SMA 5878 Functional Analysis II

Alexandre do Nascimento
Javier Cubas

Departamento de Matemática
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo

April 7, 2018
Appendix B

Weakly Compactness
In this lecture we present the proofs of Eberlein-Šmulian and Krein-Šmulian theorems.

Before we begin, remember the very useful result:

**Theorem**

*If $K$ is a convex subset of a Banach space $X$, then the closure of $K$ in the weak and strong topology coincide.*
Compactness in general topological spaces cannot be characterized with sequences as we do in metric spaces. However, in the weak topology is possible to do it as follows:

**Theorem (Eberlein-Šmulian)**

Let $W$ be a subset of a Banach space $X$. The following properties are equivalent:

(A) The closure of $W$ in the weak topology is compact in the weak topology.

(B) Every sequence of elements of $W$ has a weakly convergent subsequence in $X$.

(C) Every sequence of elements of $W$ has a limit point in $X$. 
Lemma (B.1.1)

If $X$ is a separable Banach space, then $X^*$ has a countable total subset $A^* = \{a_n^*: n \in \mathbb{N}\}$ with the property

$$\|x\|_X = \sup_{n \in \mathbb{N}} |\langle x, a_n^* \rangle|_{X^*,X^*}.$$

Proof:
Let $\{a_n\}_\mathbb{N}$ a sequence of unitary vectors which are dense in the unitary sphere of $X$. For each $n \in \mathbb{N}$, let $a_n^*$ such that

$$\langle a_n, a_n^* \rangle_{X^*,X^*} = \|a_n\|_X = \|a_n^*\|_{X^*} = 1.$$

We show that $A^* = \{a_n^*: n \in \mathbb{N}\}$ is total.
**Lemma B.1.1**

If $y \in X - \{0\}$, with $a_n^*(y) = 0$ for all $n \in \mathbb{N}$, take $x = y/\|y\|$, then $\|x\|_X = 1$ and $a_n^*(x) = 0$.

Let $\{a_{n_k}\}_{k \in \mathbb{N}}$ a subsequence of $\{a_n\}_{n \in \mathbb{N}}$ such that $a_{n_k} \xrightarrow{k \to \infty} x$, then the sequences

$$\langle a_{n_k}, a_{n_k}^* \rangle x, x^* \text{ and } \langle x, a_{n_k}^* \rangle x, x^*$$

have the same limit when $k$ goes to infinity and they are constant, i.e.,

$$\langle x, a_{n_k}^* \rangle x, x^* = 0 \text{ and } \langle a_{n_k}, a_{n_k}^* \rangle x, x^* = 1,$$

which is a contradiction, q.e.d.
Lemma B.1.1

Now, we prove the second claim. For any $x \in X$ with $\|x\|_X = 1$, there is a subsequence $\{a_{n_k}\}_{k \in \mathbb{N}}$ of $\{a_n\}_{n \in \mathbb{N}}$ such that $a_{n_k} \xrightarrow{k \to \infty} x$ and

$$1 = \|x\|_X = \sup_{\|x^*\|_{X^*} = 1} |\langle x, x^* \rangle_{X, X^*}| \geq \sup_{n \in \mathbb{N}} |\langle x, a_n^* \rangle_{X, X^*}|$$

$$= \lim_{k \to \infty} \sup_{n \in \mathbb{N}} |\langle a_{n_k}, a_n^* \rangle_{X, X^*}| \geq \lim_{k \to \infty} |\langle a_{n_k}, a_{n_k}^* \rangle_{X, X^*}| = 1$$

and consequently, for all $x \in X$,

$$\|x\|_X = \sup_{n \in \mathbb{N}} |\langle x, a_n^* \rangle_{X, X^*}|.$$
Lemma (B.1.2)

Let $X$ be a Banach space over $\mathbb{K}$ such that $X^*$ contains an countable total set. Then a weakly compact subset of $X$ is metrizable.

**Proof:** Let $A^* = \{ a_n^* : n \in \mathbb{N} \}$ be a total set of $X^*$ with $\| a_n^* \|_{X^*} = 1$ for all $n \in \mathbb{N}$ and define $d : X \times X \to \mathbb{R}^+$ the metric define by $d(x, y) = \sum_{n=0}^{\infty} 2^{-n} | \langle x - y, a_n^* \rangle_{X,X^*} |$. If $W \subset X$ is weakly compact, note that $\langle W, x^* \rangle_{X,X^*}$ is a compact subset of $\mathbb{K}$ and, of the Uniform Boundedness Principle, $W$ is a bounded set of $X$ ($\| W \|_X := \sup_{w \in W} \| w \|_X < \infty$).
Lemma B.1.2

If $W_w$ and $W_d$ denote the set $W$ with weak topology and metric topology of $d$, respectively, let $I : W_w \to W_d$ the identity operator. If $I : W_w \to W_d$ is continuous then it is homeomorphism (since that $W$ is weakly compact) and the result follows. Lack to prove that $I : W_w \to W_d$ is continuous. Indeed, given $\epsilon > 0$ let $N \in \mathbb{N}$ such that

$$
\sum_{n=N+1}^{\infty} 2^{-n} |\langle x - y, a_n^* \rangle x, x^*| \leq \sum_{n=N+1}^{\infty} 2^{-n} \| W \| x < \frac{\epsilon}{2}
$$

and $V = \{ y \in W : |\langle x - y, a_n^* \rangle x, x^*| < \frac{\epsilon}{2(N+1)}, n = 0, 1, \ldots, N \}$ be a neighborhood of $x$ in $W_w$ such that

$$d(x, y) < \epsilon, \forall y \in V,$$

This concludes the proof. $\square$
**Corollary B.1.1**

*Corollary (B.1.1)*

In the same hypothesis of the Eberlein Smulian ‘s Theorem, we have that (A) implies (B).

**Proof:**

Let \( \{w_n\}_{n \in \mathbb{N}} \) a sequence in \( W \) and \( Y := \overline{\text{span}}\{w_n : n \in \mathbb{N}\} \).

Since \( Y \) is also closed in the weak topology, see Theorem 1 and the weak topology is Hausdorff, the subset \( W \cap Y \) has compact closure in the weak topology in the Banach space \( Y \).
Corollary B.1.1

Now, $Y$ is separable and by the Lemmas B.1.1 and B.1.2, we have that $\mathcal{W} \cap Y$ with the weak topology is metrizable, and therefore $\{w_n\}_{n \in \mathbb{N}}$ has a weakly convergent subsequence to an element of $Y$, i.e.,

$$w_{n_k} \rightharpoonup y \in Y, \text{ in } Y,$$

but $X^* \subset Y^*$, then $w_{n_k} \rightharpoonup y \in X$, in the weak topology of $X$. \qed
Lemma (B.1.3)

(C) ⇒ (A).

Proof:
If every sequence of $W$ has a limit point in $X$, for a given $x^* \in X^*$ the subset $\langle W, x^* \rangle_{X,X^*}$ of $K$ has the same propriety in $K$. It follows that $\langle W, x^* \rangle_{X,X^*}$ is a bounded subset of $K$ and by Uniform boundedness Principle $W$ is bounded.
Lemma B.1.3

Let \( J : X \to X^{**} \) be the canonical application.

Since \( J(W) \) is bounded, the closure \( w^*(J(W)) \) of \( J(W) \) in the weak* topology of \( X^{**} \) is compact, Banach-Alaoglu ‘s Theorem.

We claimed its enough to show \( w^*(J(W)) \subseteq J(X) \).

(Then, of course, the case when \( X \) is reflexive is far more easier).
Lemma B.1.3

Let $x^{**} \in w^*(JW)$ and $x_1^* \in X^*$, $\|x_1^*\|_{X^*} = 1$, $w_1 \in W$ with

$$|\langle x_1^*, x^{**} - Jw_1 \rangle_{X^*, X^{**}}| < 1.$$ 

Before we proceed, let $F$ a finite dimensional subspace of $X^{**}$. The unitary sphere of $F$ is compact and therefore has a $\frac{1}{4}$-net $\{x_{1}^{**}, \ldots, x_{n}^{**}\}$.

Choose $x_p^*$ on the unitary sphere of $X^*$ such that

$$\langle x_p^*, x^{**} \rangle_{X^*, X^{**}} > \frac{3}{4}, \quad 1 \leq p \leq n.$$ 

then, for any $x^{**} \in F$ we have that

$$\max\{|\langle x_p^*, x^{**} \rangle_{X^*, X^{**}}| : 1 \leq p \leq n\} \geq \frac{1}{2}\|x^{**}\|_{X^{**}}.$$
Lemma B.1.3

Now choose $x_2^*, \ldots, x_{n_2}^*$ in $X^*$, $\|x_m^*\|_{X^*} = 1$ and

$$\max\{ |\langle x_m^*, y^{**}\rangle x^*_m, x^{**}| : 2 \leq m \leq n_2 \} \geq \frac{1}{2} \|y^{**}\|_{X^{**}}$$

for all $y^{**} \in \text{span}\{x^{**}, x^{**} - Jw_1\}$. Using again that $x^{**} \in w^*(J(W))$, choose $w_2 \in W$ such that

$$\max\{ |\langle x_m^*, x^{**} - Jw_2\rangle x^*_m, x^{**}| : 1 \leq m \leq n_2 \} < \frac{1}{2}.$$
Lemma B.1.3

Choose \( x_{n_2+1}, \ldots, x_{n_3} \) in the unitary sphere of \( X^* \) such that

\[
\max\{|\langle x_m, y^{**}\rangle x^*, x^{**}| : n_2 < m \leq n_3\} \geq \frac{1}{2}\|y^{**}\| x^{**}
\]

for all \( y^{**} \in \text{span}\{x^{**}, x^{**} - Jw_1, x^{**} - Jw_2\} \) and, using again that \( x^{**} \in \omega^*(J(W)) \) choose \( w_3 \in W \) such that

\[
\max\{|\langle x_m, x^{**} - Jw_3\rangle x^*, x^{**}| : 1 \leq m \leq n_3\} < \frac{1}{3}.
\]

This process can be continued indefinitely. Let \( \{w_n\}_{n \in \mathbb{N}} \) be the sequence resulting from this construction.
By hypothesis, there exist a point $x \in X$ which is a limit point of the sequence $\{w_n\}_{n \in \mathbb{N}}$ in the weak topology of $X$.

Since $Z = \overline{\text{span}}\{w_n : n \in \mathbb{N}\}$ is weakly closed, $x \in Z$ and take $R^{**} = \text{span}\{x^{**}, x^{**} - Jw_1, x^{**} - Jw_2, \cdots\}$.

We have that, for all $y^{**} \in R^{**}$,

$$\sup_{m \in \mathbb{N}} |\langle x_m^*, y^{**} \rangle x_m^*, x^{**}| \geq \frac{1}{2} \|y^{**}\| x^{**}$$

and therefore, for any point in the closure of $R^{**}$, in particular for $x^{**} - Jx$. 
Lemma B.1.3

Another characteristic of our construction is that

\[
|\langle x^*_m, x^{**} - Jw_n \rangle x^*, x^{**} | < \frac{1}{p}, \quad n > n_p > m.
\]

therefore, for \( n > n_p > m \),

\[
|\langle x^*_m, x^{**} - Jx \rangle x^*, x^{**} | \leq |\langle x^*_m, x^{**} - Jw_n \rangle x^*, x^{**} | + |\langle w_n - x, x^*_m \rangle x, x^* |
\]

Since \( x \) is a limit point of \( \{ w_n \}_{n \in \mathbb{N}} \) in the weak topology, given \( x^*_m \) and an integer \( N > m \) there exist \( w_n \) with \( |\langle w_n - x, x^*_m \rangle x, x^* | < \frac{1}{N} \) e \( n > n_N > m \). For this element we have

\[
|\langle x^*_m, x^{**} - Jx \rangle x^*, x^{**} | \leq |\langle x^*_m, x^{**} - Jw_n \rangle x^*, x^{**} | + |\langle w_n - x, x^*_m \rangle x, x^* | < \frac{2}{N}
\]
Lemma B.1.3

and, consequently $\langle x^*_m, x^{**} - Jx \rangle_{x^*,x^{**}} = 0$ for all $m$. As seen above

$$\sup_{m \in \mathbb{N}} |\langle x^*_m, x^{**} - Jx \rangle_{x^*,x^{**}}| \geq \frac{1}{2} \|x^{**} - Jx\|_{x^{**}}$$

and therefore $x^{**} = Jx$. This concludes the proof.
Before we begin, we present an important auxiliar result.
Lemma B.2.1

Lemma
Let $X$ be a separable Banach space and $x^{**} \in X^{**}$. Suppose for all $x^* \in X^*$ and sequence $\{x_n^*\}$ in $X^*$ which converges to $x^*$ in the weak* topology; this is, $\langle x, x_n^* \rangle \xrightarrow{n \to \infty} \langle x, x^* \rangle$ for all $x \in X$, we have that $\langle x_n^*, x^{**} \rangle \xrightarrow{n \to \infty} \langle x^*, x^{**} \rangle$. Then $x^{**} = Jx$ for some $x \in X$.

Proof: Let $\{x_j\}_{j \in \mathbb{N}}$ a dense subset of $X$. Suppose that $x^{**} \notin JX$; this is, that $d(x^{**}, JX) = d > 0$. By the Hahn-Banach Theorem, there exist $x^{***} \in X^{***}$ such that, $\|x^{***}\|_{X^{***}} = 1$, $\langle JX, x^{***} \rangle_{X^*, X^{***}} = 0$ e $\langle x^{**}, x^{***} \rangle_{X^{**}, X^{***}} = d$. Let

$$W_n = \{z^* : |\langle x_i, z^* \rangle_{X, X^*}| < 1 \text{ for } i = 1, \cdots, n\}.$$
By the Goldstine Theorem ($JX^*$ is dense in $X^{***}$ with the weak$^*$ topology of $X^{***}$), given $Jx_1, \ldots, Jx_n, x^{**} \in X^{**}$ and $\epsilon > 0$, there exist $x^* \in X^*$, $\|x^*\|_{X^*} = 1$, such that

$$|\langle x_1, x^* \rangle| = |\langle x_1, x^* \rangle x, x^* - \langle Jx_1, x^{**} \rangle x^{**}, x^{***} | = |\langle Jx_1, Jx^* - x^{***} \rangle x^{**}, x^{***} | < \epsilon,$$

$$|\langle x_n, x^* \rangle| = |\langle x_n, x^* \rangle x, x^* - \langle Jx_1, x^{**} \rangle x^{**}, x^{***} | = |\langle Jx_n, Jx^* - x^{***} \rangle x^{**}, x^{***} | < \epsilon,$$

$$|\langle x^*, x^{**} \rangle x^*, x^{**} - \langle x^{**}, x^{***} \rangle x^{**}, x^{***} | = |\langle x^{**}, Jx^* - x^{***} \rangle x^{**}, x^{***} | < \epsilon.$$
Therefore, there exist a functional 
\( x_n^* \in B_{1}^{X^*}(0) \cap \{ z^* \in X^* : |\langle z^*, x^{**} \rangle x^*, x^{**} | \geq d/2 \} \cap W_n \). The sequence \( \{ x_n^* \} \) converges to zero in the weak* topology of \( X^* \) indeed, given \( x \in X \) and \( \varepsilon > 0 \), there exist \( x_j \) with \( \|(x/\varepsilon) - x_j\| < 1 \) and

\[
|\langle x/\varepsilon, x_n^* \rangle x, x^* | \leq |\langle x/\varepsilon - x_j, x_n^* \rangle x, x^* | + |\langle x_j, x_n^* \rangle x, x^* | < 2, \quad \text{for all } n \geq j.
\]

However, \( |\langle x_n^*, x^{**} \rangle x^*, x^{**} | \geq d/2 \) and this gives us a contradiction, this concludes the proof of lemma. \( \square \)

We are now in a position to prove the Krein-Šmulian Theorem.
Using Eberlein Smulian result we the following theorem which is already known in the strong topology.

**Theorem (Krein-Šmulian)**

If $X$ is a Banach space and $K \subset X$ is weakly compact, then the closed convex hull $\overline{co}K$ of $K$ is weakly compact.
Proof of K-S Theorem

First we reduced our prove to the case that $X$ is separable. From Eberlein Smulian Theorem, its enough to show that every sequence of elements of $\overline{co}K$ has a weakly convergent subsequence.

Since each element of a sequence of $\overline{co}K$ is a convex finite linear combination of elements of $K$, the sequence will be generate by a sequence $S$ in $K$.

Let $Y$ be the closure of the subspace generate by $S$, its enough to show that $\hat{K} = K \cap Y$ is weakly compact (Theorem 1) in $Y$. 


Hence, it’s enough to consider $X$ separable and $K$ weakly compact. Define $K_w$ as the subset $K$ with the weak topology.

Consider $T : X^* \rightarrow C(K_w)$ defined by $Tx^*(k) = x^*(k)$, $k \in K$ and the dual operator of $T$, $T^* : C(K_w)^* \rightarrow X^{**}$.

Choose any element of $C(K_w)^*$, by the Riesz representation Theorem is a regular measure $\mu$, and let $\{x_n^*\}_{n \in \mathbb{N}}$ a bounded sequence that converges in the weak$^*$ topology of $X^*$ to $x^*$. 
Then, by the dominated convergence Theorem,

\[ \langle x_n^*, T^* \mu \rangle_{X^*, X^{**}} = \int x_n^*(k) \, d\mu(k) \xrightarrow{n \to \infty} \int x^*(k) \, d\mu(k) = \langle x^*, T^* \mu \rangle_{X^*, X^{**}}. \]

From Lemma B.2.1 we have that \( T^* \mu \in JX. \)

Now, take the unitary disk in \( C(K_w)^* \), which is convex and compact in weak* topology. Then, the range of the disk by \( J^{-1} T^* \) is also convex, weakly compact and contains \( K. \)

This shows that \( \overline{co}(K) \) is weakly compact and finishes the proof.

\[ \square \]