will be constructed from h in some way) then the geodesic inflexions are given by $-(F_x + pF_y)/F_p = 2q$ that is

$$-(F_x + pF_y)(1 + h_x^2 + h_y^2) = F_p(-h_y + ph_x)(h_{xx} + 2ph_{xy} + p^2 h_{yy}).$$

Proof: For any point (x, y) of the parameterising plane the normal at the corresponding point $(x, y, h(x, y))$, and the tangent direction $v(p)$ is easily computed. One can consequently write down the normal plane, and check that if the germ of the image of the curve $(x + t, y + pt + qt^2)$ was to lie, to second order, in this plane then q has the given form.

Remark 2.2.15. (1) This calculation now allows us to carry out the relevant computations for various geometric BDE's. We have seen that we may always suppose that

$(x, y, p) = (0, 0, 0)$. Moreover we are always studying the IDE in a neighbourhood of a point where $F_p = 0$. From this we deduce that the condition for a geodesic inflexion on an integral curve is $F_x + pF_y = \alpha(x, y, p)F_p$ where α, F_p lie in $\mathcal{M}(x, y, p)$. So the linear part of this condition coincides with that for $F_x + pF_y = 0$. As remarked in 2.10 above this linear part determines the relative positions of the inflexional, cuspidal sets and the separatrices (if any). In other words the models described above will apply.

(2) In the case when the direction is asymptotic we note that $q = 0$, so the 2-contact element has no quadratic part. So we have geodesic curvature zero if and only if the pull-back of the asymptotic curves have inflexions, as we have seen before.

2.2. Application: Conjugate Curve Congruence

In [18] Fletcher constructed a natural family of BDE's that links the asymptotic curves of a smooth surface in $\mathbb{R}^3$ to the lines of curvature. Take the set of all directions in all tangent planes making a fixed (signed) angle $\alpha \in [-\pi/2, \pi/2]$ with their conjugate direction with respect to the second fundamental form of the surface. Note that when $\alpha = 0$ this gives the asymptotic directions, since these are the self conjugate directions, and when $\alpha = \pm\pi/2$ it yields the principal directions. For a fixed angle α these directions are labeled $\mathcal{C}_\alpha$, and called *the conjugate curve congruence*. This family appears to be of some substantial importance. Note also that the conjugate directions, and hence this family, are important in computer vision when dealing with projections of surfaces; see [20].

These directions are given by

$$\begin{gathered}(\sin\alpha(Gm - Fn) - n\cos\alpha\sqrt{EG - F^2})dy^2 + \\ (\sin\alpha(Gl - En) - 2m\cos\alpha\sqrt{EG - F^2})dydx + \\ (\sin\alpha(Fl - Em) - l\cos\alpha\sqrt{EG - F^2})dx^2 = 0,\end{gathered}$$

where E, F, G (e, f, g) are the coefficients of the first (second) fundamental form at (x, y). Away from umbilics the special parametrisation where x-constant and y-constant curves are principal curves can be adopted and the $\mathcal{C}_\alpha$ can be simplified considerably to the form

$$\kappa_2 \cos\alpha dy^2 + (\kappa_2 - \kappa_1)\sin\alpha dydx + \kappa_1 \cos\alpha dx^2 = 0,$$

where $\kappa_i,\ i = 1, 2$ are the principal curvatures at (x, y). The stable structures and the bifurcations in the $\mathcal{C}_\alpha$ curves when α varies are described in [9].

The congruence $\mathcal{C}_\alpha$ defines at most two directions at points in $\mathbb{R}^2$. Take at each point the unit vector corresponding to a given direction. This defines a map-germ from $\mathbb{R}^2 \to S^2$. The fold singularities of this map is the *fold curve*. It turns out that the fold curve of a given direction is the locus of the geodesic inflexions of the integral curves of the other direction ([18]). It follows from Remark 2.2.15 that the linear condition determining the position of the inflexion set of the BDE of $\mathcal{C}_\alpha$ (or the *fold curve* of $\mathcal{C}_\alpha$) is that determined in the models discussed above, when using a co-ordinate chart.

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Suppose the surface is given locally in Monge form $z = h(x, y)$ with

$$h(x,y) = \tfrac{1}{2}\Sigma_{i=0}^{2}\binom{i}{2}a_i x^{2-i}y^i + \tfrac{1}{6}\Sigma_{i=0}^{3}\binom{i}{3}b_i x^{3-i}y^i + \tfrac{1}{24}\Sigma_{i=0}^{4}\binom{i}{4}c_i x^{4-i}y^i + \dots .$$

By taking $\sin\alpha_0 = a_0/\sqrt{a_0^2 + a_1^2}$, $\cos\alpha_0 = -a_1/\sqrt{a_0^2 + a_1^2}$, we fix the x-axis as an element of $\mathcal{C}_{\alpha_0}$ at the origin. The 2-jet of the BDE $\mathcal{C}_{\alpha_0}$ is then given (after scaling) by $Ap^2+2Bp+C$ with

$$\begin{aligned}
A =& 2a_1(a_0 + a_2)\\
B =& (-a_0a_2 + a_0^2 + 2a_1^2)+(2a_1b_1 + a_0b_0 - a_0b_2)x+(a_0b_1 - a_0b_3 + 2a_1b_2)y\\
C =& 2(b_0a_1 - a_0b_1)x + 2(a_1b_1 - a_0b_2)y+\\
& (a_0^3a_1 + a_0a_1^3 - a_0c_1 + c_0a_1)x^2 + 2(-a_0c_2 + a_0^3a_2 + a_1c_1 + a_1^2a_0a_2)xy\\
& +(-a_0a_1^3 + 2a_0^2a_1a_2 - a_0c_3 + a_1c_2 + a_0a_1a_2^2)y^2.
\end{aligned}$$

When $a_0^2 - a_0a_2 + 2a_1^2 = 0$ the origin is on the discriminant of the congruence $\mathcal{C}_{\alpha_0}$. We have generically a well-folded singularity if furthermore $b_0a_1 - a_0b_1 = 0$. By scaling and using the notation in Section 2 we have

$$\begin{aligned}
u &= 1 + (b_0^2 + b_1^2)/(b_1^2 - b_0b_2)\\
v &= b_1(a_0^2 + a_1^2)(a_0^3a_1 + a_0a_1^3 - a_0c_1 + c_0a_1)/2a_0(b_1^2 - b_0b_2).
\end{aligned}$$

In particular, when $\alpha = 0$ we are dealing with the asymptotic lines, and at a cusp of Gauss we have $a_0 = a_1 = b_0 = 0$ and therefore $u = 2$ as in §3.1.

When $\alpha = \pm\pi/2$ we obtain the lines of curvature. This case is dealt with in Section 4. When $0 < \alpha < \pi/2$ it is clear that all the cases in Figure 1 occur in this situation.

3. THE MORSE TYPE 1 CASE

When $F_{pp} \neq 0$ we can rewrite F, after a smooth change of co-ordinates, in the form $p^2 + f(x, y) = 0$. The discriminant of the equation is then given by $f(x, y) = 0$. When f has a Morse singularity so does F and the surface M is singular (an isolated point or a cone). We say that F is of *Morse type 1*. We showed in [9] that the generic topological models in this situation are $p^2 + (\pm x^2 \pm y^2) = 0$. (See also [21] where the behaviour of the integral curves when the discriminant is a node is studied, and applied to gas dynamics.) These phenomena occur generically in 1-parameter families. When the coefficient of x^2 above is $+1$ two well-folded foci appear on one side of the transition and none on the other. An implicit differential equation equivalent to this normal form is labelled *Foci-Morse Type 1*. When the coefficient of x^2 is -1 two well-folded saddles appear on one side of the transition and none on the other ([9]). An equation equivalent to this normal from is labelled *Saddles-Morse Type 1*. Note that we are only interested in the case when $F = 0$ is a cone, not an isolated point. Note also that since the Legendre transform is a diffeomorphism the same will be true for the surface $G = 0$.

When dealing with duality of IDEs we can only use affine changes of co-ordinates, so we can no longer write the equation in a simple from $p^2 + f(x, y) = 0$. However, since the discriminant is singular, the 1-jet of F at the origin is zero and we have the following.

Lemma 3.3.1. *Assume that the point under consideration is the origin, F has zero 1-jet at the origin and $F_{pp}(0,0,0) \neq 0$. Then Saddles and Foci Morse Type 1 equations are distinguished by the 2-jet of F. Writing the 2-jet of F at $(0,0,0)$ as*

$$j^2F = p^2 + 2(b_1x + b_2y)p + ax^2 + 2bxy + cy^2$$

then we have the following.

(i) The discriminant has a Morse singularity if and only if

$$\Delta_F = b^2 - 2bb_1b_2 + ab_2^2 - ac + b_1^2c \neq 0.$$

(ii) Assuming $\Delta_F \neq 0$, the equation is of Saddles-Morse Type 1 if $b_1^2 - a > 0$ and Foci-Morse Type 1 if $b_1^2 - a < 0$.

Proof: For an IDE with $F_{pp} \neq 0$ to be of type *Saddles/Foci-Morse Type 1*, we need the discriminant to have a Morse singularity and its branches (complexify if necessary) to be transverse to the unique direction defined by the equation at the origin. It is equivalent to *Foci-Morse Type 1* if the lifted field on the cylinder, obtained by blowing up M, has no singularities on the exceptional circle. Otherwise, it has 2 saddles singularities on this circle and the equation is equivalent to a *Saddles-Morse Type 1.*

The claim that the *Saddles/Foci-Morse Type 1* condition depends only on the 2-jet of F follows from the fact that the 2-jet of the discriminant of the equation and the condition for it to be transversal to the unique direction at the origin depend only on the 2-jet of F at the origin. The 2-jet of the discriminant is given by

$$j^2\delta_F = (b_1^2 - a)x^2 + 2(b_1b_2 - b)xy + (b_2^2 - c)y^2.$$

This has a Morse singularity if and only if

$$\Delta_F = b^2 - 2bb_1b_2 + ab_2^2 - ac + b_1^2c \neq 0.$$

The branches of the discriminant are transverse to the unique direction $(1,0)$ determined by the equation at the origin if and only if $b_1^2 - a \neq 0$. When $b_1^2 - a > 0$ the lifted field on the blow up of M has two saddles singularities on the exceptional circle. In the case $b_1^2 - a < 0$, the lifted field has no singularities on the exceptional circle.

Theorem 3.3.2. *Let $F = 0$ be an IDE with a Morse Type 1 singularity with 2-jet as above.*

(i) The Legendre transform has a Morse Type 1 singularity if and only if the separatrices of F (when they exist) are not inflexional, the condition being $a \neq 0$.

(ii) The discriminant type (a node or an isolated point) is preserved under duality if $a > 0$ and reversed otherwise.

(iii) The singularity type (saddles, foci) is preserved under duality.

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Proof: We apply the Legendre transformation to F. The 2-jet of the resulting function G is given by

$$j^2G = aP^2 + 2(b_1X - bY)P + X^2 - 2b_2XY + cY^2.$$

We have $G_{PP} \neq 0$ if and and only if $a \neq 0$. Geometrically, this has the following interpretation. When the lifted field on the blow up of the surface M has two saddle singularities on the exceptional circle, this circle is a common separatrix of the field at the two singularities. The remaining separatrices project to smooth tangential curves in the (x, y)-plane with tangent line along the unique direction defined by F at the origin, the x-axis in our setting. So the initial term of these curves, still called separatrices, is given by $y = \alpha x^2$. Substituting this in $F = 0$ yields $4\alpha^2 + 4b_1\alpha + a = 0$. The condition $a \neq 0$ means that the origin is not an inflexion point of one of these curves, i.e. they are parabolas. If $a > 0$ the separatrices bend in the same direction, otherwise they bend in opposite directions.

The 2-jet of the discriminant of G is given by

$$j^2\delta_G = (b_1^2 - a)X^2 + 2(ab_2 - bb_1)XY + (b^2 - ac)Y^2.$$

This has a Morse singularity if and only if $\Delta_G = a\Delta_F \neq 0$. The Morse type of the discriminant is preserved under duality when $a > 0$ and reversed when $a < 0$.

The branches of δ_G are transverse to the unique direction $(1, 0)$ determined by $G = 0$ at the origin if and only if $b_1^2 - a \neq 0$. We have *Saddle-Morse Type 1* equation if $b_1^2 - a > 0$ and a *Foci-Morse Type 1* equation if $b_1^2 - a > 0$. So the type of the singularity is preserved under duality.

We now need to consider the inflexion set of the integral curves of $F = 0$. We recall that this set is given by $F = F_x + pF_y = 0$. The surface $F_x + pF_y = 0$ is locally smooth as $\nabla(F_x + pF_y)(0) = (a, b, b_1)$ and $a \neq 0$. When $F = 0$ defines a cone, which we assume it does here, the surface $F_x + pF_y = 0$ generically intersects this cone in an isolated point or a pair of curves. (The surface $F_x + pF_y = 0$ is tangent to the cone when the discriminant of G has a singularity worse than Morse.) If $b_1 = 0$ the inflexion curves in the (x, y)-plane are certainly tangential. Suppose then that $b_1 \neq 0$. Then eliminating p in $F = F_x + pF_y = 0$ yields an equation in the (x, y)-plane with a 2-jet of the form

$$\delta_{IF} = a(a - b_1^2)x^2 + 2a(b - b_1b_2)xy + (b^2 + b_1^2c - 2bb_1b_2)y^2.$$

We have a non degenerate quadratic if $\Delta_{IF} = b_1^2\Delta_G \neq 0$ resulting in a pair of transverse inflexion curves if $\Delta_{IF} > 0$ and an isolated point if $\Delta_{IF} < 0$. Observe that $\Delta_{IF} = ab_1^2\Delta_F$, so Δ_{IF} and Δ_F have the same Morse type if $a > 0$ and reversed type if $a < 0$. We thus have

Proposition 3.3.3. *Let $F = 0$ be an IDE with a Morse Type 1 singularity and suppose that $j^2F = p^2 + 2(b_1x + b_2y)p + ax^2 + 2bxy + cy^2$. Assume that $a \neq 0$ and $b_1 \neq 0$. Then the inflexion set has a Morse singularity. This singularity is of the same type (a node or an isolated point) as that of the discriminant of $F = 0$ if $a > 0$ and of opposite type if $a < 0$.*

As we assume that $a(b_1^2 - a) \neq 0$ (see Lemma 3.3.1 and Theorem 3.3.2) the inflexion curves are transverse to the unique direction determined by the equation. When δ_F is a node, it separates a neighbourhood of the origin into four sectors and the integral curves of the IDE $F = 0$ lie in two of them (where $\delta_F \geq 0$). Considering one of these sectors we have several configurations depending on whether the unique direction separates the inflexion curves or not. They are separated if $a(a - b_1^2)(b^2 + b_1^2 c - 2bb_1b_2) < 0$ and not separated if $a(a - b_1^2)(b^2 + b_1^2 c - 2bb_1b_2) > 0$.

We also seek to analyse how each case dualises under the Legendre transformation. A calculation shows that the 2-jet of the equation defining the inflexion curves of G (after scaling) is given by

$$\delta_{IG} = (a - b_1^2)X^2 + 2(bb_1 - ab_2)XY + (ab_2^2 + b_1^2 c - 2bb_1b_2)Y^2.$$

We have $\Delta_{IG} = b_1^2 \Delta_F$ and the inflexion curves are separated by the unique direction if $(a - b_1^2)(ab_2^2 + b_1^2 c - 2bb_1b_2) < 0$ and not separated if $(a - b_1^2)(ab_2^2 + b_1^2 c - 2bb_1b_2) > 0$.

To aid the process of identifying all the cases we take j^2F in a simpler form. Since $b_1 \neq 0$ we can make linear changes of co-ordinates and set $b_1 = 1$ and $b_2 = 0$. Then $\Delta_F = b^2 + c - ac$ and the position of the inflexion curves of F is determined by the sign of $a(a-1)(b^2 + c)$. The position of the inflexion curves of G depends on the sign of $(a-1)c$.

Proposition 3.3.4. *Suppose that the equation $F = 0$ is of Morse Type 1, the separatrices through the origin are not inflexional and the inflexion curves are not tangential. Then the local models of the configurations of tangent lines to the cusp and inflexion curves and of the unique direction at the origin together with their corresponding duals are those in Figure 3.*

In Figure 3 the separatrices are also drawn to point out how they curve and to make the distinction between the saddle and foci type.

When $b_1 = 0$ we have the following.

Proposition 3.3.5. *Suppose that the equation $F = 0$ is of Morse Type 1, the separatrices through the origin are not inflexional but the inflexion curves are not transversal. Then the inflexion set generically has an $A_3^{\pm}$-singularity.*

The local models of the configurations of the tangent lines to the cusps and inflexion curves and of the unique direction at the origin, together with their corresponding duals are those in Figure 4.

The proof is a calculation and we omit it.

Remark 3.3.6. *In Proposition 3.3.4 and 3.3.5 the tangent lines to the curves of interest are given. One can refine this classification by considering the actual cusps and inflexion curves and the way they bend in relation to each other. This bending depends on j^3F. However their relative position only depends on j^2F.*

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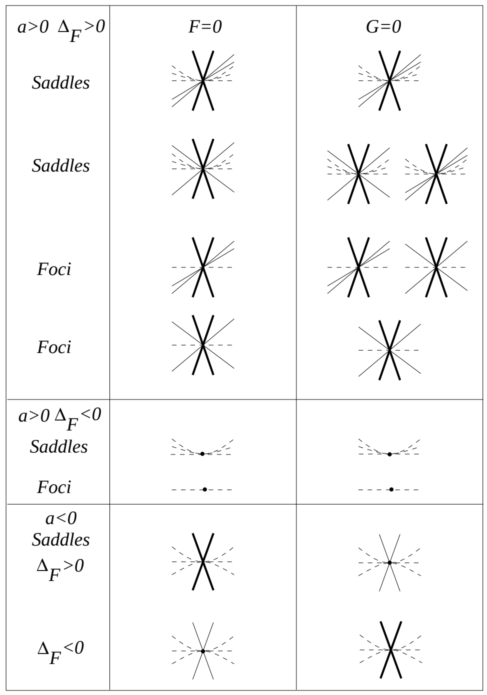

FIG. 3. Configurations of the tangent lines to the cusp set (thick) and inflexion curves (thin) (unique direction and separatrices dashed) and their duals: the inflexion set has a Morse singularity.

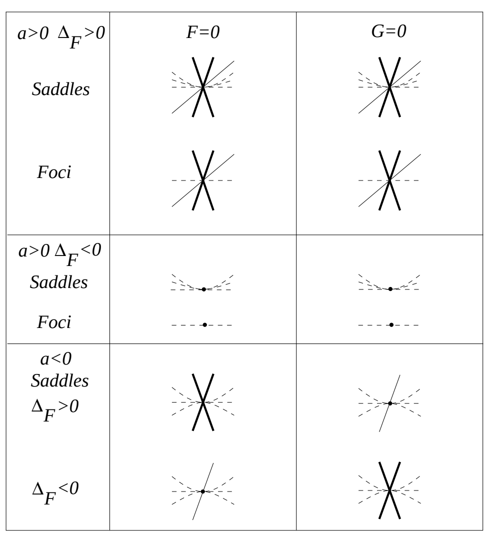

FIG. 4. Configurations of the tangent lines to the cusp (thick) and inflexion set (thin) (unique direction and separatrices dashed) and their duals: the inflexion set has an $A_3^{\pm}$-singularity.

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3.1. Application: Asymptotic curves on surfaces in $\mathbb{R}^3$

Let S be a smooth surface given locally at the origin in a Monge form $z = h(x, y)$. Assume that the origin is a parabolic point so that

$$j^4 h = a_0 y^2 + b_1 y^3 + b_2 x y^2 + b_3 x^2 y + b_4 x^3 + \sum_{i=0}^{i=4} d_i y^{4-i} x^i.$$

It is shown in [5] that when the family of height function is not versal at a cusp of Gauss, the parabolic set has a Morse singularity. This occurs when $b_3 = b_4 = 0$ and $d_5 \neq 0$. The equation of the asymptotic curves $h_{yy} dy^2 + 2h_{xy} dx dy + h_{xx} dx^2 = 0$ has then a *Morse Type 1* singularity. With the above conditions we have $j^1 h_{xy} = 2b_2 y$ so the coefficient of x vanishes. We are then in the situation of Proposition 3.3.5. Hence the inflexion set (the flecnodal curve in this case) generically has an $A_3^{\pm}$-singularity. All the configuration in Figure 4 occur. These depend on the 3-jet of the coefficients of the BDE which in turn depend on the 5-jet of h.

3.2. Application: Conjugate Curve Congruence

We consider the example in Section 2.2. It is easy to check that we can apply the results here because of Lemma 2.13. The *Morse Type 1* singularity occurs on a $\mathcal{C}_{\alpha_0}$ when in addition to the conditions for having a well-folded singularity in Section 2.2, we have $a_1 b_1 - a_0 b_2 = 0$. Then the coefficient of xp in F becomes $2(a_1 b_1 + a_0 b_0)$. This is generically non-zero for $\alpha_0 \neq 0$. So we are in the situation of Proposition 3.3.4. By varying the coefficients of f we obtain all the cases in Figure 3.

4. BDE'S WITH VANISHING COEFFICIENTS

As pointed out in the introduction, BDE's

$$a(x, y) dy^2 + 2b(x, y) dx dy + c(x, y) dx^2 = 0$$

where $a = b = c = 0$ at the origin are of interest in differential geometry and control theory. When the discriminant $\delta = b^2 - ac$ has a Morse singularity the surface M is smooth ([6]). Such BDE's are said to be of *Morse Type 2*. Topological models of these BDE's are given in [3] and [6].

Suppose given a BDE of *Morse Type 2*. Then a linear change of co-ordinates reduces the 1-jet of the coefficients of the BDE to

$$(y, b_1 x + b_2 y, \epsilon y),$$

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with $\epsilon = \pm 1$, (see [6] for the general case and [19] when $\epsilon = -1$). We shall also need to consider the 2-jet of these coefficients, so we write

$$\begin{aligned} j^2a(x,y) &= y + a_{20}x^2 + a_{21}xy + a_{22}y^2, \\ j^2b(x,y) &= b_1x + b_2y + b_{20}x^2 + b_{21}xy + b_{22}y^2, \\ j^2c(x,y) &= \epsilon y + c_{20}x^2 + c_{21}xy + c_{22}y^2. \end{aligned}$$

The lifted field ξ on M has 1 or 3 zeros (saddles or nodes) on the exceptional fibre, with the exceptional fibre being a common separatrix to all of them. The zeros are the roots of the cubic

$$\phi(p) = (F_x + pF_y)(0,0,p) = p(p^2 + 2b_2p + 2b_1 + \epsilon).$$

The eigenvalues of ξ at the zeros of ϕ are

$$\begin{aligned} \alpha_1(p) &= \; 2(p^2 + b_2p + b_1) \\ -\phi'(p) &= \; -(3p^2 + 4b_2p + 2b_1 + \epsilon) \end{aligned}$$

We showed in [6] that the topological type of the above BDE with 1-jet $(y, b_1x + b_2y, \epsilon y)$ depends only on (b_1, b_2), and all the BDE's whose corresponding (b_1, b_2) belong to the same connected component of the (b_1, b_2)-plane bounded by some exceptional curves are topologically equivalent. These exceptional curves are given by the following conditions:

(i) the discriminant fails to be Morse: $b_1 = 0$;
(ii) the cubic ϕ has a repeated root: $2b_1 + \epsilon = 0$ or $b_2^2 - 2b_1 - \epsilon = 0$;
(iii) the polynomials α_1, ϕ have a common root: $b_1 = 0$ if $\epsilon = -1$, and b_1=0 or $b_2^2 = (b_1+1)^2$ if $\epsilon = +1$.

Now on the surface M at the zeros of ξ we know that $F_x + pF_y = 0$. Generally this will be a surface which will cut the surface M transversely. If the three roots of $\phi = 0$ are distinct the condition for non-transverse intersection is that $F_x(0,0,p)$ and $F_y(0,0,p)$ (and consequently $\phi(p)$) have a common root, that is $F = 0$ is not smooth. So we deduce the following:

PROPOSITION 4.4.1. *(i) At each zero of ξ we expect three curves: the p-axis, the other separatrix and the curve corresponding to inflexions. Generically these will have distinct tangents at the zeros. Projecting back down to the (x, y)-plane we are essentially blowing down, so there will be 1 or 3 curves of inflexions 2-point tangent to the corresponding projected separatrices (those distinct from the p-axis). The configuration of these curves depend on the 2-jet of the coefficients a, b, c of the BDE.*

(ii) We can get all possible cases (3 cases when the cubic has 1 root and 18 cases when it has 3 roots) by varying the coefficients a_{ij}, b_{ij}, c_{ij}.

We now consider the Legendre transformation, which of course generally will not be a BDE. Taking an affine chart $p = dy/dx$ of the projective line, we obtain an equation of the form

$$G = a(P, XP - Y)X^2 + 2b(P, XP - Y)X + c(P, XP - Y).$$

(We shall not need to consider the other chart $q = dx/dy$ as no singularity of ξ arises at the point at infinity.) As any value of p is a solution of the original equation $F = 0$, the equation $G = 0$ should be studied in a neighbourhood of the X-axis; the Legendre transform has naturally blown-up the original BDE. We observe that the X-axis is an integral curve of G, that is $G(X, 0, 0) = 0$.

THEOREM 4.4.2. *Suppose given a BDE $F = 0$ of Morse Type 2 with coefficients (a, b, c) with 1-jet $(y, b_1x + b_2y, \epsilon y)$, and (b_1, b_2) off the exceptional curves described above. Then,*

(i) $G_P(X, 0, 0) \neq 0$ if and only if $\phi(X) \neq 0$. So away from the roots of the cubic ϕ on the X-axis, the equation $G = 0$ can be reduced to an ordinary differential equation.

(ii) At the zeros of $\phi(X)$, we generically have $G_{PP}(X, 0, 0) \neq 0$. Furthermore, $G = 0$ has a well-folded singularity at these points. The well-folded singularity is of type saddle (resp. node) if the root of ϕ is a saddle (resp. node) of the field ξ. The X-axis is a common separatrix to all the well-folded singularities.

(iii) The inflexion set of G consists of the X-axis (infinitely degenerate inflexions) when $\epsilon = -1$, and of the union of the X-axis with two smooth curves intersecting this axis transversally at two points distinct from the zeros of ϕ when $\epsilon = +1$.

(iv) The bending of the cusp curve at $(s, 0, 0)$, with s a zero of ϕ, is determined by the sign of $b_1G_{PP}(s, 0, 0)$. The bending of the non-flat separatrix at such a point is determined by the sign of $(\alpha_1(s)+\phi'(s))G_{PP}(s, 0, 0)$, provided $\alpha_1(s)+\phi'(s) \neq 0$. So the relative bending of the cusp curve and the non-flat separatrix depends only on $b_1(\alpha_1(s) + \phi'(s))$. We have $\alpha_1(s) + \phi'(s) = 0$ at a root of ϕ if and only if

$$(3b_1 + 2\epsilon)^2 - 4b_2^2(b_1 + \epsilon) = 0$$

which gives another exceptional set in the (b_1, b_2)-plane.

(v) $G_{PP} = 0$ at s, a zero of ϕ, if and only if the integral curve through the origin of $F = 0$ with a slope s is inflexional. We have

$$\begin{aligned}\tfrac{1}{2}G_{PP}(X, 0, 0) &= a_{22}X^4 + (a_{21} + 2b_{22})X^3 + (a_{20} + 2b_{21} + c_{22})X^2 \\ &\quad +(2b_{20} + c_{21})X + c_{22},\end{aligned}$$

that is, $G_{PP}(X, 0, 0)$ depends only on the 2-jet of the coefficients of the initial BDE. In particular, the bending of the cusp curves at two distinct roots of ϕ depends on the 2-jet of F, and all possible combinations occur by varying the coefficients a_{ij}, b_{ij}, c_{ij}.

(vi) The partition of the (b_1, b_2)-plane into the regions where the relative bending of the cusp curve and the non-flat separatrix at a zero of ϕ is constant is given in Figure 5.

The proof follows from relatively straightforward calculations.

4.1. Application: Asymptotic lines of surfaces in $\mathbb{R}^3$

The coefficients of the BDE giving the asymptotic lines vanish at flat umbilics. These points occur generically in 1-parameter families of surfaces. If the surface is given locally in Monge form $z = h(x, y)$, the origin is a flat umbilic if $j^2h = 0$. Then $h = C(x, y) + h.o.t$ where C is a cubic in (x, y). In general, this cubic is non degenerate, so by linear changes

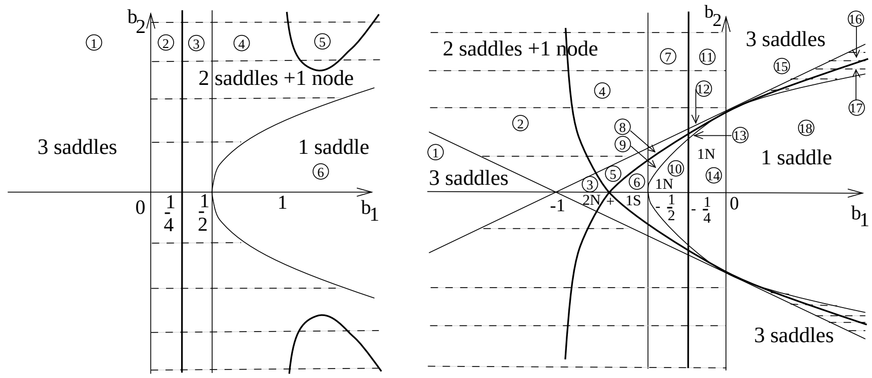

FIG. 5. Partition of the (b_1, b_2) plane ($\epsilon = -1$ left, $\epsilon = +1$ right) by the exceptional curves.

of co-ordinates we may set $C(x, y) = \frac{1}{6}y^3 \pm \frac{1}{2}x^2y$. We have an elliptic flat umbilic ($-$ case) if the cubic has 3 roots and a hyperpolic flat umbilic ($+$ case) if it has 1 root.

The BDE $h_{yy}dy^2 + 2h_{xy}dxdy + h_{xx}dx^2 = 0$ is of *Morse Type 2* at a flat umbilic. The lifted field has 3 saddles at an elliptic umbilic and 1 saddle at a hyperbolic umbilic (see [6]). The bending of the non-flat separatrix and the cusp curve of the Legendre transform G is as in Region 1 (Figure 5, left) for the elliptic flat umbilic and Region 18 (Figure 5, right) for the hypebolic flat umbilic.

We deduce the following from the results in Section 2.1 and Proposition 4.4.1.

PROPOSITION 4.4.3. *There are 3 flecnodal curves at an elliptic flat umbilic and 1 at a hyperbolic flat umbilic. These curves are tangent to the separatrices of the BDE of the asymptotic lines.*

4.2. Application: Sub-parabolic lines of surfaces in $\mathbb{R}^3$

Geodesic inflexions on the lines of curvatures of a smooth surface $S \subset \mathbb{R}^3$ are an important robust feature of the surface. Let p_0 be a point on S which is not an umbilic. The locus of points where the principal curvature is extremal along the other line of curvature is called the *sub-parabolic line* in [12], [24], [22]. It can be characterised, via a computation of the first and second fundamental forms of the focal set, as the locus of points on the surface whose image is the parabolic curve on the focal set [17], [11]. However, as in the case of ridges, the sub-parabolic lines were first and are best described using singularity theory, and an associated family of maps, the folding maps, [24]. A point is sub-parabolic if and only if folding the surface along a the normal plane that contains a principal direction induces a map-germ with a singularity of type $S_{\geq 2}$. The folding map also reveals some fascinating geometry of the surface and its focal set at umbilics. At such points all directions are principal so folding the surface in any normal plane induces a map with a singularity of type cross-cap or worse. On the projective line $\mathbb{R}P^1$ of such directions, there may be 3 or 1 directions where the singularity is of type S_2 (resp. B_2), *i.e.* there are 3 or 1 sub-parabolic lines (resp. ridges) through an umbilic [12]. It turns out, and we shall deduce this from Proposition 4.4.1 and our analysis of the Legendre transform, that the configuration of these sub-parabolic lines is closely related to that of the lines of curvature [12], [22].

A result in [22] characterises the sub-parabolic line as the locus of points where the other lines of curvature have a geodesic inflexion. Using the results in Section 2.1, Proposition 4.4.1 and Theorem 4.4.2 we deduce the following.

PROPOSITION 4.4.4. *(1) The linear part of the inflexion set of the BDE of the lines of curvature of a smooth surface in* $\mathbb{R}^3$ *corresponds to the linear part of the sub-parabolic lines.*

(2) There are 3 (at a Star or Monstar) or 1 (at a Lemon) sub-parabolic lines through a generic umbilic, and these curves are tangent to the corresponding lines of curvatures through the umbilic point. (Compare [12], [22], [24].)

We observe that the discriminant of the BDE of the lines of curvature consists of the umbilic points. So the 1-jet of the coefficients of this BDE are of the form $(y, b_1x+b_2y, -y)$ at an umbilic point, where b_1, b_2 depend only on the coefficients of the cubic part of the function defining the surface in Monge form at the umbilic. All the cases in Figure 5 left occur for the Legendre transform of this BDE.

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